1. **Diagrammatic understanding of BCS instability of Fermi liquid theory.**

   (a) Recall that only the four-fermion interactions with special kinematics are marginal. Keeping only these interactions, show that cactus diagrams (like this: \( \frac{\mathcal{V}}{\mathcal{V}} \)) dominate.

   (b) To sum the cacti, we can make bubbles with a corrected propagator. Argue that this correction to the propagator is innocuous and can be ignored.

   (c) Armed with these results, compute diagrammatically the Cooper-channel susceptibility (two-particle Green’s function),

   \[ \chi(\omega_0) \equiv \left\langle T \psi_{\mathbf{k},\omega_3,\downarrow}^{\dagger} \psi_{-\mathbf{k},\omega_4,\uparrow}^{\dagger} \psi_{-\mathbf{p},\omega_1,\downarrow} \psi_{\mathbf{p},\omega_2,\uparrow} \right\rangle \]

   as a function of \( \omega_0 \equiv \omega_1 + \omega_2 \), the frequencies of the incoming particles. Think of \( \chi \) as a two point function of the Cooper pair field \( \Phi = \epsilon_{\alpha\beta} \psi_{\alpha}^{\dagger} \psi_{\alpha} \) at zero momentum.

   Sum the geometric series in terms of a (one-loop) integral kernel.

   (d) Do the integrals. In the loops, restrict the range of energies to \( |\omega| < E_D \) (or \( |\epsilon(k)| < E_D \)), the Debye energy, since it is electrons with these energies which experience attractive interactions.

   Consider for simplicity a round Fermi surface. For doing integrals of functions singular near a round Fermi surface, make the approximation \( \epsilon(k) \approx v_F (|k| - k_F) \), so that \( d^d k \approx k_F^{d-1} \frac{d^d \Omega}{v_F^d} d\Omega_{d-1} \).

   (e) Show that when \( V < 0 \) is attractive, \( \chi(\omega_0) \) has a pole. Does it represent a bound-state? Interpret this pole in the two-particle Green’s function as indicating an instability of the Fermi liquid to superconductivity. Compare the location of the pole to the energy \( E_{BCS} \) where the Cooper-channel interaction becomes strong.

   (f) **Cooper problem.** [optional] We can compare this result to Cooper’s influential analysis of the problem of two electrons interacting with each other.
in the presence of an inert Fermi sea. Consider a state with two electrons
with antipodal momenta and opposite spin
\[ |\psi\rangle = \sum_k a_k \psi_{k,\uparrow} \psi_{-k,\downarrow}^\dagger |F\rangle \]
where \( |F\rangle = \prod_{k<k_F} \psi_{k,\uparrow}^\dagger \psi_{k,\downarrow}^\dagger |0\rangle \) is a filled Fermi sea. Consider the Hamiltonian
\[ H = \sum_k \epsilon_k \psi_{k,\sigma}^\dagger \psi_{k,\sigma} + \sum_{k,k'} V_{k,k'} \psi_{k,\sigma}^\dagger \psi_{k',\sigma} \psi_{k',\sigma'}^\dagger \psi_{k,\sigma'}^\dagger. \]
Write the Schrödinger equation as
\[ (\omega - 2\epsilon_k) a_k = \sum_{k'} V_{k,k'} a_{k'}. \]
Now assume (following Cooper) that the potential has the following form:
\[ V_{k,k'} = V w_k^* w_{k'}, \quad w_k = \begin{cases} 1, & 0 < \epsilon_k < E_D \\ 0, & \text{else} \end{cases} \]
Defining \( C \equiv \sum_k \omega_k^* a_k \), show that the Schrödinger equation requires
\[ 1 = V \sum_k |w_k|^2 \omega - 2\epsilon_k. \]
Assuming \( V \) is attractive, find a bound state. Compare (14) to the condition for a pole found from the bubble chains above.

2. **Topological terms in QM.**

The purpose of this problem is to demonstrate that total derivative terms in the action (like the \( \theta \) term in QCD) do affect the physics.

The euclidean path integral for a particle on a ring with magnetic flux \( \theta = \int \vec{B} \cdot d\vec{a} \) through the ring is given by
\[ Z = \int [D\phi] e^{-\frac{\beta}{\hbar} \int_0^\beta d\tau \left( \frac{m}{2} \dot{\phi}^2 - \frac{\phi^2}{4\pi^2} \right) + \phi + 2\pi}. \]
Here
\[ \phi \equiv \phi + 2\pi \]
is a coordinate on the ring. Because of the identification (15), \( \phi \) need not be a single-valued function of \( \tau \) – it can wind around the ring. On the other hand, \( \phi \) is single-valued and periodic and hence has an ordinary Fourier decomposition. This means that we can expand the field as
\[ \phi(\tau) = \frac{2\pi}{\beta} Q\tau + \sum_{\ell \in \mathbb{Z} \setminus 0} \phi_\ell e^{\frac{i2\pi}{\beta} \ell \tau}. \]
(a) Show that the $\dot{\phi}$ term in the action does not affect the classical equations of motion. In this sense, it is a topological term.

(b) Using the decomposition (16), write the partition function as a sum over topological sectors labelled by the winding number $Q \in \mathbb{Z}$ and calculate it explicitly.

[Hint: use the Poisson resummation formula
\[ \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2} t n^2 + i z n} = \sqrt{\frac{2\pi}{t}} \sum_{\ell \in \mathbb{Z}} e^{-\frac{1}{2\pi} (z - 2\pi \ell)^2} . \]

(c) Use the result from the previous part to determine the energy spectrum as a function of $\theta$.

(d) Derive the canonical momentum and Hamiltonian from the action above and verify the spectrum.

(e) Consider what happens in the limit $m \to 0, \theta \to \pi$ with $X \equiv \frac{\theta - \pi}{m} \sim \beta^{-1}$ fixed. Interpret the result as the partition function for a spin $1/2$ particle. What is the meaning of the ratio $X$ in this interpretation?


(a) A useful device is the integral representation of the grassmann delta function. Show that
\[ -\int d\bar{\psi}_1 e^{-\bar{\psi}_1 (\psi_1 - \psi_2)} = \delta(\psi_1 - \psi_2) \]
in the sense that $\int d\psi_1 \delta(\psi_1 - \psi_2)f(\psi_1) = f(\psi_2)$ for any grassmann function $f$. (Notice that since the grassmann delta function is not even, it matters on which side of the $\delta$ we put the function: $\int d\psi_1 f(\psi_1)\delta(\psi_1 - \psi_2) = f(-\psi_2) \neq f(\psi_2)$.)

(b) Recall the resolution of the identity on a single qbit in terms of fermion coherent states
\[ \mathbb{I} = \int d\bar{\psi}d\psi \ e^{-\bar{\psi}\psi} \ket{\psi} \bra{\bar{\psi}} . \tag{4} \]
Show that $\mathbb{I}^2 = \mathbb{I}$. (The previous part may be useful.)

(c) In lecture I claimed that a representation of the trace of a bosonic operator was
\[ \text{tr}A = \int d\bar{\psi}d\psi \ e^{-\bar{\psi}\psi} \bra{\bar{\psi}} A \ket{\psi} , \]
and the minus sign in the bra had important consequences. (Here $\bra{-\bar{\psi}} c^\dagger = \bra{-\bar{\psi}} (-\bar{\psi})$.)
Check that using this expression you get the correct answer for
\[
\text{tr}(a + bc^\dagger c)
\]
where \(a, b\) are ordinary numbers.

(d) Prove the identity (20) by expanding the coherent states in the number basis.

4. **Fermionic coherent state exercise.**

Consider a collection of fermionic modes \(c_i\) with quadratic hamiltonian \(H = \sum_{ij} h_{ij} c_i^\dagger c_j\), with \(h = h^\dagger\).

(a) Compute \(\text{tr} e^{-\beta H}\) by changing basis to the eigenstates of \(h_{ij}\) (the single-particle hamiltonian) and performing the trace in that basis: \(\text{tr}... = \prod \sum_{n_x} c_{\epsilon x} = 0,1 \ldots\).

(b) Compute \(\text{tr} e^{-\beta H}\) by coherent state path integral. Compare!

(c) [super bonus problem] Consider the case where \(h_{ij}\) is a random matrix. What can you say about the thermodynamics?