Non-equilibrium dynamics of neutrinos in a medium

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The equilibration of neutrinos immersed in a thermal matter background is explored from a field-theoretic perspective. First, a non-equilibrium effective action is formulated by integrating out the bath degrees of freedom. It is shown that this effective action begets a stochastic Langevin equation of motion, which is then solved. The treatment presented in this paper was developed by D. Boyanovsky and C. M. Ho.

I. INTRODUCTION

Neutrinos have acquired a reputation for leaving only the faintest of traces as they travel, wraithlike, through the Earth. But for all the difficulty of discerning the effects of neutrinos in terrestrial settings, they nonetheless play a critical role in more extreme astrophysical environments such as supernovae and the early universe. In these cases an understanding of how the neutrinos interact with the media through which they propagate is vital.

The literature on the subject tells a story of increasingly sophisticated treatments. At the most basic level the richness of neutrino dynamics originates from neutrinos' characteristic mismatch between mass and interaction eigenstates, which in turn gives rise to the phenomenon of neutrino oscillation. Another layer of depth is added when interactions with a matter background are present, as the evolution of the neutrino system then becomes dictated by a competition between oscillation and scattering-induced decoherence. This interplay is described quantitatively by the quantum kinetic equations, which remain a topic of active research [1].

In this paper I will review a derivation due to Boyanovsky and Ho [2, 3] of the neutrino quantum kinetics that prevail in the early universe. The derivation benefits from certain simplifications: The matter background is taken to be isotropic and homogeneous, the spin and the fermionic nature of the neutrinos are neglected, and only two flavor eigenstates are assumed to exist. What this calculation does capture, however, is the generic behavior of a system whose relaxation to equilibrium is informed by both oscillation and decoherence.

In Section II I will define a model for the neutrino/bath system and derive a non-equilibrium effective action. I will then use the effective action in Section III to produce an equation of motion that can be interpreted as a stochastic Langevin equation. In Section IV, finally, I will solve this equation, enabling an explicit computation of the system's approach to equilibrium.

II. THE NON-EQUILIBRIUM EFFECTIVE ACTION

The system we are studying consists of neutrinos with two flavors ($\alpha = e, \mu$) immersed in a thermal bath of charged leptons. Two interaction types are present: Neutrinos can scatter *off of* their respective charged lepton or, with the emission of a W boson, they can scatter *into* their respective charged lepton. The Lagrangian that encapsulates such a system is

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\left\{\partial_\mu \Phi^T \partial^\mu \Phi - \Phi^T \mathbb{M}^2 \Phi\right\} + \mathcal{L}_0[W,\chi] \\ &+ GW\Phi^T\chi + G\phi_e^2\chi_e^2 + G\phi_\mu^2\chi_\mu^2, \end{aligned} \tag{1}$$

where $\mathcal{L}_0[W,\chi]$ is the free Lagrangian for the matter-background fields, G is the coupling constant, $\mathbb{M}^2$ is the 2×2 neutrino mass matrix with mass-squared entries, W is the vector-boson field, and Φ and χ are the neutrino and charged-lepton field doublets, respectively.

Our strategy will be to trace over the bath degrees of freedom in the partition function in order to obtain a non-equilibrium effective action, as the latter will enable us, ultimately, to solve the equation of motion obeyed by the neutrino fields. The first step is to state the partition function itself:

$$Z = \int D\Phi_i D\Phi_i' D\Phi^\pm D\chi DW \rho_{\Phi,i} e^{iS_J}, \tag{2}$$

where $\rho_{\Phi,i}$ is the initial density matrix of the neutrino fields and Φ_i and Φ_i' give the initial values of $\Phi^\pm$, the neutrino fields under forward (reverse) time evolution. S_J is shorthand for the non-equilibrium action in the presence of sources for the neutrino fields,

$$\begin{aligned}S_J =& \int_{t=t_i}^{\infty} d^4x \left\{\mathcal{L}_0(\Phi^+) + J_\Phi^+\Phi^+ - \mathcal{L}_0(\Phi^-) - J_\Phi^+\Phi^-\right\} \\ &+ \int_{\mathcal{C}} d^4x \left\{\mathcal{L}_0\left[\chi, W\right] + GW\phi_\alpha\chi_\alpha + G\phi_\alpha^2\chi_\alpha^2\right\}. \end{aligned} \tag{3}$$

The contour $\mathcal{C}$ along which the time integration is performed is shown in Fig. 1. The narrative behind this form for the path integral is the following: By evolving the system (with field Φ^+) from t_i to ∞ and then (with field Φ^-) from ∞ back to t_i, we are able to generate the correlation functions for the neutrino fields while simultaneously stipulating that the χ, W fields are in thermal equilibrium. The latter is accomplished through the final leg of the contour C, in which we evolve the bath fields along a Euclidean path to $t_i - i\beta$. Along the Euclidean path the interaction terms vanish, consistent with the bath being in thermal equilibrium.

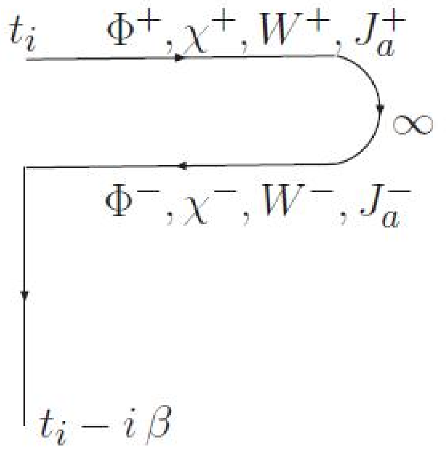

FIG. 1: The contour $\mathcal{C}$ from Eq. (3), with $\chi^\pm$, $W^\pm$ set to zero and $J_a^\pm \equiv J_\Phi^\pm$. Diagram taken from [3].

Tracing over the bath fields in Eq. (2) yields

$$\int D\chi DW e^{i\int_{\mathcal{C}} d^4x\{\mathcal{L}_0[\chi,W]+GW\phi_\alpha\chi_\alpha+G\phi_\alpha^2\chi_\alpha^2\}} = \left\langle e^{iG\int_{\mathcal{C}} d^4x\{W\phi_\alpha\chi_\alpha+\phi_\alpha^2\chi_\alpha^2\}}\right\rangle_0 \mathrm{Tr}e^{-\beta H_0[\chi,W]}. \quad (4)$$

We have effectively factorized Z into a part that enforces the thermal equilibrium of the matter background and a part that describes how the neutrinos, through their interactions with this background, approach equilibrium as well. Accordingly, the expectation value in the latter part is evaluated with respect to the free-field equilibrium density matrix of the background fields. This expectation value can be expanded in the coupling constant G to yield the influence functional

$$\begin{aligned} S_{\text{if}} =& G\int_{\mathcal{C}} d^4x\phi_\alpha^2(x)\left\langle\chi_\alpha^2(x)\right\rangle_0 + \\ & i\frac{G^2}{2}\int_{\mathcal{C}} d^4x\int_{\mathcal{C}} d^4x'\phi_\alpha(x)\phi_\beta(x')\left\langle\mathcal{O}_\alpha(x)\mathcal{O}_\beta(x')\right\rangle_0, \end{aligned} \quad (5)$$

where $\mathcal{O}_\alpha = W\chi_\alpha$. In the expansion we have dropped the contribution from ϕ_α^4 even though it is of order G^2; this term corresponds to neutrino–neutrino interactions and, although it can give rise to a number of interesting phenomena, is for our purposes an unnecessary complication.

Having traced over the bath degrees of freedom and formulated the influence functional, we can return to the partition function in Eq. (2) in order to deduce that the non-equilibrium effective action is

$$S_{\text{eff}} = \int_{t=t_i}^{\infty} d^4x\left\{\mathcal{L}_0(\Phi^+) - \mathcal{L}_0(\Phi^-)\right\} + S_{\text{if}}. \quad (6)$$

We have thus integrated out the bath fields, relegating their presence to the equilibrium expectation values that appear in the influence functional.

III. THE STOCHASTIC LANGEVIN EQUATION

The next step in the derivation is to extract an equation of motion from Z. To this end, it will be convenient to switch to Wigner coordinates, defined by

$$\Psi = \frac{1}{2}\left(\Phi^+ + \Phi^-\right), \quad R = \Phi^+ - \Phi^-. \quad (7)$$

After some manipulation (including a Fourier transform of the fields) the effective action can be rewritten as

$$\begin{aligned} S_{\text{eff}} =& \int_{t=t_i}^{\infty} dt\int d^3\vec{k}\left\{-R^T(-\vec{k},t)\left[\ddot{\Psi}(\vec{k},t) + \left(k^2\mathbb{I} + \mathbb{M}^2 + \mathbb{V}\right)\Psi(\vec{k},t)\right]\right\} \\ & + i\int_{t=t_i}^{\infty} dt\int_{t'=t_i}^{\infty} dt'\int d^3\vec{k}\left\{\frac{1}{2}R^T(-\vec{k},t)\mathcal{K}(\vec{k},t-t')R(\vec{k},t') + R^T(-\vec{k},t)i\Sigma^R(\vec{k},t-t')\Psi(\vec{k},t')\right\} \\ & + \int d^3xR_0^T(\vec{x})\dot{\Psi}_0(\vec{x}). \end{aligned} \quad (8)$$

Without working through the details of the calculation leading to this result, one can rationalize the form of Eq. (8) along the following lines. The square-bracketed term in the first line represents (loosely speaking) the evolution of the Ψ fields arising from their kinetic energy, their mass, and the matter potential $\mathbb{V}$ set up by the coupling of the neutrino field to $\langle\chi^2\rangle$. The last of these is akin to the index of refraction of the medium in which the neutrinos propagate; the diagram that produces it is shown in Fig. 2.

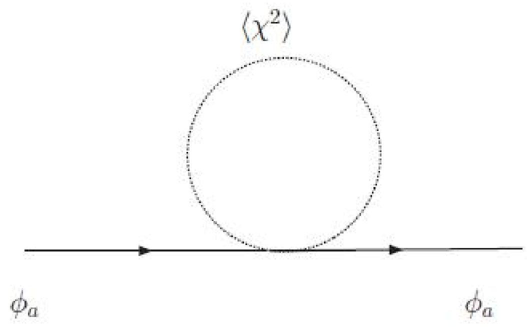

FIG. 2: Order-G one-loop self-energy corresponding to the matter potential $\mathbb{V} = G\langle\chi^2\rangle$. Diagram taken from [2].

The second line of Eq. (8) contains two new objects, $\mathcal{K}$ and $i\Sigma^R$, which are generated by the order-G^2 term in the influence functional from Eq. (5). Defining these objects here would require a lengthy detour, so I opt instead for presenting several facts (sans proof) that may allow the reader to see their conceptual significance without having to wade through the notational minutiae. It turns out that $\mathcal{K}$ and $i\Sigma^R$ obey a fluctuation–dissipation relation of the form

$$\tilde{\mathcal{K}}(\vec{k}, k^0) = \mathrm{Im}\tilde{\Sigma}^R(\vec{k}, k^0)\coth\left[\frac{\beta k^0}{2}\right]. \tag{9}$$

In this equation the tildes indicate that $\mathcal{K}(\vec{k}, t-t')$ and $i\Sigma^R(\vec{k}, t-t')$ have been Fourier-transformed. The meaning of Eq. (9) can be brought out by noting that $\mathrm{Im}\tilde{\Sigma}^R$ is the neutrino self-energy and its components are given by the cut discontinuity of the diagram in Fig. 3. $\tilde{\mathcal{K}}$, on the other hand, is related to a stochastic noise source, which is made evident by observing that the term in Eq. (8) that contains $\mathcal{K}$ can be recast as

$$\begin{aligned}\exp&\left\{-\frac{1}{2}\int dt\int dt' R^T(-\vec{k},t)\mathcal{K}(\vec{k},t-t')R(\vec{k},t')\right\}\\ &\int D\xi\exp\bigg\{-\frac{1}{2}\int dt\int dt'\xi^T(\vec{k},t)\mathcal{K}(\vec{k},t-t')\xi(-\vec{k},t')\\ &+i\int dt\xi(-\vec{k},t)R(\vec{k},t)\bigg\}.\end{aligned} \tag{10}$$

This equality can be confirmed by completing the square on the right-hand side and performing the Gaussian integral. What it reveals is that $\mathcal{K}$ permits interpretation as the two-point correlation function of a noise source ξ coupled to the field R. Referring back to Eq. (9), it is apparent that the neutrino self-energy (that is, the dissipation in the system) is related to the fluctuations of ξ.

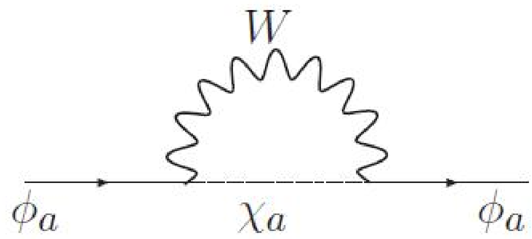

FIG. 3: Order-G^2 one-loop self-energy, with cut discontinuity corresponding to $\mathrm{Im}\tilde{\Sigma}^R$. Diagram taken from [2].

Substituting Eq. (10) into Eq. (8) allows for a drastic improvement in the tractability of the partition function. The power of this technique comes from the fact that introducing ξ replaces the term in S_{eff} that is quadratic in R with one that is linear in R. As a consequence, the path integral in Z over the field R can be explicitly computed, yielding a delta function whose argument is

$$\begin{aligned}\ddot{\Psi}(\vec{k},t) &+ \left(k^2\mathbb{I}+\mathbb{M}^2+\mathbb{V}\right)\Psi(\vec{k},t)\\ &+\int_0^t dt'\Sigma(\vec{k},t-t')\Psi(\vec{k},t') = \xi(\vec{k},t).\end{aligned} \tag{11}$$

(The Σ appearing here is defined such that $\Sigma^R(\vec{k}, t-t') = \Sigma(\vec{k}, t-t')\Theta(t-t')$.) Eq. (11) is a stochastic Langevin equation of motion satisfied by the field Ψ. It is useful at this point to note (again without proof) that the equal-time expectation values of Φ with respect to the initial density matrix of the system are the same as the equal-time expectation values of Ψ with respect to *both* the initial density matrix of the system *and* the noise distribution function. This is a long-winded way of saying that the evolution of the neutrinos themselves is found by taking the expectation value of the field Ψ that solves Eq. (11) with given initial conditions.

IV. SOLVING THE EQUATION OF MOTION

The solution to Eq. (11) is most easily obtained after a Laplace transform, which converts the Langevin equation into an algebraic matrix equation. The result is

$$\begin{aligned}\Psi(\vec{k},t) =&\dot{G}(\vec{k},t)\Psi_0(\vec{k})+G(\vec{k},t)\Pi_0(\vec{k})\\ &+\int_0^t dt' G(\vec{k},t')\xi(\vec{k},t-t'),\end{aligned} \tag{12}$$

where G is the anti-Laplace transform of

$$\tilde{G}(\vec{k},s) = \left[\left(s^2+k^2\right)\mathbb{I}+\mathbb{M}^2+\mathbb{V}+\tilde{\Sigma}(\vec{k},s)\right]^{-1}, \tag{13}$$

where I am now using tildes to denote Laplace-transformed functions. When we compute the expectation value of Ψ, the final term on the right-hand side of Eq. (12) will vanish because $\langle\xi\rangle = 0$. We can therefore conclude that G contains all of the physical information about the evolution of $\langle\Psi\rangle$.

Once G is known, Eq. (12) makes finding a solution trivial. But determining G in the first place is tricky, and I will omit the details except to say that, in anti-Laplace-transforming Eq. (13), one must insist that the poles of $\tilde{G}$ are complex: $\omega_{1,2}(k) = \pm\Omega_{1,2}(k) + i\Gamma_{1,2}/2$. Allowing the poles to pick up an imaginary component is tantamount to assuming that the poles are resonances of the Breit–Wigner form, which in turn reflects the physical reality that the neutrino quasi-particles in the medium have a finite lifetime.

Having made the foregoing remarks, I will now bypass the (non-trivial) procedure of solving for G and jump directly to the solution for $\langle\Psi\rangle$. Suppose we are interested in the transition probability $\Psi_e \to \Psi_\mu$, given a system in which all of the neutrinos are initially in the electron flavor state. The result in this case is particularly manageable:

$$\left|\left\langle \Psi_\mu(\vec{k},t)\right\rangle\right|^2 = \frac{\sin^2 2\theta_m}{4}\left[e^{-\Gamma_1 t} + e^{-\Gamma_2 t} - 2e^{-\frac{1}{2}(\Gamma_1+\Gamma_2)t}\cos\left(2\Delta\Omega t\right)\right]\left|\Psi_{0,e}(\vec{k})\right|^2 . \tag{14}$$

There are four $\vec{k}$-dependent parameters governing the transition probability: the two relaxation rates $\Gamma_{1,2}$, the oscillation frequency in matter $2\Delta\Omega \equiv \Omega_2 - \Omega_1$, and the mixing angle in matter θ_m. As indicated above, the first three of these correspond to the complex poles of $\tilde{G}$. The fourth parameter is set by the residues of these poles; it coincides exactly with the in-medium mixing angle obtained by the more traditional Schrodinger-like treatment [4].

Eq. (14) highlights the important features that are uncovered by the field-theoretic derivation developed in [2, 3]. In the absence of damping ($\Gamma_{1,2} = 0$), we recover the usual formula for neutrino oscillation in matter. Once damping is introduced, however, a novel behavior emerges: Evidently the equilibration of the neutrinos with the thermal bath is set by the interplay of the two distinct time scales associated with Γ_1 and Γ_2. Fundamentally, this finding echoes the fact that there are two ways for the system to equilibrate — electron-neutrinos can interact with electrons or muon-neutrinos can interact with muons — and that these two mechanisms are connected, of course, by oscillation.

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