1. Brain-warmer: chiral anomaly in two dimensions.

Consider a massive relativistic Dirac fermion in 1+1 dimensions, with

\[ S = \int dx dt \bar{\psi} (i \gamma^\mu (\partial_\mu + e A_\mu) - m) \psi. \]

By heat-kernel regularization of its expectation value, show that the divergence of the axial current \( j_5^\mu \equiv i \bar{\psi} \gamma^\mu \gamma^5 \psi \) is

\[ \partial_\mu j_5^\mu = 2im \bar{\psi} \gamma^5 \psi + \frac{e}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}. \]

2. Where to find a Chern-Simons term.

Consider a field theory in \( D = 2 + 1 \) of a massive Dirac fermion, coupled to a background \( U(1) \) gauge field \( A \):

\[ S[\psi, A] = \int d^3x \bar{\psi} (i \slashed{D} - m) \psi \]

where \( D_\mu = \partial_\mu - i A_\mu \).

(a) Convince yourself that the mass term for the Dirac fermion in \( D = 2 + 1 \) breaks parity symmetry. That is, parity takes \( m \rightarrow -m \). (Note that the definition of a parity transformation in \( d \) spatial dimensions is an element of \( O(d, 1) \) that’s not in \( SO(d, 1) \), i.e. one with \( \det(g) = -1 \).)

(b) We would like to study the effective action for the gauge field that results from integrating out the fermion field

\[ e^{-S_{\text{eff}}[A]} = \int [D\psi] e^{-S[\psi, A]}. \]

Focus on the term quadratic in \( A \):

\[ S_{\text{eff}}[A] = \int d^Dq A_\mu(q) \Pi^{\mu\nu}(q) A_\nu(-q) + ... \]

We can compute \( \Pi^{\mu\nu} \) by Feynman diagrams. Convince yourself that \( \Pi \) comes from a single loop of \( \psi \) with two \( A \) insertions.
(c) Evaluate this diagram using dim reg near $D = 3$. Show that, in the low-energy limit $q \ll m$ (where we can’t make on-shell fermions),

$$\Pi^{\mu\nu} = \frac{a m}{|m|} \epsilon^{\mu\nu\rho} q_\rho + \ldots$$

for some constant $a$. Find $a$. Convince yourself that in position space this is a Chern-Simons term with level $k = \frac{1}{2} \frac{m}{|m|}$.

(d) [bonus] Redo this calculation by doing the Gaussian path integral over $\psi$.

3. A bit more about Chern-Simons theory.

Consider again $U(1)$ gauge theory in $D = 2+1$ dimensions with the Chern-Simons action

$$S[a] = \frac{k}{4\pi} \int_\Sigma a \wedge da.$$  

(Here I’ve changed the name of the dynamical gauge field to a lowercase $a$ to distinguish it from the electromagnetic field $A$ which will appear anon.)

(a) Show that the Chern-Simons action is gauge invariant under $a \to a + d\lambda$, as long as there is no boundary of spacetime $\Sigma$. Compute the variation of the action in the presence of a boundary of $\Sigma$.

(b) [bonus] Actually, the situation is a bit more subtle than the previous part suggests. The actual gauge transformation is

$$a \to g^{-1} a g + \frac{1}{i} g^{-1} d g$$

which reduces to the previous if we set $g = e^{i \lambda}$. That expression, however, ignores the global structure of the gauge group (e.g. in the abelian case, the fact that $g$ is a periodic function). Consider the case where spacetime is $\Sigma = S^1 \times S^2$, and consider a large gauge transformation:

$$g = e^{i n \theta}$$

where $\theta$ is the coordinate on the circle. Show that the variation of the CS term is $\frac{k}{4\pi} \int g^{-1} \partial g \wedge f$ (where $f = da$). Since the action appears in the path integral in the form $e^{i S}$, convince yourself that the path integrand is gauge invariant if

1. $\int_\Gamma f \in 2\pi \mathbb{Z}$ for all closed 2-surfaces $\Gamma$ in spacetime, and
2. $k \in \mathbb{Z}$.

The first condition is called flux quantization, and is closely related to Dirac’s condition.
In the case where $\mathbf{G}$ is a non-abelian lie group, the argument for quantization of the level ($k$) is more straightforward. Show that the variation of the CS Lagrangian

$$L_{CS} = \frac{k}{4\pi} \text{tr} \left( a \wedge da + \frac{2}{3} a \wedge a \wedge a \right)$$

under $a \to gag^{-1} - \partial g g^{-1}$ is

$$L_{CS} \to L_{CS} + \frac{k}{4\pi} d\text{tr} dgg^{-1} \wedge a + \frac{k}{12\pi} \text{tr} (g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg).$$

The integral of the second term over any closed surface is an integer. Conclude that $e^{iS_{CS}}$ is gauge invariant if $k \in \mathbb{Z}$.

If there is a boundary of spacetime, something must be done to fix up this problem. Consider the case where $\Sigma = \mathbb{R} \times \text{UHP}$ where $\mathbb{R}$ is the time direction, and $\text{UHP}$ is the upper half-plane $y > 0$. One way to fix the problem is simply to declare that the would-be gauge transformations which do not vanish at $y = 0$ are not redundancies. This means that they represent physical degrees of freedom.

The exterior derivative on this spacetime decomposes into $d = \partial_t dt + \tilde{d}$ where $\tilde{d}$ is just the spatial part, and similarly the gauge field is $a = a_0 dt + \tilde{a}$.

Let us choose the gauge $a_0 = 0$. We must still impose the equations of motion for $a_0$ (in the path integral it is a Lagrange multiplier). Solve this equation, and evaluate the action for the resulting solution.

We can also add local terms at the boundary to the action. Consider adding $\Delta S = g \int_{\partial \Sigma} \tilde{a}_x^2$ (for some coupling constant $g$). In the presence of such a boundary term, find the equations of motion for the boundary degrees of freedom.

Interpretation: the Chern-Simons theory on a space with boundary necessarily produces a chiral edge mode.

Suppose we had a system in $2+1$ dimensions with a gap to all excitations, which breaks parity symmetry and time-reversal invariance, and involves a conserved current $J^\mu$, with

$$0 = \partial^\mu J_\mu.$$  \hspace{1cm} (1)

Solve this equation by writing $J^\mu = \epsilon^{\mu \nu \rho} \partial_\nu a_\rho$ in terms of a one-form $a = a_\mu dx^\mu$. Guess the leading terms in the action for $a_\mu$ in a derivative expansion.

Now suppose the current $J^\mu$ is coupled to an external electromagnetic field $A_\mu$ by $S \ni \int J^\mu A_\mu$. Ignoring the Maxwell term for $a$, compute the Hall conductivity, $\sigma^{xy}$, which is defined by Ohm’s law $J^x = \sigma^{xy} E^y$. 

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4. **An application of the anomaly to a theory without gauge fields.**

Consider a 1+1d theory of Dirac fermions coupled to a background scalar field $\theta$ as follows:

$$
\mathcal{L} = \bar{\Psi} \left( i\partial_\tau + me^{i\theta\gamma^5} \right) \Psi.
$$

We wish to ask: if we subject the fermion to various configurations of $\theta(x)$ (such as a domain wall where $\theta(x) = \pi + \theta(x)$) what does the fermion number do in the groundstate?

(a) Convince yourself that when $\theta$ is constant

$$
\langle j^\mu \rangle = 0
$$

where $j^\mu = \bar{\Psi} \gamma^\mu \Psi$ is the fermion number current.

(b) Minimally couple the fermion to a background gauge field $A_\mu$. Let $e^{i\Gamma[A,\theta]} = \int [d\Psi] e^{iS}$. Convince yourself that the term linear in $A$ in $\Gamma[A,\theta] = \text{const} + \int A_\mu J^\mu + \mathcal{O}(A^2)$ is the vacuum expectation value of the current $\langle j^\mu \rangle = J^\mu$.

(c) Show that by a local chiral transformation $\Psi \rightarrow e^{i\theta(x)\gamma^5/2} \Psi$ we can remove the dependence on $\theta$ from the mass term.

(d) Where does the theta-dependence go? Use the 2d chiral anomaly to relate $\langle j^\mu \rangle$ to $\partial \theta$. Notice that the result is independent of $m$. [This relation was found by Goldstone and Wilczek. The associated physics is realized in Polyacetylene.]

(e) Show that a domain wall where $\theta$ jumps from 0 to $\pi$ localizes *fractional* fermion number.

(f) [bonus problem] Consider the Dirac hamiltonian in the presence of such a soliton. Show that there is a localized mode of zero energy.