University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 230 Quantum Phases of Matter, Spr 2024
Assignment 1 – Solutions

Due 11pm Thursday, April 11, 2024

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1. **Simple stabilizer codes.**

   I've mentioned the toric code as an important solvable example of topological order. We can solve it because it is an example of what is called a *stabilizer code*. This means that the systems is made from a bunch of qubits and all the terms in the Hamiltonian are made of Pauli Xs and Zs[1] and all commute with each other. This problem is a warmup problem to get used to this idea.

   (a) Consider the Hamiltonian on two qbits

   $$-H = X_1 X_2 + Z_1 Z_2.$$

   Show that the terms commute and that the groundstate is

   $$\frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

   (b) Consider the (non-local) Hamiltonian on N qbits

   $$H_{\rm GHZ} = -X_1 \cdots X_N - \sum_{i=1}^{N-1} Z_i Z_{i+1}. \quad (1)$$

   Show that all the terms commute. Show that the groundstate is (the 'GHZ state' or 'cat state')

   $$\frac{|00...0\rangle + |11...1\rangle}{\sqrt{2}}.$$

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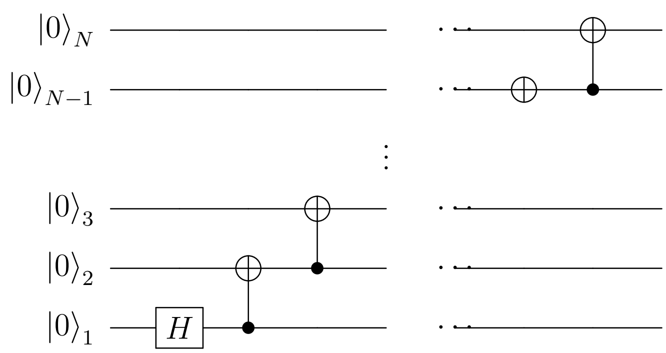
[1] A comment about notation: the notation $\sigma^x_\ell, \sigma^z_\ell$ is pretty terrible (at least for someone with deteriorating eyesight like me) because the crucial information (x or z) is hidden in the superscript. Much better is to write

$$\sigma^x_\ell \equiv X_\ell, \sigma^z \equiv Z_\ell.$$

Also, I use $|0\rangle$, $|1\rangle$ to denote the ± 1 eigenstates of Z, and $|\pm\rangle$ to denote the ± 1 eigenstates of X.

(c) [bonus] Show that the following circuit U produces the GHZ state from the product state $|0\rangle^{\otimes N}$.



Let me explain the notation. Each horizontal line represents a qubit. H represents the 'hadamard gate' acting on one qubit by $H|\uparrow\rangle = |+\rangle, H|\downarrow\rangle = |-\rangle$, *i.e.*

$$H = |+\rangle\langle\uparrow| + |-\rangle\langle\downarrow|. \tag{2}$$

The vertical line segments represent the 'control-X gate' that acts on two qubits by

$$\mathsf{CX} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X. \tag{3}$$

(The first qubit is the one with the black dot, and the second is the one with the circled +.)

Notice that this circuit has depth N.

One way to do it is to find the image under conjugation by U of the 'stabilizers' Z_i of the product state $\otimes_i |0\rangle_i$.

(d) [bonus] What state does U produce from $|1\rangle_1 \otimes |0\rangle^{\otimes N-1}$?

(e) [bonus] Find the result of feeding the Hamiltonian $-\sum_i Z_i$ (whose groundstate is the product state $|0\rangle^{\otimes N}$) through the circuit, *i.e.* what is

$$U\left(-\sum_i Z_i\right)U^\dagger \quad ?$$

Hint: use the rules for the action of CX by conjugation given in lecture.

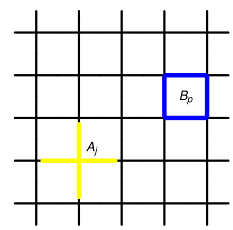
$$H_{\text{GHZ}} = U\left(-\sum_i Z_i\right)U^\dagger.$$

2. **Gauge theory can emerge from a local Hilbert space.**

The Hilbert space of a gauge theory is a funny thing: states related by a gauge transformation are physically equivalent. In particular, it is not a tensor product over independent local Hilbert spaces associated with regions of space. Because of this, there is much hand-wringing about defining entanglement in gauge theory. The following is helpful for thinking about this. It is a realization of $\mathbb{Z}_2$ lattice gauge theory, beginning from a model with no redundancy in its Hilbert space. In this avatar it is due to Kitaev and is called the *toric code*.

To define the Hilbert space, put a qubit on every link ℓ of a lattice, say the 2d square lattice, so that $\mathcal{H} = \otimes_\ell \mathcal{H}_\ell$. Let $\sigma^x_\ell, \sigma^z_\ell$ be the associated Pauli operators, and recall that $\{\sigma^x_\ell, \sigma^z_\ell\} = 0$. $\mathcal{H}_\ell = \text{span}\{|\sigma^z_\ell = 1\rangle, |\sigma^z_\ell = -1\rangle\}$ is a useful basis for the Hilbert space of a single link.

One term in the hamiltonian is associated with each site $j \to A_j \equiv \prod_{l \in j} \sigma^z_l$ and one with each plaquette $p \to B_p \equiv \prod_{l \in \partial p} \sigma^x_l$, as indicated in the figure at right.



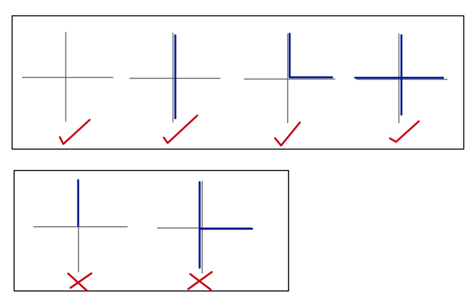
$$\mathbf{H} = -\Gamma_e \sum_j A_j - \Gamma_m \sum_p B_p.$$

(a) Show that all these terms commute with each other.

Any pair of A_j and B_p share zero or two links.

(b) The previous result means we can diagonalize the Hamiltonian by diagonalizing one term at a time. Let's imagine that $\Gamma_e \gg \Gamma_m$ so we'll minimize the 'star' terms A_j first. Which states satisfy the 'star condition' $A_j = 1$? In the σ^x basis there is an extremely useful visualization: we say a link l of $\hat{\Gamma}$ is covered with a segment of string (an electric flux line) if ${\sigma^z}_l = -1$ (so the electric field on the link is $\mathbf{e}_l = 1$) and is not covered if ${\sigma^z}_l = +1$ (so the electric field on the link is $\mathbf{e}_l = 0$):  $\equiv (\sigma^z_\ell = -1)$. Draw all possible configurations incident on a single vertex j and characterize which ones satisfy $A_j = 1$.

In the figure at right, we enumerate the possibilities for a 4-valent vertex. $A_j = -1$ if a flux line ends at j.



So the subspace of $\mathcal{H}$ satisfying the star condition is spanned by closed-string states, of the form $\sum_{\{C\}} \Psi(C) |C\rangle$.

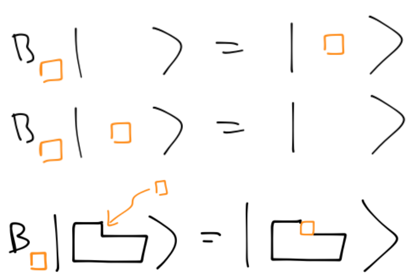
The σ^z term penalizes string configurations according to their length. This is just the role of the E^2 term in the Maxwell action – electric flux costs energy.

(c) [bonus] What is the effect of adding a term $\mathbf{\Delta H} = \sum_\ell g\sigma^x$? Convince yourself that in the limit $\Gamma_e \gg \Gamma_m$, for energies $E \ll \Gamma_e$, this is identical to $\mathbb{Z}_2$ lattice gauge theory, where $A_j = 1$ is a discrete version of the Gauss law constraint.

You can get help from section V.E of this Kogut review.]

(d) Set $g = 0$ again. In the subspace of solutions of the star condition, find the groundstate(s) of the plaquette term. First consider a simply-connected region of lattice, then consider periodic boundary conditions.

Now we look at the action of B_p on this subspace of states:



$$B_p |C\rangle = |C + \partial p\rangle$$

The condition that $B_p |\text{gs}\rangle = |\text{gs}\rangle$ is a homological equivalence. In words, the eigenvalue equation $\mathbf{B}_\square = 1$ says $\Psi(C) = \Psi(C')$ if C' and C can be continuously deformed into each other by attaching or removing plaquettes. To see how to make this connection with homology more explicit see Appendix A of these notes.

Here is a punchline to this problem. If the lattice is simply connected – if all curves are the boundary of some region contained in the lattice – then this means the groundstate

$$|\text{gs}\rangle = \sum_C |C\rangle$$

is a uniform superposition of all loops.

If the space has non-contractible loops, then the eigenvalue equation does not determine the relative coefficients of loops of different topology! On a space with $2g$ independent non-contractible loops (such as a closed surface with g handles), there are 2^{2g} independent groundstates.

No local operator mixes these groundstates. This makes the topological degeneracy stable to local perturbations of the Hamiltonian. They are connected by the action of V, W – Wilson loops:

$$W_C = \prod_{\ell \in C} \sigma^x, \quad V_{\check{C}} = \prod_{\ell \perp \check{C}} \sigma^z.$$

They commute with $\mathbf{H}_{\text{TC}}$ and don't commute with each other (specifically W_C anticommutes with $V_{\check{C}}$ if C and $\check{C}$ intersect an odd number of times).

These are the promised operators (called $\mathcal{F}_{x,y}$ above) whose algebra is represented on the groundstates.

For more of this story see section 2.2 of these notes.

3. **Groundstate degeneracy and 1-form symmetry algebra.**

(a) Suppose we have a system with Hamiltonian H with string operators W_C and $V_{\check{C}}$ supported on closed curves, and commuting with H, and satisfying $W^N = V^N = 1$.

In all parts of this problem you should make the assumption that the string operators are *deformable*: W_C acts in the same way as $W_{C+\partial p}$ on groundstates.

Suppose $[W_C, W_{C'}] = 0, [V_{\check{C}}, \check{C}']$ for all curves but

$$W_C V_{\check{C}} = \omega^{\#\left(C \cap \check{C}\right)} V_{\check{C}} W_C$$

where $\omega \equiv e^{\frac{2\pi \mathbf{i}}{N}}$ and $\#\left(C \cap \check{C}\right)$ is the number of intersection points of the curves. How many groundstates does such a system have on the two-torus (that is, with periodic boundary conditions on both spatial directions)?

This is what happens in the $\mathbb{Z}_N$ toric code.

For each non-contractible cycle $C_{x,y}$ of the torus, we get a pair of string operators $W_C, V_{\check{C}}$, with

$$W_{C_x} V_{\check{C}_y} = \omega V_{\check{C}_y} W_{C_x}, \quad W_{C_y} V_{\check{C}_x} = \omega^{-1} V_{\check{C}_x} W_{C_y}$$

– note that the orientation of the intersection matters now. Let's diagonalize W_{C_x} and W_{C_y}. Their eigenvalues are roots of unity. Starting from a state $|(1,1)\rangle$ with eigenvalues $(1,1)$, the action of $V_{\check{C}_y}^n V_{\check{C}_x}^m$ generates

$$|(n, -m)\rangle = V_{\check{C}_y}^n V_{\check{C}_x}^m \, |(1,1)\rangle$$

with the eigenvalues ω^n and ω^{-m} under W_{C_x} and W_{C_y}. Since they have different eigenvalues (for $n, m \in \{0...N-1\}$)), they are linearly independent. This gives N^2 groundstates as the minimal representation of this algebra. On a genus g Riemann surface, with g conjugate pairs of cycles, we would find N^{2g} groundstates.

(b) Now suppose in a different system we have just one set of string operators W_C satisfying

$$W_C W_{C'} = \omega^{\#(C \cap C')} W_{C'} W_C,$$

with the same definitions as above. How many groundstates does this system have on the two-torus?

This is what happens in the Laughlin fractional quantum Hall state with filling fraction $\frac{1}{N}$.

Now we get just one conjugate pair of operators on the torus:

$$W_{C_x} W_{C_y} = \omega W_{C_y} W_{C_x}.$$

Acting on an eigenstate $|1\rangle$ of W_{C_x} with eigenvalue 1, $W_{C_y}^n$ generates

$$|n\rangle = W_{C_y}^n |1\rangle$$

with W_{C_x}-eigenvalue ω^n. Therefore the minimal representation has N states. On a genus-g Riemann surface, there would be N^g states.

(c) [Bonus problem] Redo the previous problems for a genus g Riemann surface, *i.e.* the surface of a donut with g handles.

For the topological order in part a (which is called $\mathbb{Z}_N$ gauge theory), there are two conjugate pairs of operators for each handle, and therefore there are N^{2g} groundstates. For the topological order in part b (which is called $\mathsf{U}(1)_N$ (this is pronounced '$\mathsf{U}(1)$ level N') FQHE), there is only one conjugate pair for each handle, and hence N^g groundstates.

4. **Simplicial homology and the toric code.** [Bonus]

The toric code is a physical realization of *homology*, a construction that extracts topological invariants of topological spaces. This problem explains the relation.

(a) The toric code we've discussed so far has qbits on the links $\ell \in \Delta_1(\mathbf{\Delta})$ of a graph $\mathbf{\Delta}$. But the definition of the Hamiltonian involves more information than just the links of the graph: we have to know which vertices v lie at the boundaries of each link ℓ, and we have to know which links are boundaries of which faces. The Hamiltonian has two kinds of terms: a 'plaquette' operator $B_p = \prod_{\ell \in \partial p} X_\ell$ associated with each 2-cell (plaquette) $p \in \Delta_2(\mathbf{\Delta})$. and 'star' operators, $A_s = \prod_{\ell \in \partial^{-1}(s)} Z_\ell$, associated with each 0-cell (site) $s \in \Delta_0(\mathbf{\Delta})$. Here I've introduced some notation that will be useful, please be patient: Δ_k denotes a collection of k-dimensional polyhedra which I'll call k-simplices or more accurately k-cells – k-dimensional objects making up the space. (It is important that each of these objects is topologically a k-ball.) This information constitutes (part of) a *simplicial complex*, which says how these parts are glued together:

$$\Delta_d \xrightarrow{\partial} \Delta_{d-1} \xrightarrow{\partial} \cdots \Delta_1 \xrightarrow{\partial} \Delta_0 \quad (4)$$

where ∂ is the (signed) boundary operator. For example, the boundary of a link is $\partial \ell = s_1 - s_0$, the difference of the vertices at its ends. The boundary of a face $\partial p = \sum_{\ell \in \partial p} \ell$ is the (oriented) sum of the edges bounding it. By $\partial^{-1}(s)$ I mean the set of links which contain the site s in their boundary (with sign).

Think of this collection of objects as a triangulation (or more generally some chopping-up) of a smooth manifold X. Convince yourself that the sequence of maps (4) is a complex in the sense that $\partial^2 = 0$ (mod two).

(b) [not actually a question] This means that the simplicial complex defines a set of homology groups, which are topological invariants of X, in the following way. (It is homology and not cohomology because ∂ decreases the degree k). To define these groups, we should introduce one more gadget, which is a collection of vector spaces over some ring R (for the ordinary toric code, $R = \mathbb{Z}_2$)

$$\Omega_p(\mathbf{\Delta}, R), \ p = 0...d \equiv \dim(X)$$

basis vectors for which are p-simplices:

$$\Omega_p(\mathbf{\Delta}, R) = \text{span}_R\{\sigma \in \Delta_p\}$$

– that is, we associate a(n orthonormal) basis vector to each p-simplex (which I've just called σ), and these vector spaces are made by taking linear combinations of these spaces, with coefficients in R. Such a linear combination of p-simplices is called a *p-chain*. It's important that we can *add* (and subtract) p-chains, $C + C' \in \Omega_p$. A p-chain with a negative coefficient can be regarded as having the opposite orientation. We'll see below how better to interpret the coefficients.

The boundary operation on Δ_p induces one on Ω_p. A chain C satisfying $\partial C = 0$ is called a *cycle*, and is said to be *closed*.

So the pth homology is the group of equivalence classes of p-cycles, modulo boundaries of $p + 1$ cycles:

$$H_p(X, R) \equiv \frac{\ker (\partial : \Omega_p \to \Omega_{p-1}) \subset \Omega_p}{\text{Im} (\partial : \Omega_{p+1} \to \Omega_p) \subset \Omega_p}$$

This makes sense because $\partial^2 = 0$ – the image of $\partial : \Omega_{p+1} \to \Omega_p$ is a subset of $\ker (\partial : \Omega_p \to \Omega_{p-1})$. It's a theorem that the dimensions of these groups are the same for different (faithful-enough) discretizations $\mathbf{\Delta}$ of X. Furthermore, their dimensions (as vector spaces over R) $b_p(X)$ contain (much of[2]) the same information as the Betti numbers defined by de Rham cohomology. For more information and proofs, see the great book by Bott and Tu, *Differential forms in algebraic topology*.

[2] I don't want to talk about torsion homology.

(c) A state of the toric code on a cell-complex $\mathbf{\Delta}$ can be written (for the hamiltonian described above, this is in the basis where Z_ℓ is diagonal) as an element of $\Omega_1(X, \mathbb{Z}_2)$,

$$|\Psi\rangle = \sum_C \Psi(C) |C\rangle$$

where C is an assignment of an element of $\mathbb{Z}_2$ in X (the eigenvalue of Z_ℓ). For the case of $\mathbb{Z}_2$ coefficients, $1 = -1 \mod 2$ and we don't care about the orientations of the cells. Show that the conditions for a state $\Psi(C)$ to be a groundstate of the toric code ($A_s |\Psi\rangle = |\Psi\rangle \, \forall s$ and $B_p |\Psi\rangle = |\Psi\rangle \, \forall p$) are exactly those defining an element of $H_1(X, \mathbb{Z}_2)$.

(d) Consider putting a spin variable on the p-simplices of $\mathbf{\Delta}$. More generally, let's put an N-dimensional hilbert space $\mathcal{H}_N \equiv \text{span}\{|n\rangle, n = 1..N\}$ on each p-simplex, on which act the operators

$$\mathbf{X} \equiv \sum_{n=1}^{N} |n\rangle \langle n| \omega^n = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \omega & 0 & \dots \\ 0 & 0 & \omega^2 & \dots \\ 0 & 0 & 0 & \ddots \end{pmatrix}, \qquad \mathbf{Z} \equiv \sum_{n=1}^{N} |n\rangle \langle n+1| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \\ 1 & 0 & 0 & \dots \end{pmatrix}$$

where $\omega^N = 1$ is an nth root of unity. If you haven't already, check that they satisfy the clock-shift algebra: $\mathbf{ZX} = \omega \mathbf{XZ}$. For $N = 2$ these are Pauli matrices and $\omega = -1$.

Consider the Hamiltonian

$$\mathbf{H}_p = -J_{p-1} \sum_{s \in \Delta_{p-1}} A_s - J_{p+1} \sum_{\mu \in \Delta_{p+1}} B_\mu - g_p \sum_{\sigma \in \Delta_p} \mathbf{Z}_\sigma$$

with

$$A_s \equiv \prod_{\sigma \in \partial^{-1}(s) \subset \Delta_p} \mathbf{Z}_\sigma$$

$$B_\mu \equiv \prod_{\sigma \in \partial \mu} \mathbf{X}_\sigma \ .$$

This is a lattice version of p-form $\mathbb{Z}_N$ gauge theory, at a particular, special point in its phase diagram (the RG fixed point for the deconfined phase).

Show that

$$0 = [A_s, A_{s'}] = [B_\mu, B_{\mu'}] = [A_s, B_\mu], \quad \forall s, s', \mu, \mu'$$

so that for $g_p = 0$ this is solvable.

(e) Show that the groundstates of $\mathbf{H}_p$ (with $g_p = 0$) are in one-to-one correspondence with elements of $H_p(\mathbf{\Delta}, \mathbb{Z}_N)$.

Here's the solution: Suppose $J_{p-1} \gg J_{p+1}$ so that we should satisfy $A_s = 1$ most urgently. This equation is like a gauss law, but instead of flux *lines* in the $p = 1$ case, we have flux *sheets* for $p = 2$ or ... whatever they are called for larger p. The condition $A_s = 1$ means that these sheets satisfy a conservation law that the total flux going *into* the $p - 1$ simplex vanishes. So a basis for the subspace of states satisfying this condition is labelled by configuration of closed sheets. For $N = 2$ there is no orientation, and each p-simplex is either covered ($\mathbf{Z}_\sigma = -1$) or not ($\mathbf{Z}_\sigma = 1$) and the previous statement is literally true. For $N > 2$ we have instead sheet-nets (generalizing string nets), with $N - 1$ non-trivial kinds of sheets labelled by $k = 1...N - 1$ which can split and join as long as they satisfy

$$\sum_{\sigma \in v(s)} k_\sigma = 0 \text{ mod } N, \forall s. \tag{5}$$

This is the Gauss law of p-form $\mathbb{Z}_N$ gauge theory.

The analog of the plaquette operator B_μ acts like a kinetic term for these sheets. In particular, consider its action on a basis state for the $A_s = 1$ subspace $|C\rangle$, where C is some collection of (N-colored) closed p-sheets – by an N-colored p-sheet, I just mean that to each p-simplex we associate an integer k_σ (mod N), and this collection of integers satisfies the equation (5).

The action of the plaquette operator in this basis is

$$B_\mu |C\rangle = |C + \partial\mu\rangle$$

Here $C + \partial\mu$ is another collection of p-sheets differing from C by the addition (mod N) of a sheet on each p-simplex appearing in the boundary of μ. The eigenvalue condition $B_\mu = 1$ then demands that the groundstate wavefunctions $\Psi(C) \equiv \langle C|\text{groundstate}\rangle$ have equal values for chains C and $C' = C + \partial\mu$. But this is just the equivalence relation defining the pth homology of $\mathbf{\Delta}$. Distinct, linearly-independent groundstates are the labelled by p-homology classes of $\mathbf{\Delta}$. More precisely, they are labelled by homology with coefficients in $\mathbb{Z}_N$, $H_p(\mathbf{\Delta}, \mathbb{Z}_N)$.