University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2017
Assignment 9

Due 12:30pm Wednesday, June 7, 2017

1. **An application of the anomaly to a theory without gauge fields.**

   Consider a 1+1d theory of Dirac fermions coupled to a background scalar field θ as follows:
$$\mathcal{L} = \bar{\Psi}\left(\partial\!\!\!/ + m e^{\mathbf{i}\theta\gamma^5}\right)\Psi.$$
   We wish to ask: if we subject the fermion to various configurations of $\theta(x)$ (such as a domain wall where $\theta(x) = \pi\theta(x)$) what does the fermion number do in the groundstate?

   (a) Convince yourself that when θ is constant
$$\langle j^\mu \rangle = 0$$
   where $j^\mu = \bar{\Psi}\gamma^\mu\Psi$ is the fermion number current.

   (b) Minimally couple the fermion to a *background* gauge field A_μ. Let $e^{\mathbf{i}\Gamma[A,\theta]} = \int [d\Psi] e^{\mathbf{i}S}$. Convince yourself that the term linear in A in $\Gamma[A,\theta] = \text{const} + \int A_\mu J^\mu + \mathcal{O}(A^2)$ is the vacuum expectation value of the current $\langle j^\mu \rangle = J^\mu$.

   (c) Show that by a local chiral transformation $\Psi \to e^{\mathbf{i}\theta(x)\gamma^5/2}$ we can remove from the action the position dependence from θ.

   (d) Where does the theta-dependence go? Use the 2d chiral anomaly to relate $\langle j^\mu \rangle$ to $\partial\theta$. Notice that the result is independent of m. [This relation was found by Goldstone and Wilczek. The associated physics is realized in Polyacetylene.]

   (e) Show that a domain wall where θ jumps from 0 to π localizes *fractional* fermion number.

   (f) [bonus problem] Consider the Dirac hamiltonian in the presence of such a soliton. Show that there is a localized mode of zero energy.

2. **T-duality: not just for the free theory.** [Polchinski problem 8.3] Here is a path integral derivation of T-duality which is more general than just a single free boson.

   Consider the sigma model whose action is

$$S(\partial X, Y) = S(Y) + \frac{1}{4\pi\alpha'} \int d^2z \left(\delta^{ab} G_{XX}(Y) \partial_a X \partial_b X + \left(\delta^{ab} G_{\mu X} + \epsilon^{ab} B_{\mu X} \right) \partial_a X \partial_b Y^\mu \right) \ .$$

   Here Y^μ are a bunch of coordinates on which the background fields G, B may depend in arbitrarily complicated ways. X only appears through its derivatives.

   (a) Show that by replacing $\partial_\mu X$ by $\partial_\mu X + A_\mu$ we arrive at a theory with an invariance under local shifts of $X \to X + \alpha(x)$.

   (b) Add a 2d θ term $\mathbf{i}\phi F_{\mu\nu}$, with $F = dA$ and the angle ϕ a dynamical field. Show that the path integral over ϕ undoes the previous step and returns us to the original model. Hint: use the gauge $\partial_\mu A^\mu = 0$.

   (c) Instead choose the gauge $X = 0$ and do the integral over A_μ. Identify ϕ as the T-dual variable. To get the period right, you need to think about non-perturbative parts of the gauge field path integral.

3. **T-duality as EM duality of 0-forms.**

   In this problem we will contextualize the form of the T-duality map

$$\phi(z, \bar{z}) = \phi_L(z) + \phi_R(\bar{z}) \mapsto \tilde{\phi}(z, \bar{z}) \equiv \phi_L(z) - \phi_R(\bar{z})$$

   in terms of more general duality maps on form fields.

   Consider a massless p-form field a in D (euclidean) dimensions, more specifically, on $\mathbb{R}^D$. We will treat it classically. Suppose its eom are

$$\mathrm{d} \star \mathrm{d}a = 0 \ .$$

   [1]

[1] By this notation, I mean the following. The exterior derivative of a p-form is a $p+1$ form:

$$(\mathrm{d}a)_{\mu_1 \cdots \mu_{p+1}} = \left(\partial_{\mu_1} a_{\mu_2 \cdots \mu_{p+1}} \pm \text{perms} \right) \frac{1}{(p+1)!}$$

The Hodge dual of a k-form is a $d-k$ form:

$$(\star\omega_k))_{\mu_1 \cdots \mu_{d-k}} \equiv \epsilon_{\mu_1 \cdots \mu_d} (\omega_k)^{\mu_{d-k+1} \cdots \mu_d} \ .$$

This equation says $\star \mathrm{d}a$ is closed, which on $\mathbb{R}^D$ which has no nontrivial topology, this means it is exact: we can define $\star \mathrm{d}a = \mathrm{d}\tilde{a}$.

For abelian gauge theory in $D = 4$ show that this map $a \to \tilde{a}$ takes $(E, B) \to (\tilde{E}, \tilde{B}) = (B, -E)$.

Show that the map between ϕ and $\tilde{\phi}$ is of this form, if we regard ϕ as a 0-form potential.

For help see this paper by Chris Beasley.

4. **$\mathsf{SU}(2)$ current algebra from free scalar.**

Consider again a compact free boson $\phi \simeq \phi + 2\pi$ in $D = 1 + 1$ with action

$$S[\phi] = \frac{R^2}{8\pi} \int \mathrm{d}x\mathrm{d}t \partial_\mu \phi \partial^\mu \phi. \tag{1}$$

[Notice that if we redefine $\tilde{\phi} \equiv R\phi$ then we absorb the coupling R from the action $S[\tilde{\phi}] = \frac{1}{8\pi} \int \mathrm{d}x\mathrm{d}t \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi}$ but now $\tilde{\phi} \simeq \tilde{\phi} + 2\pi R$ has a different period – hence the name 'radius'.[2]]

So: there is a special radius (naturally called the $\mathsf{SU}(2)$ radius) where new operators of dimension $(1, 0)$ and $(0, 1)$ appear, and which are charged under the boson number current $\partial_\pm \phi$. Their dimensions tell us that they are (chiral) currents, and their charges indicate that they combine with the obvious currents $\partial_\pm \phi$ to form the (Kac-Moody-Bardakci-Halpern) algebra $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$.

Here you will verify that the model (1) does in fact host an $SU(2)_L \times SU(2)_R$ algebra involving *winding modes* – configurations of ϕ where the field winds around its target space circle as we go around the spatial circle. We'll focus on the holomorphic (R) part, $\phi(z) \equiv \phi_R(z)$; the antiholomorphic part will be identical, with bars on everything.

Define

$$J^\pm(z) \equiv :e^{\pm i\phi(z)}:, \qquad J^3 \equiv i\partial\phi(z).$$

The dots indicate a normal ordering prescription for defining the composite operator: no wick contractions between operators within a set of dots.

(a) Show that $J^3, J^\pm$ are single-valued under $\phi \to \phi + 2\pi$.

(b) Compute the scaling dimensions of $J^3, J^\pm$. Recall that the scaling dimension Δ of a holomorphic operator in 2d CFT can be extracted from its two-point correlation function:

$$\left\langle \mathcal{O}^\dagger(z)\mathcal{O}(0) \right\rangle \sim \frac{1}{z^{2\Delta}} \ .$$

[2] Relative to the notation I used in lecture, I have set $\pi T \equiv R^2$. A note for the string theorists: I am using units where $\alpha' = 2$.

For free bosons, all correlation functions of composite operators may be computed using Wick's theorem and

$$\langle \phi(z)\phi(0)\rangle = -\frac{1}{R^2}\log z.$$

Find the value of R where the vertex operators $J^\pm$ have dimension 1.

(c) Defining $J^\pm \equiv \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)$ show that the operator product algebra of these currents is

$$J^a(z)J^b(0) \sim \frac{k\delta^{ab}}{z^2} + i\epsilon^{abc}\frac{J^c(0)}{z} + ...$$

with $k = 1$. This is the level-$k = 1$ SU(2)Kac-Moody-Bardakci-Halpern algebra.

(d) [Bonus tedium] Defining a mode expansion for a dimension 1 operator,

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

show that

$$[J^a_m, J^b_n] = i\epsilon^{abc}J^c_{m+n} + mk\delta^{ab}\delta_{m+n}$$

with $k = 1$, which is an algebra called Affine $SU(2)$ at level $k = 1$. Note that the $m = 0$ modes satisfy the ordinary $SU(2)$ lie algebra.

For hints (and some applications in string theory) see problem 5 here.