1. **Topology brain warmer.**

Compute the de Rham cohomology of the \( d \)-torus. Hint: you can choose \( p \)-chains of the form

\[
A_{i_1 \cdots i_p}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p}
\]

where \( x^i \simeq x^i + 1 \) are periodically identified coordinates, and \( A_{i_1 \cdots i_p} \) is a single-valued (i.e. periodic) function. Notice that \( x^i \) is not a single-valued function.

2. **Coherent state quantization brain-warmers.**

(a) Start with first order action

\[
S = \int dt \; \tilde{z}_\alpha^\dagger \dot{z}_\alpha.
\]

Show that the Hamiltonian is \( H = 0 \).

(b) Check the completeness relation in the spin 1/2 coherent state basis.

(c) Show that different spinor representations, i.e. different choices of \( \psi \) in

\[
z = \begin{pmatrix}
e^{i(\psi+\varphi/2)} \cos \theta/2 \\
e^{i(\psi-\varphi/2)} \sin \theta/2
\end{pmatrix}
\]

shift the coefficient of the total derivative \( \dot{\varphi} \) part of the WZW functional.

3. **Topological terms in QM.** [from Abanov]

The euclidean path integral for a particle on a ring with magnetic flux \( \theta = \int \vec{B} \cdot d\vec{a} \) through the ring is given by

\[
Z = \int [D\phi] e^{-\int_0^\beta d\tau (\frac{m}{2} \dot{\phi}^2 - \frac{\pi}{2} \dot{\phi}^2)}.
\]

Here

\[
\phi \equiv \phi + 2\pi
\]

is a coordinate on the ring. Because of the identification (1), \( \phi \) need not be a single-valued function of \( \tau \) – it can wind around the ring. On the other hand, \( \dot{\phi} \) is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as

\[
\phi(\tau) = \frac{2\pi}{\beta} Q \tau + \sum_{\ell \in \mathbb{Z}} \phi_\ell e^{i\frac{2\pi}{\beta} \ell \tau}.
\]
(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (2), write the partition function as a sum over topological sectors labelled by the \textit{winding number} $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula
\[ \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2} t n^2 + izn} = \sqrt{\frac{2\pi}{t}} \sum_{\ell \in \mathbb{Z}} e^{-\frac{1}{2t}(z-2\pi\ell)^2}. \]

(c) Use the result from the previous part to determine the energy spectrum as a function of $\theta$.

(d) [more optional] Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

(e) Consider what happens in the limit $m \to 0, \theta \to \pi$ with $X \equiv \frac{\theta - \pi}{m} \sim \beta^{-1}$ fixed. Interpret the result as the partition function for a spin 1/2 particle. What is the meaning of the ratio $X$ in this interpretation?

An important lesson here is that total derivative terms in the action do affect the physics.

4. Geometric Quantization of the 2-torus.

Redo the analysis that we did in lecture for the two-sphere for the case of the two-torus, $S^1 \times S^1$. The coordinates on the torus are $(x, y) \simeq (x + 2\pi, y + 2\pi)$; use $Nd\alpha \wedge d\beta$ as the symplectic form. Show that the resulting Hilbert space represents the Heisenberg algebra

\[ e^{ix} e^{iy} = e^{iy} e^{ix} e^{\frac{2\pi i}{N}}. \]

(I am using boldface letters for operators.) The irreducible representation of this algebra is the same Hilbert space as a particle on a periodic one-dimensional lattice with $N$ sites.

5. Particle on a sphere with a monopole inside.

Consider a particle of mass $m$ and electric charge $e$ with action

\[ S[\vec{x}] = \int dt \left( \frac{1}{2} m \dot{\vec{x}}^2 + e \dot{\vec{x}} \cdot \vec{A}(\vec{x}) \right) \]

constrained to move on a two sphere of radius $r$ in three-space, $\vec{x}^2 = r^2$. Suppose further that there is a \textit{magnetic monopole} inside this sphere: this means that $4\pi g = \int_{S^2} \vec{B} \cdot d\vec{a} = \int_{S^2} F$, where $F = dA$. (Since the particle lives only at $\vec{x}^2 = r^2$, the form of the field in the core of the monopole is not relevant here.)
(a) Find an expression for $A = A_i dx^i = A_\theta d\theta + A_\phi d\phi$ such that $F = dA$ has flux $4\pi g$ through the sphere.

(b) Show that the Witten argument gives the Dirac quantization condition $2eg \in \mathbb{Z}$.

(c) Take the limit $m \to 0$. Count the states in the lowest Landau level. Compare with the calculation in lecture for coherent state quantization of a spin-$s$. 