Quantization of the WZW Action in Chiral Perturbation Theory

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Witten's framework for the Wess-Zumino chiral effective action, as described in [3], is reveiwed.

INTRODUCTION

This note reviews Witten's discussion of the effective field theory for QCD in [3]. It begins with the necessary background on chiral perturbation theory, as presented in [2] and a myriad of other literature. We find that the lowest order action has too much symmetry, necessitating the addition of a "WZW term". After studying a simple analogue of our problem, we exhibit Witten's geometric derivation of the quantization of the WZW action [3].

CHIRAL SYMMETRY BREAKING

The Lagrangian density for QCD is given in terms of $D_\mu = \partial_\mu - igA_\mu$ and $F_{\mu\nu} = i[D_\mu, D_\nu]/g$ as

$$\mathcal{L} = -\tfrac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{q}i\not{D}q. \tag{1}$$

The quark field $q = (q_1, \ldots, q_{n_f})^T$ is taken to have n_f massless flavors in the fundamental representation of the gauge group $SU(n_c)$. The spinors may be resolved into chiral components $q = q_R + q_L$, where $q_{R/L} = P_{R/L}q$ and $P_{R/L} = (1 \pm \gamma^5)/2$. Recalling $\{\gamma^5, \gamma^\mu\} = 0$, we find that $\overline{q}_L i\not{D} q_R = q^\dagger P_L \gamma^0 i D_\mu \gamma^\mu P_R q = \overline{q} i\not{D} P_L P_R q = 0$ and similarly $\overline{q}_R i\not{D} q_L = 0$. We may therefore rewrite (1) as

$$\mathcal{L} = -\tfrac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{q}_R i\not{D} q_R + \overline{q}_L i\not{D} q_L. \tag{2}$$

This is manifestly invariant under the independent global transformations $q_{R/L} \to g_{R/L}q_{R/L}$ for $g_{R/L} \in U(n_f)_{R/L}$. The flavor chiral symmetry group may be decomposed as

$$SU(n_f)_R \otimes SU(n_f)_L \otimes U(1)_V \otimes U(1)_A.$$

The vector subgroup V corresponds to $g_R = g_L$, and the axial subgroup A corresponds to $g_R = g_L^\dagger$.

To spontaneously break the $SU(n_f)_A$ symmetry, the condensate $\overline{q}q$ acquires a nonzero vacuum expectation value $\langle 0|\overline{q}q|0\rangle = 2vn_f$. Since $\overline{q}q = \overline{q}_L q_R + \overline{q}_R q_L$, which follows from $\overline{q}_R q_R = q^\dagger P_R \gamma^0 P_R q = \overline{q} P_L P_R q = 0$ and $\overline{q}_L q_L = 0$, it is clear that the vacuum is invariant under vector but not axial rotations. This results in the chiral symmetry breakdown $SU(n_f)_R \otimes SU(n_f)_L \to SU(n_f)_V$.

In reality, chiral symmetry is only approximate, as it is explicitly broken by quark masses. The broken symmetry generators are accompanied by light "pseudo-Goldstone" bosons. $SU(2)$ flavor symmetry is better than $SU(3)$ due to the higher s quark mass, which is better than $SU(4)$ due to the much higher c quark mass. This explains the mass gap between the pion triplet and the four kaons and eta, and between this octet and next lightest mesons.

EFFECTIVE FIELD THEORY

We may construct a low-energy effective Lagrangian from the spacetime fluctuations of $\langle\overline{q}q\rangle$. To that end, we define $\langle\overline{q}_{Lj}(x)q_{Ri}(x)\rangle = vu_{ij}(x)$, or simply $u = \langle q_R\overline{q}_L\rangle/v$ with the spinor and color degrees of freedom traced out. We can write the $SU(n_f)$ matrix u as $u(x) = e^{2i\phi(x)/f}$, where $\phi(x) = t^a\phi^a(x)$. The Lie generators t^a are in the fundamental representation, the meson fields ϕ^a are the $n_f^2 - 1$ Goldstone bosons arising from the spontaneously broken axial symmetry, and the pion decay constant f sets the scale of the effective theory at $4\pi f \sim 1$ GeV.

The effective theory must share the same symmetries as (2). Notice $u \to g_R u g_L^\dagger$ when $q_{R/L} \to g_{R/L}q_{R/L}$. It is therefore useful to define $r_\mu = iu\partial_\mu u^\dagger$ and $l_\mu = iu^\dagger\partial_\mu u$. By cyclicity of the trace, the trace of a product of r_μ's or l_μ's is invariant under chiral rotations. By differentiating $uu^\dagger = u^\dagger u = 1$, note that $r_\mu = r_\mu^\dagger$ and $l_\mu = l_\mu^\dagger$. It is also clear from the exponential expansion of u and cyclicity of the trace that $\frac{1}{2}f\operatorname{tr} r_\mu = \operatorname{tr}(uu^\dagger\partial_\mu\phi) = \partial_\mu \operatorname{tr}\phi = 0$ and similarly $\operatorname{tr} l_\mu = 0$. The lowest order effective Lagrangian which respects chiral symmetry is therefore

$$\mathcal{L}_\sigma = \tfrac{1}{4}f^2\operatorname{tr}(r_\mu r^\mu) = \tfrac{1}{4}f^2\operatorname{tr}(\partial_\mu u\partial^\mu u^\dagger). \tag{3}$$

Taking $\operatorname{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$, the prefactor $\frac{1}{4}f^2$ is required to correctly normalize the kinetic terms. Indeed, we can see by expanding the exponential that

$$\begin{aligned}\mathcal{L}_\sigma &= \operatorname{tr}(\partial_\mu\phi\partial^\mu\phi) + \tfrac{1}{3}\operatorname{tr}([\phi,\partial_\mu\phi][\phi,\partial^\mu\phi])/f^2 + \cdots \\ &= \tfrac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \tfrac{1}{6}f^{abe}f^{cde}\phi^a\partial_\mu\phi^b\phi^c\partial^\mu\phi^d/f^2 + \cdots.\end{aligned}$$

The Lagrangian (3) is also invariant under $u \to u^T$, which represents charge conjugation if we appropriately identify the particles in terms of the fields. For $n_f = 2$, we have $q = (u, d)^T$ and $2t^a = \sigma^a$ with σ^a the Pauli matrices, and we define

$$2\phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}.$$

For $n_f = 3$, we have $q = (u, d, s)^T$ and $2t^a = \lambda^a$ with λ^a the Gell-Mann matrices, and we define

$$2\phi = \begin{pmatrix} \pi^0 + \eta/\sqrt{3} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \eta/\sqrt{3} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\,\overline{K}^0 & -2\eta/\sqrt{3} \end{pmatrix}.$$

The final discrete symmetry we need to check is parity invariance. Under parity, $\vec{x} \to -\vec{x}$ and $q \to \gamma^0 q$ so that

$$q_R\overline{q}_L \to P_R\gamma^0 qq^\dagger\gamma^0 P_L\gamma^0 = \gamma^0 P_L qq^\dagger P_R = (q_R\overline{q}_L)^\dagger.$$

Hence $u \to u^\dagger$, or equivalently $\phi^a \to -\phi^a$. That is, the Goldstones are pseudoscalars. Although (3) is invariant under both $\vec{x} \to -\vec{x}$ and $\phi^a \to -\phi^a$, (2) is invariant only under their combined action of parity. To describe an interaction like $K^+K^- \to \pi^0\pi^+\pi^-$ in which an even number of mesons decays into an odd number, we need to add a term to $\mathcal{L}_\sigma$ which lifts the redundant symmetry.

MAGNETIC MONOPOLE ANALOGY

An analagous situation occurs in the case of a particle constrained to move on a two-sphere of unit radius. The Lagrangian is $L_f = \frac{1}{2}m\dot{x}^2 + \lambda(x^2-1)$ with $x = \sqrt{x_i x_i}$, which yields the Euler-Lagrange equations $m\ddot{x}_i = 2\lambda x_i$. Contracting with x_i gives $2\lambda = m\ddot{x}_i x_i = -m\dot{x}^2$, where we used the constraint $x = 1$ and its second derivative, so the equations of motion are $m\ddot{x}_i + m\dot{x}^2 x_i = 0$. L_f is invariant under both $t \to -t$ and $x_i \to -x_i$. Suppose we want it to be invariant only under the combined action $t \to -t$, $x_i \to -x_i$. We begin with a simple modification of the equations of motion which realizes our goal,

$$m\ddot{x}_i + m\dot{x}^2 x_i = g\varepsilon_{ijk}\dot{x}_j x_k. \tag{4}$$

In order to derive (4) from an action, introduce the vector potential $A_i = \varepsilon_{ijk}n_j x_k/(x(x+n_l x_l))$, where $\vec{n}$ is any unit vector. Then $\varepsilon_{ijk}\partial_j A_k = x_i$ so that

$$g\varepsilon_{ijk}\dot{x}_j x_k = g(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\dot{x}_j\partial_k A_l = g(\dot{x}_j\partial_i A_j - \dot{A}_i),$$

which would be the Euler-Lagrange terms associated to $g\dot{x}_i A_i$ but for the singularity at $x_i = -n_i$. An action for the closed trajectory ∂D is therefore $S_f + \Phi_\pm$, where

$$\Phi_\pm = \pm g\int_{D^\pm} d\Sigma = g\int_{\partial D} dx_i A_i \tag{5}$$

by Stokes' theorem. Here ∂D bounds the region D^+ in a counterclockwise manner, and D^- in a clockwise manner.

This action is well-defined by the lefthand side of (5). The trade-off is that it is nonlocal and ambiguous. The path integral is consistent only if $e^{i\Phi_+} = e^{i\Phi_-}$. That is,

$$\Phi_+ - \Phi_- = g\int_{S^2} d\Sigma = 2\pi n \tag{6}$$

for integer n. The integral is simply 4π so that $g = n/2$. This is in fact the Dirac quantization condition, as A_i is the potential due to a magnetic monopole at the origin.

QUANTIZATION OF THE WZW ACTION

A local $SU(n_f)_R$ transformation $u(x) \to e^{i\epsilon(x)}u(x)$, where $\epsilon(x) = t^a\epsilon^a(x)$, yields the infinitesimal variations $\delta u = i\epsilon u$ and $\delta r_\mu = \partial_\mu\epsilon + i[\epsilon, r_\mu]$ for small ϵ. Thus

$$\delta\,\mathrm{tr}(r_{\mu_1}\cdots r_{\mu_n}) = \sum_{k=0}^{n-1}\sigma^k\,\mathrm{tr}((\partial_{\mu_1}\epsilon)r_{\mu_2}\cdots r_{\mu_n}) \tag{7}$$

by cyclicity of the trace, where $\sigma = (\mu_1\cdots\mu_n)$ permutes the indices. Using (7) and integrating by parts gives us $\delta S_\sigma = -\frac{1}{2}f^2\int d^4x\,\mathrm{tr}(\epsilon\partial_\mu r^\mu)$. Since $\partial_\mu r^\mu$ is traceless, the $n_f^2 - 1$ conditions $\delta S_\sigma = 0$ are equivalent to the equation of motion $\frac{1}{2}f^2\partial_\mu r^\mu = 0$. Now the modification

$$\tfrac{1}{2}f^2\partial_\mu r^\mu = 5g\varepsilon_{\mu\nu\lambda\sigma}r^\mu r^\nu r^\lambda r^\sigma \tag{8}$$

violates $\vec{x} \to -\vec{x}$ and $\phi^a \to -\phi^a$ while respecting their combination. The former sends $r_\mu \to r^\mu$, and the righthand side picks up a minus sign since $\varepsilon_{\mu\nu\lambda\sigma} = -\varepsilon^{\mu\nu\lambda\sigma}$. The latter sends $r_\mu \to l_\mu = -u^\dagger r_\mu u$, so the lefthand side picks up a minus sign after multiplying through by u on the left and $u^\dagger$ on the right.

An action which reproduces (8) is $S_\sigma + \Gamma_\pm$, where

$$\Gamma_\pm = \pm g\int_{D^\pm} d^5x\,\varepsilon_{ijklm}\,\mathrm{tr}(r^i r^j r^k r^l r^m). \tag{9}$$

$\Gamma_\pm$ vanishes for $n_f = 2$, so assume $n_f \geq 3$. Here $D^\pm$ are complementary five-discs bounded by spacetime, which we imagine as compactified into a four-sphere. Note that

$$\varepsilon_{ijklm}\partial^i(r^j r^k r^l r^m) = \varepsilon_{ijklm}\partial^i(\partial^j u\partial^k u^\dagger\partial^l u\partial^m u^\dagger) = 0$$

by antisymmetry of ε_{ijklm} and equality of mixed partials. Then by (7), integration by parts, and Stokes' theorem,

$$\begin{aligned}\delta\Gamma_\pm &= \pm 5g\int_{D^\pm} d^5x\,\varepsilon_{ijklm}\partial^i\,\mathrm{tr}(\epsilon r^j r^k r^l r^m)\\ &= 5g\int d^4x\,\varepsilon_{\mu\nu\lambda\sigma}\,\mathrm{tr}(\epsilon r^\mu r^\nu r^\lambda r^\sigma).\end{aligned}$$

As $\varepsilon_{\mu\nu\lambda\sigma}r^\mu r^\nu r^\lambda r^\sigma$ is traceless by antisymmetry of $\varepsilon_{\mu\nu\lambda\sigma}$ and cyclicity of the trace, $\delta(S_\sigma + \Gamma_\pm) = 0$ implies (8).

By analogy with (6), we need $e^{i\Gamma_+} = e^{i\Gamma_-}$. That is,

$$\Gamma_+ - \Gamma_- = g\int_{S^5} d^5x\,\varepsilon_{ijklm}\,\mathrm{tr}(r^i r^j r^k r^l r^m) = 2\pi n \tag{10}$$

for integer n. The integral is $480\pi^3$ times the winding number of the map $u\colon S^5 \to SU(3)$ [1]. It follows that $g = n/(240\pi^2)$. Using $\varepsilon_{ijklm}\partial^i(\partial^j\phi\partial^k\phi\partial^l\phi\partial^m\phi) = 0$ and Stokes' theorem, we obtain the leading order WZW term

$$\begin{aligned}\Gamma_\pm &= \pm g\frac{2^5}{f^5}\int_{D^\pm} d^5x\,\varepsilon_{ijklm}\partial^i\,\mathrm{tr}(\phi\partial^j\phi\partial^k\phi\partial^l\phi\partial^m\phi) + \cdots\\ &= \frac{2n}{15\pi^2 f^5}\int d^4x\,\varepsilon_{\mu\nu\lambda\sigma}\,\mathrm{tr}(\phi\partial^\mu\phi\partial^\nu\phi\partial^\lambda\phi\partial^\sigma\phi) + \cdots.\end{aligned}$$

It turns out that a correct description of QCD couplings to QED fixes $n = \pm n_c$, where n_c is the number of colors.

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