Morse Homology and RG Flows

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In this paper, I discuss Morse Homology and illustrate how it may be a useful and natural mathematics for studying the topology of the theory space of RG flows. It is meant to be a survey of the mathematical concepts and results, rather than a development of the topic or even a detailed explanation of the topic.

INTRODUCTION

A recently developed method to study the space of RG flows uses tools from Morse homology [1]. This novel method of studying the theory space structure may provide fruitful for a number of reasons, perhaps the most important being that it may allow for a deeper understanding of the C-theorem in any dimension.

I aim to develop the motivation and some of the background necessary to understand Morse homology. Along the way, I point out theorems and corollaries that make this math well defined and suitable to be used for the purpose of studying RG flows. As such, the majority of the paper is focused on the mathematics. However, the reader may be able to make many connections to RG flows along the way if they have a sufficient background.

In this paper all functions f will be assumed to be smooth functions over a manifold M such that $f : M \to \mathbb{R}$. To guide your intuition, f should be thought of as a height function, so that level sets on M are given by $f^{-1}(c)$, where c is the "height". Figure 1 depicts this situation.

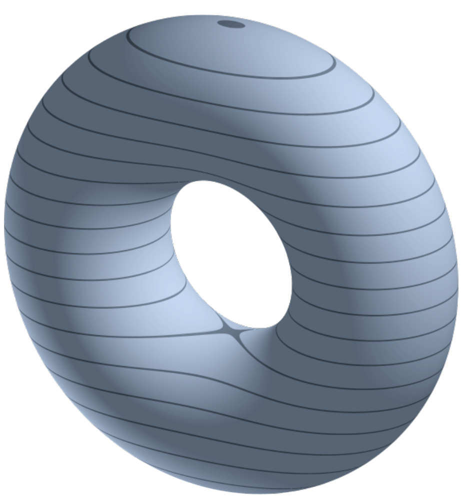

FIG. 1: Torus with level sets drawn in. The upmost points correspond to the largest heights. Image from [2].

CLASSICAL MORSE THEORY

Definition: The set of **critical points** of a function f is defined to be the set $Cr(f) \equiv \{p \in M \mid df_p = 0\}$. The function is said to be a **Morse function** $\Leftrightarrow$ $\forall\ p \in Cr(f) \Rightarrow |H_p(f)| \neq 0$. That is, the Hessian based at point p of the function f is non-degenerate. [5]

Morse Lemma: Let p be a critical point of a Morse function f on an n-dimensional manifold. Then locally,

$$f = f(p) - (y^1)^2 - (y^2)^2 - \ldots - (y^{\lambda_p})^2 + (y^{\lambda_p+1})^2 + \ldots + (y^n)^2 \quad (1)$$

λ_p happens to be an invariant over the set of all Morse functions on M, so already we have some notion of topological information. λ_p is called the index of the Morse function and happens to be the number of negative eigenvalues of $H_p(f)$ (i.e. the number of ways to independently descend the manifold at point p.) In Figure 1 the topmost point has $\lambda_p = 2$; the two interior points have $\lambda_p = 1$; the bottommost point has $\lambda_p = 0$.

Since each critical point of a Morse function is non-degenerate, we can decompose the tangent space of a manifold into subspaces constructed by taking linear combinations of both the negative and positive eigenvectors independently:

$$T_p M = T_p^s M \oplus T_p^u M \quad (2)$$

where $T_p^s M = span\ \{v_{\lambda_i} \mid \lambda_i > 0\}$;
$T_p^u M = span\ \{v_{\lambda_i} \mid \lambda_i < 0\}$.

Notice that $dim(T_p^u M) = \lambda_p$ and $dim(T_p^s M) = n - \lambda_p$

Corollary: The set of non-degenerate critical points are isolated. This may not be the case if degeneracy is allowed.

Corollary: A Morse function on a compact manifold has a finite number of critical points. This will be useful when constructing a well defined boundary operator. In particular, it forces the coefficients of the terms in the boundary operator to be finite. If degeneracy is allowed, care has to be taken (refer to the Section on Morse-Bott Homology).

MORSE-SMALE HOMOLOGY

Define a function φ to be a gradient flow of a Morse function which satisfies:

1. $\varphi : \mathbb{R} \times M \to M$
2. $\frac{\partial}{\partial t}\varphi(t,x) = -\nabla f(\varphi(t,x))$
3. $\varphi(0,*) = id_M$

Definition: the **unstable** and **stable** manifolds of a manifold M are constructed by taking limits of φ:

$$W_p^u = \{x \in M \mid \lim_{t\to-\infty} \varphi(t,x) = p\} \tag{3}$$

$$W_p^s = \{x \in M \mid \lim_{t\to+\infty} \varphi(t,x) = p\} \tag{4}$$

Since $TW_p^u \cong T_p^u M \Rightarrow dim(W_p^u) = \lambda_p$. It is useful to think of W_p^u as a λ_p dimensional open ball, especially if you want to think of the stable and unstable manifolds as giving a CW-structure over M. It is also useful to think of the unstable manifold as the collection of flow lines *flowing out of a critical point* and the stable manifold as the collection of flow lines *flowing in to a critical point*. So, flows will start at unstable manifolds and flow to stable manifolds. Figure 2 depicts this situation.

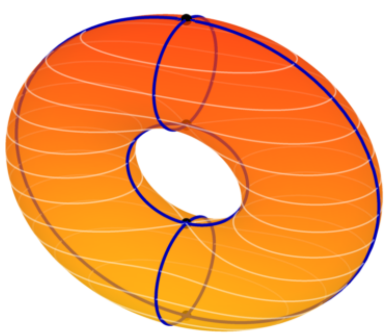

FIG. 2: Torus with level sets (gray) and a few examples of the stable and unstable manifolds. Image from [3].

Definition: If the stable and unstable manifolds of a Morse function all intersect transversally, the function is called **Morse-Smale**.

Intuitively, this means that at any point, the tangent spaces of the stable and unstable manifolds have enough information to construct the entire tangent space at that point. For example, the stable/unstable manifolds drawn for the height function in Figure 2 don't quite satisfy the transversality condition. The metric here can be perturbed by imagining slightly tilting the torus. Figure 3 depicts this.

It turns out that Morse-Smale functions flow out of unstable critical points to stable points of strictly lower indexes. This fact can be proven using the following corollary and is important for constructing a well defined boundary operator. The following corollary itself can be proven using the transversality condition of the stable and unstable manifolds and by a simple set theoretic argument.

Corollary: For Morse-Smale functions, and for flows starting at critical point p and ending on critical point q:

$$dim(W_p^u \cap W_q^s) = \lambda_p - \lambda_q \tag{5}$$

Kupka-Smale Theorem: For any metric on a manifold M, the set of smooth Morse-Smale functions is dense in the space of all smooth functions on M. Also, one can always find a Riemannian metric on M to make a Morse function into a Morse-Smale function.

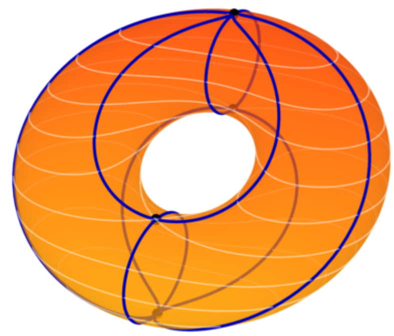

FIG. 3: "Tilted torus" with level sets (gray) and a few examples of transverse stable and unstable manifolds. Image from [3].

Definition: For flows going from $p \to q$ the **Moduli Space** is defined as:

$$M(p,q) = (W_p^u \cap W_q^s)/\mathbb{R} \tag{6}$$

Corollary: $\lambda_p - \lambda_q = 1 \Rightarrow M(p,q)$ is a compact 0-dimensional manifold. A classification theorem states these manifolds are finite (i.e. there is a finite number of flows going from two critical points which differ in index by 1).

Definition: The boundary operator can now be defined. $C_k \equiv$ free Abelian group generated by critical points of index k. Then, $\partial_k : C_k(f) \to C_{k-1}(f)$. Specifically:

$$\partial_k(p) = \sum_{q \in Cr_{k-1}(f)} \#M(p,q)q\ ,\ p \in C_k(f) \tag{7}$$

$\#M(p,q) \in \mathbb{Z}$ is defined to be the signed number of flow lines from $p \to q$.

One of the final ingredients into showing this is a well defined homology is to show that $\partial^2 \equiv 0$. I consider points p and q such that $\lambda_p = i+2$ and $\lambda_q = i$. Rather than go through the steps of the derivation, I hope to give an intuition, albeit a very tenuous one, for why this is true. We need to rely on a few results:

- $M(p,q)$ can be compactified to $\overline{M(p,q)}$
- The boundary of $\overline{M(p,q)}$ turns out to be equivalent to the action of ∂^2. In other words, $\partial^2 p \sim \#\partial\overline{M(p,q)}$
- $\overline{M(p,q)}$ is an oriented 1-manifold with boundary, so its signed number of boundary points is zero

Morse Homology Theorem: The homology of the Morse-Smale chain complex $(C_*(f), \partial_*)$ is isomorphic to the singular homology on M.

Corollary: $M = \coprod_p W_p^u = \coprod_p W_p^s$. i.e. the stable and unstable manifolds contain all the topological information of M.

MORSE-BOTT HOMOLOGY

I will briefly mention a generalization of Morse-Smale Homology to the case where the critical points are non-isolated. In other words, f will have an infinite number of critical points. In this case, the Morse Homology theorem does not apply because the boundary operator is no longer well defined. Now, $Cr(f) = \coprod_i C_i$ where C_i are connected submanifolds of M such that $df|_{C_i} \equiv 0$. If $H_p(f)$ is non-degenerate in the normal direction for all $p \in C_i$ then f is called **Morse-Bott**.

By the Kupka-Smale theorem, a Morse-Bott function can be perturbed to a Morse-Smale function, which means the Morse Homology theorem can be applied to this perturbed function! Refer to [4] for the explicit form of the perturbed function.

RG FLOWS

Finally we come to RG flows and how they fit into the picture. I will discuss the math as presented above, and make no modifications in the math in attempting to bridge the gap between the homology and physics.

$\partial_t \varphi(t,x)$ should be taken to be analogous to the usual beta function, so if we are to directly apply this math, we are actually assuming that RG flows are determined by the gradient flow of some function. To see this, consider condition (2) of φ:

$$\frac{\partial}{\partial t}\varphi(t,x) = -\nabla f(\varphi(t,x))$$

The dimension of W_p^u can be thought of as the number of relevant operators at a critical point in the theory space of RG flows. Indeed, each relevant operator will flow from a UV $\rightarrow$ IR point without losing the majority of its information.

So, we should be able to count the number of flows between two points, that differ in their number of relevant operators by 1, by considering the moduli space of the flows. This would give us information on π_0 of the RG space.

When we have an instance of a conformal manifold, we actually need to use the tools provided by Morse-Bott theory. In this case, each C_i should be thought of as a conformal manifold.

Acknowledgements I would like to thank Justin Roberts and Benjamin Grinstein for their insights into these topics.

[1] S. Gukov, "Counting RG Flows," arXiv:1503.01474 (2015)
[2] Wikipedia, "Morse theory"
[3] Wikipedia, "Morse-Smale system"
[4] D. Hurtubise, "Three Approaches To Morse-Bott Homology," arXiv:1208.5066 (2013)
[5] Recall that the Hessian is the matrix of second derivatives of the function evaluated at a point. A non-degenerate Hessian is just the statement that we can definitively say what the curvature of the manifold is at every critical point.