

Physics 239 Spring 2016 Assignment 6

Due 11am Thursday, May 19, 2016

1. Stationary states of unital channels.

Check that the unital condition $\sum_i \mathcal{K}_i \mathcal{K}_i^\dagger = \mathbb{1}$ implies that the uniform density matrix $\mathbf{u} \equiv \mathbb{1}/|\mathcal{H}|$ is a fixed point of the associated quantum channel.

$$\mathcal{E}(\mathbf{u}) = \mathbf{u}.$$

2. Random quantum expanders. [somewhat open-ended and numerical]

Consider the family of quantum channels of the form

$$\rho \mapsto \mathcal{E}_\chi(\rho) = \sum_{i=1}^{\chi} p_i \mathbf{U}_i \rho \mathbf{U}_i^\dagger$$

with $\{\mathbf{U}_i\}$ a collection of unitaries. Such a channel is called a *quantum expander*. Show that such a channel is unital.

Sample χ random unitaries from the Haar measure on $\mathbf{U}(d)$ e.g. in Mathematica¹. (You can take $p_i = 1/\chi$ for definiteness if you wish.)

Sample a random initial density matrix².

Consider the rate at which repeated action of the channel \mathcal{E}_χ , $\rho_n = \mathcal{E}^n(\rho)$ mixes the initial state ρ as a function of χ (and d). We can use the von Neumann entropy as a measure of this mixing. Make some plots and some estimates.

If n is very large, how many terms do I actually need to include in the sum in

$$\mathcal{E}^n(\rho) = \sum_{i_1 \dots i_n} p_{i_n} \dots p_{i_1} \mathbf{U}_{i_1} \dots \mathbf{U}_{i_n} \rho \mathbf{U}_{i_n}^\dagger \dots \mathbf{U}_{i_1}^\dagger \quad ?$$

Consider the eigenstates (eigenoperators) of the (super)operator \mathcal{E}_χ . Can you show that any state orthogonal (in the Hilbert-Schmidt norm) to $\mathbb{1}$ has a an eigenvalue less than 1?

¹ Haar measure means the measure which is invariant under the group action. I did this by choosing a $d \times d$ complex matrix X with entries chosen from the gaussian distribution (which is indeed invariant under $\mathbf{U}(d)$) and then taking $Y = X + X^\dagger$ to make it hermitian, and then using the matrix U which diagonalizes Y .

² I did this by choosing a complex matrix X with entries chosen from the gaussian distribution, and then taking $Y = X + X^\dagger$ to make it hermitian and then taking $Z = Y^2$ to make it positive and then taking $\rho = Z/\text{tr}Z$ to make it a density matrix. What distribution did I use?

3. Scramble.

For this problem \mathcal{H}_A has dimension d .

- (a) **Warmup.** The set of linear operators $\text{End}(\mathcal{H}_A)$ is itself a Hilbert space with the Hilbert-Schmidt inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \text{tr} \mathbf{A}^\dagger \mathbf{B}$. Find an orthogonal basis $\{\mathbf{U}_a\}$ for this space (over \mathbb{C}) whose elements are themselves unitary operators, $\text{tr} \mathbf{U}_a^\dagger \mathbf{U}_b = d\delta_{ab}$.

[Hint: consider the algebra generated by the unitaries \mathbf{X}, \mathbf{Z} on the qdit teleportation problem on the previous problem set.]

Bonus: For the case of $|A| = 2^k$ find such a basis whose elements square to one.

- (b) Consider a maximally entangled state $|\Phi\rangle \equiv \frac{1}{\sqrt{d}} \sum_i |ii\rangle \in \mathcal{H}_A \otimes \mathcal{H}_A$. Show that the d^2 maximally entangled states

$$|\Phi_a\rangle \equiv \mathbf{U}_a \otimes \mathbb{1} |\Phi\rangle$$

form an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_A$.

- (c) Check your answers to the previous two parts for the case of qubits $d = 2$. Make a basis of product states from linear combinations of maximally entangled states.
- (d) For an arbitrary operator $\mathbf{A} \in \text{End}(A)$. find $\{p_a, \mathbf{U}_a\}$ with p_a probabilities and \mathbf{U}_a unitary such that the associated channel *scrambles* \mathbf{A} in the sense that

$$\sum_a p_a \mathbf{U}_a \mathbf{A} \mathbf{U}_a^\dagger = \frac{\text{tr} \mathbf{A}}{d} \mathbb{1}.$$

- (e) Use the previous result and the concavity of the entropy to show that the uniform state $\mathbf{u} = \mathbb{1}/d$ has the maximum von Neumann entropy on A .
- (f) Bonus problem: for the case where \mathbf{A} is Hermitian, find a set of only d unitaries which scramble \mathbf{A} .

4. Turtles all the way down. [open-ended]

A question you may have about our discussion of polarization-damping as a model of decoherence is: why does the environment reset to the reference state $|0\rangle_E$?

We can postpone the question a bit by coupling the environment to its own environment, according to an amplitude damping channel. On the previous problem set, you saw that the result of the repeated action of such a channel can set $\rho_E = |0\rangle\langle 0|$. This statement in turn assumes a forgetful meta-environment. A thermodynamic limit is required to postpone the question indefinitely. Construct such a thermodynamic limit.

5. **Two faces of strong subadditivity.** Show that for any state ρ_{ABC} on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$,

$$S(A) + S(B) \leq S(AC) + S(BC)$$

and that this statement is equivalent to strong subadditivity in the form

$$S(A'C') + S(A'B') \geq S(A') + S(A'B'C') \quad \forall A'B'C' .$$

Hint: use the Araki-Lieb purification trick.

6. **Consequences of SSA for mutual information.** Prove that SSA implies

$$I(A : B) + I(A : C) \leq 2S(A) .$$

Is the analogous inequality for Shannon entropies true?

Find an example of a state where $I(A : B) > S(A)$.