

Quantum Field Theory C (215C) Spring 2013 Assignment 7

Posted May 28, 2015

Due 11am, **Tuesday, June 9, 2015**

Problem Set 7

1. **Large- N saddle points** [from Halpern]

Consider the partition function for an N -vector of scalar fields in D dimensions

$$Z = \int [D\phi] e^{iS[\phi]}, \quad S[\vec{\phi}] = \int d^D x \left(\frac{1}{2} \partial\phi^a \partial\phi^a - NV \left(\frac{\vec{\phi}^2}{N} \right) \right)$$

with a general 2-derivative $O(N)$ -invariant action. We're going to do this path integral by saddle point, which is a good idea at large N . As usual, the constant prefactors in Z drop out of physical ratios so you should ignore them.

(a) Change variables to the $O(N)$ singlet field $\zeta \equiv \vec{\phi}^2/N$ by inserting the identity

$$1 = \int [D\zeta] \delta \left[\zeta - \frac{\vec{\phi}^2}{N} \right]$$

into the path integral representation for Z . Represent the functional delta function as

$$\delta \left[\zeta - \frac{\vec{\phi}^2}{N} \right] = \int [D\sigma] e^{i \int d^D x \sigma (\vec{\phi}^2 - \zeta N)}.$$

Do the integral over ϕ^a to obtain

$$Z = \int [D\zeta D\sigma] e^{iN S_{\text{eff}}[\zeta, \sigma]}.$$

Determine $S_{\text{eff}}[\zeta, \sigma]$.

(b) The integrals over ζ, σ have a well-peaked saddle point at large N . Obtain the coupled large- N saddle point equations for the saddle point configurations ζ_0, σ_0 , and in particular the equation

$$\zeta_0(x) = \left(\frac{\mathbf{i}}{-\square - 2V'(\zeta_0)} \right)_{xx}$$

(the subscript denotes a matrix element of the position-space operator).

(c) [more optional] Show that

$$\frac{\delta}{\delta\sigma(x)} \text{tr} \log(-\square + \sigma) = \left(\frac{1}{-\square + \sigma} \right)_{xx}$$

by Taylor expansion.

(d) At large N , we know that

$$\zeta_0(x) \stackrel{N \rightarrow \infty}{\equiv} \left\langle \frac{\vec{\phi}^2(x)}{N} \right\rangle = \zeta_0, \text{ constant.}$$

Use this to show that the saddle point equation is the *gap equation*

$$\zeta_0 = \int \bar{d}^D k_E \frac{1}{k_E^2 + 2V'(\zeta_0)}$$

which determines ζ_0 , the expectation value of the order parameter $\langle \vec{\phi}^2/N \rangle$.

(e) What class of diagrams did you just sum?

(f) **Toy model emphasizing the similarity to our discussion of BCS.** You might be worried that in the discussion of BCS I didn't introduce the analog of ζ (which is the actual pair field). In this small, very optional, sub-problem I want to show that what we did there is the same as what we did here. Consider the integral

$$Z = \int d^2x e^{-a|x|^2 - u|x|^4};$$

here x is a proxy for the fermion field ψ , but it is just a complex number. Now we insert

$$1 = \int d^2\zeta \delta[\zeta - x^2] \delta[\bar{\zeta} - \bar{x}^2] = \int d^2\zeta \int d^2\sigma e^{i\bar{\sigma}(\zeta - x^2) + i\sigma(\bar{\zeta} - \bar{x}^2)}$$

to find

$$Z = \int d^2\zeta \int d^2\sigma e^{-u|\zeta|^2 + i(\sigma\bar{\zeta} + \bar{\sigma}\zeta)} \int d^2x e^{-a|x|^2 - i\bar{\sigma}x^2 - i\sigma\bar{x}^2}$$

Do the x integral and find

$$Z = \int d^2\zeta \int d^2\sigma e^{-u|\zeta|^2 + i(\sigma\bar{\zeta} + \bar{\sigma}\zeta) - \log(a^2 - |\sigma|^2)}.$$

Now we can do the ζ integral to get

$$Z = \int d^2\sigma e^{-\frac{1}{u}|\sigma|^2 - \log(a^2 - |\sigma|^2)}$$

which is what we would have found using the usual HS trick as described in lecture. The relationship between ζ and σ on-shell (*i.e.* at the saddle point of the (gaussian) ζ integral) is $\zeta \sim i\sigma$, but really they are conjugate variables. Now do the σ integral by saddle point to find the analog of the gap equation for this 0+0 dimensional QFT.

(g) [super-optional] Show that the BCS interaction is a marginally relevant perturbation of the Fermi liquid fixed point(s). (For help, see the papers by Polchinski or by Shankar cited in the lecture notes.)

2. **Chiral anomaly in two dimensions.** [extra credit]

Consider a massive relativistic Dirac fermion in 1+1 dimensions, with

$$S = \int dxdt \bar{\psi} (\mathbf{i}\gamma^\mu (\partial_\mu + eA_\mu) - m) \psi.$$

By heat-kernel regularization of its expectation value, show that the divergence of the axial current $j_\mu^5 \equiv \mathbf{i}\bar{\psi}\gamma^m u\gamma^5\psi$ is

$$\partial_\mu j_\mu^5 = 2\mathbf{i}m\bar{\psi}\gamma^5\psi + \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$

3. **Topological terms in QM** [extra credit] [from Abanov]

The euclidean path integral for a particle on a ring with magnetic flux $\theta = \int \vec{B} \cdot d\vec{a}$ through the ring is given by

$$Z = \int [D\phi] e^{-\int_0^\beta d\tau (\frac{m}{2}\dot{\phi}^2 - \mathbf{i}\frac{\theta}{2\pi}\dot{\phi})}.$$

Here

$$\phi \equiv \phi + 2\pi \tag{1}$$

is a coordinate on the ring. Because of the identification (1), ϕ need not be a single-valued function of τ – it can wind around the ring. On the other hand, $\dot{\phi}$ is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

$$\phi(\tau) = \frac{2\pi}{\beta} Q\tau + \sum_{\ell \in \mathbb{Z}} \phi_\ell e^{\mathbf{i}\frac{2\pi}{\beta}\ell\tau}. \tag{2}$$

- (a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.
- (b) Using the decomposition (2), write the partition function as a sum over topological sectors labelled by the *winding number* $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula

$$\sum_{n \in \mathbb{Z}} e^{-\frac{1}{2}tn^2 + \mathbf{i}zn} = \sqrt{\frac{2\pi}{t}} \sum_{\ell \in \mathbb{Z}} e^{-\frac{1}{2t}(z - 2\pi\ell)^2}.]$$

- (c) Use the result from the previous part to determine the energy spectrum as a function of θ .
- (d) [more optional] Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

- (e) Consider what happens in the limit $m \rightarrow 0, \theta \rightarrow \pi$ with $X \equiv \frac{\theta - \pi}{M} \sim \beta^{-1}$ fixed. Interpret the result as the partition function for a spin 1/2 particle. What is the meaning of the ratio X in this interpretation?

The lesson here is that (even in this simple QM example) total derivative terms in the action do affect the physics.

4. **Geometric Quantization of the 2-torus.** [extra credit]

Redo the analysis that we did in lecture for the two-sphere for the two-torus, $S^1 \times S^1$. The coordinates on the torus are $(x, y) \simeq (x+2\pi, y+2\pi)$; use $Ndx \wedge dy$ as the symplectic form. Show that the resulting Hilbert space represents the heisenberg algebra

$$e^{i\mathbf{x}} e^{i\mathbf{y}} = e^{i\mathbf{y}} e^{i\mathbf{x}} e^{\frac{2\pi i}{N}}.$$

(I am using boldface letters for operators.) The irreducible representation of this algebra is the same Hilbert space as a particle on a periodic one-dimensional lattice with N sites.

5. **Particle on a sphere with a monopole inside.** [extra credit]

Consider a particle of mass m and electric charge e with action

$$S[\vec{x}] = \int dt \left(\frac{1}{2} m \dot{\vec{x}}^2 + e \dot{\vec{x}} \cdot \vec{A}(\vec{x}) \right)$$

constrained to move on a two sphere of radius r in three-space, $\vec{x}^2 = r^2$. Suppose further that there is a *magnetic monopole* inside this sphere: this means that $4\pi g = \int_{S^2} \vec{B} \cdot d\vec{a} = \int_{S^2} F$, where $F = dA$. (Since the particle lives only at $\vec{x}^2 = r^2$, the form of the field in the core of the monopole is not relevant here.)

- Find an expression for $A = A_i dx^i = A_\theta d\theta + A_\varphi d\varphi$ such that $F = dA$ has flux $4\pi g$ through the sphere.
- Show that the Witten argument gives the Dirac quantization condition $2eg \in \mathbb{Z}$.
- Take the limit $m \rightarrow 0$. Count the states in the lowest Landau level. It should agree with the result from lecture.

6. **Coherent state quantization** [extra credit]

- Start with first order action $S = \int dt z_\alpha^\dagger \dot{z}_\alpha$. Show that the Hamiltonian is $\mathbf{H} = 0$.
- Check the completeness relation in the spin 1/2 coherent state basis.
- Show that different spinor representations, *i.e.* different choices of ψ in

$$z = \begin{pmatrix} e^{i(\psi + \varphi/2)} \cos \theta/2 \\ e^{i(\psi - \varphi/2)} \sin \theta/2 \end{pmatrix}$$

shift the coefficient of the total derivative $\dot{\varphi}$ part of the WZW functional.