Conformal Bootstrapping

Mark Derdzinski[1]

[1]*Department of Physics, University of California at San Diego, La Jolla, CA 92093*

In studying conformal field theories (CFTs), we often would like to identify the properties of the critical points, such as the critical exponents. Techniques to determine the critical exponents in certain dimensions based on universality do not exploit the conformal symmetry, and have certain ambiguities. Here we explore the "conformal bootstrap" technique to extract operator dimensions from operator product expansions. While the technique is old, recent progress has been made in its application to extract the critical exponents of some CFTs and classify critical points based on OPEs. [1]

INTRODUCTION

Among the many reasons to study conformal field theory (CFT), the conformal symmetry is integral in the theory of critical phenomena. Why conformal symmetry emerges at the critical point is not fully understood but appears to be a genuine feature of criticality, and so we assume the conformal invariance of the critical point. Studying the properties of critical points is key to understanding critical phenomena in both classical and quantum systems, and the most important parameters characterizing the critical points are the critical exponents.

Consider a prototypical $O(n)$ ferromagnet in $D = 3$ dimensions, with the partition function

$$Z = \exp\left[-\frac{1}{T}\sum_{\langle ij\rangle}\sigma_i \cdot \sigma_j\right]$$

where σ_i are unit N-component vectors arranged on a lattice (for example, in the simplest case where $N = 1$, $\sigma = \pm 1$, we recover the Ising model). The two-point spin correlation functions behave like a power law at short distances and exponentially decay far away [1]:

$$\langle\sigma(r)\sigma(0)\rangle = \begin{cases} r^{-2\Delta_\sigma}, & \text{if } r \ll \xi \\ r^{-(D-1)/2}\exp(-r/\xi), & r \gg \xi \end{cases}$$

ξ is the correlation length at temperature T. At $T = T_c$, ξ diverges and the correlator is scale-invariant [1]. The parameter Δ_σ is referred to as the *spin field dimension* (and corresponds to the critical exponent η often found in literature). There is also an analogously defined *energy field dimension* Δ_ϵ:

$$\langle\epsilon(r)\epsilon(0)\rangle = r^{-2\Delta_\epsilon}$$

where ϵ is the energy field, and the lowest-dimension scalar operator which is a singlet of the global symmetry. For example, $\mathbb{Z}_2$ is the global symmetry for the Ising model, where ϵ is also the only *relevant* singlet scalar since we only need to tune one parameter for criticality, T (in other systems with multicritical behavior, multiple scalars of this type could exist). Determining the critical

exponents is of great interest in studying critical behavior since they parametrize the physics of systems near criticality, and are believed to be universal.

DETERMINING CRITICAL EXPONENTS

Previous techniques for computing 3D critical exponents rely heavily on universality, *i.e.* that models with different UV structure reduce to the same CFT at the IR critical point (with the same critical exponents), provided they share the same global structure (global symmetries, dimension, etc.). Critical exponents can be measured experimentally by taking any real system with the required symmetry, such as a magnet, and recording the critical properties [2]. Numerically, the critical exponents can also be computed on a lattice simulation of such systems [3].

Calculating the critical exponents (that is, without experiment or numerical simulation) has traditionally been done using the ε-expansion, by extending the theory to $4 - \varepsilon$ dimensions and expanding perturbatively in ε. Assuming the line of fixed points continuously connects to the 3D fixed point as $\varepsilon \to 1$, the 3D critical exponents can be extracted. However, the perturbative series in these calculations is only asymptotically convergent – for $\varepsilon = 1$, the divergence appears after only a few terms, and the resummation procedures required to improve the precision have a theoretical uncertainty associated with free parameters.

USING CONFORMAL SYMMETRY

In the $D = 2$ case, there is a classification of critical points based on the conformal symmetry [1], which allows the 2D ising model critical exponents to be calculated exactly:

$$\Delta_\sigma = 1/8, \ \Delta_\epsilon = 1 \ (\text{2D Ising})$$

Using the conformal symmetry to extract the critical exponents appears to be a powerful tool, except that the conformal symmetry is only present at the critical point

– because it does not appear along the RG flow, it is left unused in the $4 - \varepsilon$ expansion calculations.

The idea of using CFT techniques and operator product expansions (OPEs) to classify fixed points is not new [4, 5], but the practical implementation has only appeared recently. The crux of the strategy is to use consistency conditions and the conformal crossing symmetry to turn the calculation into an optimization problem, which can then be solved by numerical techniques with high precision.

Conformal symmetry implies that the four-point correlator must have the form

$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle = x_{12}^{-2\Delta_\sigma} x_{34}^{-2\Delta_\sigma} g(u,v)$$

where $x_{ij} = |x_i - x_j|$, and $g(u,v)$ is a function of the ratios $u = (x_{12}^2 x_{34}^2)/(x_{13}^2 x_{24}^2)$, $v = (x_{14}^2 x_{23}^2)/(x_{13}^2 x_{24}^2)$. The correlator can then be evaluated using an OPE

$$\sigma(x_1)\sigma(x_2) = \sum_{\mathcal{O}} \lambda_{\mathcal{O}} C(x_1 - x_2, \partial_{x_2})\mathcal{O}(x_2)$$

where the sum is over primary conformal operators $\mathcal{O}$, and the coefficient functions $C_{12} = C(x_1 - x_2, \partial_{x_2})$ are fixed by the conformal group. The OPE can be applied to both pairs of the four point function and reduce it to a sum of two point functions, acted on by the coefficient operators C_{12} and C_{34}. Since only identical fields will have nonzero two point functions, only diagonal terms contribute, and $g(u,v)$ can be written in the following fashion:

$$g(u,v) = \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 g_\Delta^{(l)}(u,v)$$

where the ingredients $g_\Delta^{(l)}$, called *conformal blocks*, are completely fixed functions that depend only on (the dimension and spin of) operator $\mathcal{O}$.

The "bootstrap" condition can now be formulated, using the crossing symmetry of the four point function. By exchanging $1 \leftrightarrow 3$, we must have $v^{\Delta_\sigma} g(u,v) = u^{\Delta_\sigma} g(v,u)$, in other words,

$$g(u,v) = (u/v)^{\Delta_\sigma} g(v,u) \tag{1}$$

Combined with the OPE, this is a non-trivial statement which can also be interpreted as the associativity of the operator algebra (Figure 1). By implementing the condition first on fields with smallest dimension Δ_σ, we can find constraints for intermediate states and apply them to their four point functions, thus 'bootstrapping' through the conformal blocks.

RECENT DEVELOPMENTS USING CONFORMAL BOOTSTRAPPING

The reason the bootstrap condition did not immediately lead to solutions for higher dimension CFTs is
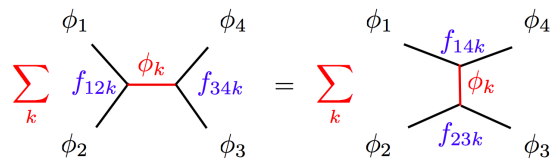

FIG. 1: The bootstrap condition, interpreted as associativity of the operator algebra [6].

twofold. At the time of its inception, conformal blocks were still poorly understood and represented in terms of complicated integrals and power expansions. Second, the bootstrap condition (Eq. 1) is difficult – it is a a functional equation, in a space of infinitely many operator dimensions and OPE coefficients. Only relatively recently have results been obtained finding concrete expressions for the conformal blocks and extracting the critical exponents out of the bootstrap condition.

In different dimensions have been formulated in recent years [7, 8]. For example, the conformal blocks in $D = 4$ are

$$g_\Delta^{(l)}(u,v) = \frac{(-1)^l}{2^l} \frac{z\bar{z}}{z - \bar{z}} [k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})] \ (D = 4)$$

$$k_\beta(z) = x_2^{\beta/2} F_1(\beta/2, \beta/2, \beta; z), u = z\bar{z}, v = (1-z)(1-\bar{z}).$$

The constraints imposed on Δ_ϵ as a function of Δ_σ can be extracted from the conformal blocks by applying the bootstrap condition numerically [9].

In the last few years, the technique has been extended to the 3D case, where the critical exponents of the 3D Ising model have been extracted using bootstrap techniques (Figure 2) [6]. The agreement with previous efforts (from experiment and MC simulations) is good evidence that the 3D Ising model contains a full conformal symmetry (a sort of bootstrapping of the justification for using the technique to begin with). This approach also has advantages over other methods – contraints introduced at each step are completely rigorous up to numerical error, compared to ε-expansion where the error in summation can only be estimated, and in lattice calculations where discretizing introduces an uncertainty. Conformal bootstrap techniques will continue to improve as the bootstrap condition is extended to other correlators, and the numerical techniques used to bootstrap through conformal blocks are refined.

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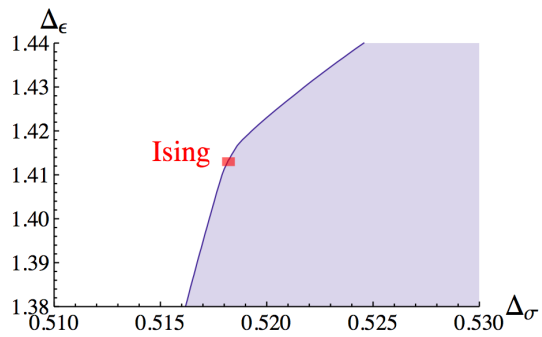
FIG. 2: Shaded part of $\Delta_\sigma, \Delta_\epsilon$ plane allowed by the bootstrap condition [6]. The kink in the boundary is very close to the known 3D Ising model dimensions. The red rectangle represents the error on the dimensions.

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