1. **Anomaly cancellation in the Standard Model.** If we try to gauge a chiral symmetry (such as hypercharge in the Standard Model (SM)), it is important that it is actually a symmetry, *i.e.* is not anomalous. In $D = 3 + 1$, a possible anomaly is associated with a choice of three currents, out of which to make a triangle diagram. We’ll call a “$G_1 G_2 G_3$ anomaly” the diagram with insertions of currents for $G_1, G_2$ and $G_3$. Generalizing a little, we showed that the divergence of the current for $G_1$ is

$$\partial_{\mu} J^{A_{\mu}}_1 = \frac{1}{32\pi^2} e^{\mu\nu\rho\sigma} F^{2B}_{\mu\nu} F^{3C}_{\rho\sigma} \sum_{f} (-1)^{f} \text{tr}_{R(f)} \{T_{A}^{1}, T_{B}^{2}\} T^{C}_{3}.$$  

The sum is over each Weyl fermion, $R(f)$ is its representation under the combined group $G_1 \times G_2 \times G_3$, and $T_{1}^{A}$ are a basis of generators of the Lie algebra of $G_1$ etc. in the representation of the field $f$. By $(-1)^{f}$ I mean $\pm$ for left- and right-handed fermions respectively.

We consider the possibilities in turn.

(a) Convince yourself that the divergence of the $U(1)_{Y}$ hypercharge current gets a contribution of the form

$$\partial_{\mu} J^{\mu}_{Y} = \left( \sum_{\text{left}} Y^{3}_{l} - \sum_{\text{right}} Y^{3}_{r} \right) \frac{g^{2}}{32\pi^2} e^{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma}$$

from the triangle with three insertions of the current itself (here $B$ is the hypercharge gauge field strength). The sum on the RHS is over all left- and right-handed Weyl spinors weighted by the cube of their hypercharge. Check that this sum evaluates to zero in the SM.

(b) Show that any anomaly of the form $SU(N)U(1)^{2}$ or $SU(N)G_1 G_2$ is zero.

(c) (Easy) Convince yourself that there is no $SU(3)_{3}$ anomaly for QCD.

(d) Check that there is never an $SU(2)_{3}$ anomaly. (Hint: the generators satisfy $\{\tau^{a}, \tau^{b}\} = 2\delta^{ab}$.)

(e) Show that the $SU(3)^2 U(1)_{Y}$ anomaly demands that $2Y_{Q} - Y_{u} - Y_{d} = 0$. Check that this is true in the SM.
(f) Show that a necessary condition for hypercharge to not have an anomaly with the Electroweak gauge bosons on the RHS is \( Y_L + 3Y_Q = 0 \), where \( Y_L \) and \( Y_Q \) are the hypercharges of the left-handed leptons and quarks. Check that this works out in the SM.

(g) There is another kind of anomaly called a gravitational anomaly. This is a violation of current conservation in response to coupling to curved space. An example is of the form

\[
\partial_\mu j^\mu_Y = a \text{tr} R \wedge R
\]

where \( R \) is a two-form related to the curvature of spacetime (analogous to the field strength \( F \)). The coefficient \( a \) is proportional to \( \sum_{\text{left}} \text{tr} Y_i - \sum_{\text{right}} \text{tr} Y^c_i \). Check that this too vanishes for hypercharge in the Standard Model.

These conditions, plus the assumption that the right-handed neutrino is neutral, actually determine all the hypercharge assignments.

2. **Right-handed neutrinos.**

Consider adding a right-handed singlet (under all gauge groups) neutrino \( N_R \) to the Standard Model. It may have a majorana mass \( M \); and it may have a coupling \( g_\nu \) to leptons, so that all the dimension \( \leq 4 \) operators are

\[
\mathcal{L}_N = \bar{N}_R i \gamma \partial N_R - \frac{M}{2} \bar{N}_R^c N_R - \frac{M}{2} \bar{N}_R N_R^c + (g_\nu \bar{N}_R H^T_i L_j \epsilon^{ij} + h.c.)
\]

where \( N_R^c = C (\bar{N}_R)^T \) is the the charge conjugate field, \( C = i \gamma_2 \gamma_0 \) (in the Dirac representation), \( H \) is the Higgs doublet, \( L \) is the left-handed lepton doublet, containing \( \nu_L \) and \( e_L \). Take the mass \( M \) to be large compared to the electroweak scale. Integrate out the right-handed neutrinos at tree level. [Hint: you may find it useful to work in terms of the Majorana field

\[
N \equiv N_R + N_R^c
\]

which satisfies \( N = N^c. \)]

Show that the leading term in the expansion in \( 1/M \) is a dimension-5 operator made of Standard Model fields. Explain the consequences of this operator for neutrino physics, assuming a vacuum expectation value for the Higgs field.

Place a bound on \( M \) assuming that the observed neutrinos have masses \( m_\nu < 0.5 \) eV.

Here’s an example which illustrates the manipulations we did in describing the BCS phenomenon. Now that we’ve learned about fermionic path integrals, consider the partition function for an $N$-vector of fermionic spinor fields in $D$-dimensions:

$$Z = \int [d\psi d\bar{\psi}] e^{iS[\psi]}, \quad S[\psi] = \int d^D x \left( \bar{\psi}^a i \sigma^a \psi^a - \frac{g}{N} (\bar{\psi}^a \psi^a)^2 \right).$$

(a) At the free fixed point, what is the dimension of the coupling $g$ as a function of the number of spacetime dimensions $D$? Show that it is classically marginal in $D = 2$, so that this action is (classically) scale invariant.

(b) We will show that this model in $D = 2$ exhibits dimensional transmutation in the form of a dynamically generated mass gap. Here are the steps: first use the Hubbard-Stratonovich trick to replace $\psi^4$ by $\sigma \psi^2 + \sigma^2$ in the action, where $\sigma$ is a scalar field. Then integrate out the $\psi$ fields. Find the saddle point equation for $\sigma$; argue that the saddle point dominates the integral for large $N$. Find a translation invariant saddle point. Plug the saddle point configuration of $\sigma$ back into the action for $\psi$ and describe the resulting dynamics.

4. Polyacetylene returns.

On HW01 problem 4, you may have wondered what is the connection between the field theory we were studying (a scalar coupled to fermions in $D = 2$) and polyacetylene. I’d like to explain that connection a bit.

Consider an extension of the model above to include also phonon modes, i.e. degrees of freedom encoding the positions of the ions in the solid. (Again we ignore the spins of the electrons for simplicity.)

$$H = -t \sum_n (1 + u_n) c_n^\dagger c_{n+1} + h.c. + \sum_n K (u_n - u_{n+1})^2 \equiv H_F + H_E.$$  

Here $u_n$ is the deviation of the $n$th ion from its equilibrium position (in the $+x$ direction), so the second term represents an elastic energy.

(a) (The part with the free massless Dirac field you had time to do on HW2.)

(b) Consider a configuration

$$u_n = \phi (-1)^n$$  

where the ions move closer in pairs. Compute the electronic spectrum. (Hint: this enlarges the unit cell. Define $c_{2n} \equiv a_n, c_{2n+1} \equiv b_n$, and solve in...
Fourier space, \( a_n \equiv \oint dke^{2\imath kn}a_k \). You should find that when \( \phi \neq 0 \) there is a gap in the electron spectrum (unlike \( \phi = 0 \)). Expand the spectrum near the minimum gap and include the effects of the field \( \phi \) in the continuum theory.

(c) **Peierls’ instability.** Compute the groundstate energy of the electrons \( H_F \) in the configuration (1), at half-filling (i.e. the number of electrons is half the number of available states). Check that you recover the previous answer when \( \phi = 0 \). Interpret the answer when \( \phi = 1 \).

Compute \( H_E \) in this configuration, and minimize the sum of the two as a function of \( \phi \).

(d) You should find that the energy is independent of the sign of \( \phi \). This means that there are two groundstates. We can consider a domain wall between a region of + and a region of −. Show that this domain wall carries a fermion mode whose energy lies in the bandgap and which carries fermion number \( \frac{1}{2} \).

(e) Verify the result of the previous part by diagonalizing the relevant tight-binding matrix.

(f) Time-reversal played an important role here. If we allow complex hopping amplitudes, we can make a domain wall without midgap modes. Explain this from field theory. Bonus: explain this from the lattice hamiltonian.