University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2017
Assignment 7

Due 12:30pm Wednesday, May 24, 2017

1. **More about 0+0d field theory.**

   Here we will study a bit more some field theories with no dimensions at all, that is, integrals.

   Consider the case where we put a label on the field: $q \to q_a, a = 1..N$. So we are studying

   $$Z = \int \int_{-\infty}^{\infty} \prod_a dq_a \; e^{-S(q)}.$$

   Let

   $$S(q) = \frac{1}{2} q_a K_{ab} q_b + T_{abcd} q_a q_b q_c q_d$$

   where T_{abcd} is a collection of couplings.

   (a) Show that the propagator has the form:

   $$a - - - - - - b = \left(K^{-1}\right)_{ab} = \sum_k \phi_a(k)^\star \frac{1}{k} \phi_b(k)$$

   where $\{k\}$ are the eigenvalues of the matrix K and $\phi_a(k)$ are the eigenvectors in the a-basis.

   (b) Show that in a diagram with a loop, we must sum over the eigenvalue label k. (For definiteness, consider the order-g correction to the propagator.)

   (c) Consider the case where $K_{ab} = t\left(\delta_{a,b+1} + \delta_{a+1,b}\right)$, with periodic boundary conditions: $a + N \equiv a$. Find the eigenvalues. Show that in this case if

   $$T_{abcd} q_a q_b q_c q_d = \sum_a g q_a^4$$

   the k-label is conserved at vertices, *i.e.* the vertex is accompanied by a delta function on the sum of the incoming eigenvalues.

   (d) (Bonus question) What is the more general condition on T_{abcd} in order that the k-label is conserved at vertices?

   (e) (Bonus question) Study the physics of the model described in 1c.

Back to the case without labels.

(f) By a change of integration variable show that

$$Z = \int_{-\infty}^{\infty} dq\ e^{-S(q)}$$

with $S(q) = \frac{1}{2}m^2q^2 + gq^4$ is of the form

$$Z = \frac{1}{\sqrt{m^2}} \mathcal{Z}\left(\frac{g}{m^4}\right).$$

This means you can make your life easier by setting $m = 1$, without loss of generality.

(g) Convince yourself (*e.g.* with Mathematica) that the integral really is expressible as a Bessel function.

(h) It would be nice to find a better understanding for why the partition function of (0+0)-dimensional ϕ^4 theory is a Bessel function. Then find a Schwinger-Dyson equation for this system which has the form of Bessel's equation for

$$K(y) \equiv \frac{1}{\sqrt{y}} e^{-a/y} \mathcal{Z}(1/y)$$

for some constant a. (Hint: I found it more convenient to set $g = 1$ for this part and use $\xi \equiv m^2$ as the argument. If you get stuck I can tell you what function to choose for the 'anything' in the S-D equation.)

(i) Make a plot of the perturbative approximations to the 'Green function' $G \equiv \langle q^2 \rangle$ as a function of g, truncated at orders 1 through 6. Plot them against the exact answer.

(j) (Bonus problem) Show that $c_{n+1} \sim -\frac{2}{3}nc_n$ at large n (by brute force or by cleverness).

2. **Combinatorics from 0-dimensional QFT.**

This problem may not be fully de-bugged yet. Let me know if you run into trouble.

Catalan numbers $C_n = \frac{(2n)!}{n!(n+1)!}$ arise as the answer to many combinatorics problems (beware: there is some disagreement about whether this is C_n or C_{n+1}).

One such problem is: count random walks on a 1d chain with $2n$ steps which start at 0 and end at 0 without crossing 0 in between.

Another such problem is: in how many ways can $2n$ (distinguishable) points on a circle be connected by chords which do not intersect within the circle.

Consider a zero-dimensional QFT with the following Feynman rules:

- There are two fields h and l.
- There is an $\sqrt{t}h^2l$ vertex in terms of a coupling t.
- The bare l propagator is 1.
- The bare h propagator is 1.
- All diagrams can be drawn on a piece of paper without crossing.[1]
- There are no loops of h.

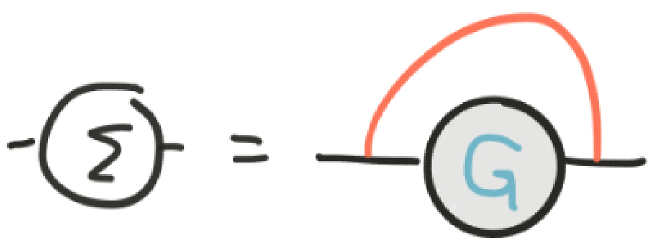
The last two rules can be realized from a lagrangian by introducing a large N (below).

(a) Show that the full green's function h is

$$G(t) = \sum_n t^n C_n$$

the generating function of Catalan numbers.

(b) Argue by diagrams for the Schwinger-Dyson equation



where Σ is the 1PI self-energy of h.

(c) Solve this equation for the generating function $G(t)$.

(d) For what combinatorics problem is the full h green's function $G(t) = \frac{1}{1-\Sigma(t)}$ the generating function?

[1] An annoying extra rule: All the l propagators must be on one side of the h propagators. You'll see in part 2f how to justify this.

(e) If you are feeling ambitious, add another coupling N^{-1} which counts the crossings of the l propagators. The resulting numbers can be called Touchard-Riordan numbers.

(f) How to realize the no-crossings rule? Consider

$$L = \frac{\sqrt{t}}{\sqrt{N}} l_{\alpha\beta} h_\alpha h_\beta + \sum_{\alpha,\beta} l_{\alpha\beta}^2 + \sum_{\alpha} h_\alpha^2$$

where $\alpha, \beta = 1 \cdots N$. By counting index loops, show that the dominant diagrams at large N are the ones we kept above. Hint: to keep track of the index loops, introduce ('t Hooft's) double-line notation: since l is a matrix, it's propagator looks like: $\begin{matrix} \alpha - - - - - - - - \alpha \\ \beta - - - - - - - - \beta \end{matrix}$, while the h propagator is just one index line α______α, and the vertex is ___!!___. If you don't like my ascii diagrams, here are the respective pictures: $\langle l_{\alpha\beta} l_{\alpha\beta} \rangle =$  ,

$\langle h_\alpha h_\alpha \rangle =$  and the hhl vertex is:  .

(g) Use properties of Catalan numbers to estimate the size of non-perturbative effects in this field theory.

(h) There are many other examples like this. Another similar one is the relationship between symmetric functions and homogeneous products. A more different one is the enumeration of planar graphs. For that, see BIPZ.

3. **Pair production in a uniform electric field (Schwinger effect)**

This problem is also brand new. Please come ask me if something is confusing.

We wish to estimate the rate of charged particle anti-particle production from the vacuum in the presence of a ginormous background uniform electric field. For simplicity we'll consider sometimes charged massive scalars,

$$S = \int d^4x D_\mu \Phi^\star D^\mu \Phi - m^2 \Phi^\star \Phi, \quad D_\mu \Phi = (\partial_\mu - \mathbf{i} e A_\mu) \Phi.$$

Notice that this action is quadratic in Φ, so we should be able to make progress.

(a) We would like to compute the *vacuum persistence amplitude*, whose square is the probability *not* to create particles. In terms of interaction picture quantities, this is an element of the S-matrix:

$$S_{00} = \langle 0 | \mathcal{T} e^{\mathbf{i} \int_{-\infty}^{\infty} V} | 0 \rangle .$$

Think about trying to evaluate this by Wick contractions in scalar QED or ordinary QED (treating the gauge field as a background), where $V = \int d^3x \Phi^\dagger A_\mu \partial^\mu \Phi$ or $\int d^3x \bar{\Psi} \not{A} \Psi$ respectively.

(b) Convince yourself by the exponentiation of the disconnected diagrams that

$$S_{00} = e^{\mathbf{i}\int d^4x w(x)} \overset{\text{uniform } E}{=} e^{\mathbf{i}VTw/2}.$$

(c) Convince yourself that $\operatorname{Im} w$ can be interpreted as the probability density per unit time per unit volume for pair creation.

(d) [not really a question] Now we try to calculate w. It is actually just the effective Lagrangian from integrating out the charged fields in the backround electric field, just as in our path integral calculation of the chiral anomaly. For the case of Dirac fermions,

$$e^{\mathbf{i}VTw[A]} = \int [d\psi] e^{\mathbf{i}\int d^4x \bar{\Psi}\left(\mathbf{i}\not{D} - m\right)\Psi} = \det\left(\not{D} - m\right) = e^{\frac{1}{2}\operatorname{tr}\ln\left(\not{D}^2 - m^2\right)}$$

so that

$$w = \frac{\mathbf{i}}{2} \int_0^\infty \frac{dT}{T} e^{-\mathbf{i}Tm^2} \langle x | e^{-\mathbf{i}\not{D}^2 T} | x \rangle .$$

Here we used the identity

$$\ln \frac{B}{A} = \int_0^\infty \frac{dT}{T} \left(e^{-T(B+\mathbf{i}\epsilon)} - e^{-T(A+\mathbf{i}\epsilon)} \right)$$

where A is a constant we can ignore. We can massage this expression to arrive at the answer below, but it will be useful to think about it from the point of view of a single particle.

(e) The only connected diagram is a single particle loop, in the background of A. We could evaluate this in the same way as in the anomaly calculation in lecture. Instead, because the process only ever involves one particle, we can use the single-particle path integral.

The next few lines are instructions about how to arrive at a correct expression for that integral. If you wish, skip to (2). (An alternate route from here to (2) can be found in Itzykson-Zuber, page 193-4.) For a relativistic scalar particle, this is

$$w[A] = \int [dx] e^{-m\int ds \sqrt{\dot{x}^2} - \mathbf{i}e\int ds A_\mu \dot{x}^\mu}. \tag{1}$$

The square root is bad, let's get rid of it by introducing an auxiliary variable (worldline metric) e:

$$w[A] \simeq \int [dx de] e^{-\int ds\left(m^2 e + \frac{\dot{x}^2}{2e} - \mathbf{i}A_\mu \dot{x}^\mu\right)}.$$

Solving the equations of motion for e and plugging back in gives back (1). This path integral has a gauge redundancy of worldline reparametrizations, $s \to s(\tau), e \to e\frac{d\tau}{ds}$. Fix this redundancy by setting $e =$ constant. Actually the value of this constant is meaningful (it is the proper time duration T of the path) and we must integrate over it. The resulting integral is[2]

$$w[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(0)=x(T)} [dx] e^{-\int_0^T ds \left(\frac{1}{2}\dot{x}^2 + \mathbf{i}eA\dot{x}\right)}. \tag{2}$$

We consider periodic orbits because we are studying vacuum matrix elements. [3] It is convenient to rescale $T \to m^2 T, s = Tu$ so that the period is 1:

$$w[A] = \int_0^\infty \frac{dT}{T} e^{-T} \int_{x(0)=x(1)} [dx] e^{-\int_0^1 du \left(\frac{m^2}{2T}\dot{x}^2 + \mathbf{i}eA\dot{x}\right)}. \tag{3}$$

Do the T integral by saddle point.

Now focus on constant electric field, in the gauge $A_3 = Ex^0$. Find the conditions for a saddle point of the X integral. Contract with $\dot{x}^\mu$ to show that $\dot{x}^2 \equiv a^2$ is a constant (to be determined). Solve the equation (hint: cyclotron orbits, $x^3(u) = \frac{m}{eE}\cos\frac{eEa}{m}u, x^0(u) = \frac{m}{eE}\sin\frac{eEa}{m}u$). Show that periodicity requires $a = \frac{m}{eE}2\pi n, n \in \mathbb{Z}$.

Find the on-shell action. The solutions for $n > 1$ are multiple-instantons, saddle points with smaller action than $n = 1$. Arrive at an expression of the form

$$\mathrm{Im}\, w \sim \sum_{n=1} f(n) e^{-n\frac{m^2\pi}{|eE|}}$$

where $f(n)$ is a function to be determined by studying the gaussian fluctuations about the saddles.

(f) Argue that our saddle point approximations were a good idea when $m^2 \gg eE$. This in fact that case for electric fields humans can make, if m is the mass of the electron.

(g) Why is this a contribution to the *imaginary* part? Here we need to include the fluctuations about the saddle point. They contribute a factor of $\frac{1}{\sqrt{\det_{ij} S_{ij}}}$ where S_{ij} denotes the matrix of second derivatives of the action. If this thing has a negative eigenvalue (rather, an odd number of them) the $\sqrt{\det}$ will be imaginary. See if you can verify the existence of a negative eigenvalue here.

[2]We can slimily justify the $1/T$ in the measure by demanding that the result is scale invariant when $m = 0$. If you like worrying about this kind of factor, consider learning about perturbative string theory.

[3]Actually: you can ask where did the factor of VT go. We really have a marked point on our trajectory – the particle propagates from x to x, and we have to integrate over x.

This process hasn't been observed in vacuum so far, because we don't know how to make a big enough electric field to make the probability for production appreciable over human time and length scales. However, there are analogous phenomena of dielectric breakdown in material insulators. See here for a discussion of the analogy.