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Physics 230 Quantum Phases of Matter, Spr 2024
Assignment 3 – Solutions

Due 11pm Thursday, April 25, 2024

Problems about Abelian Chern-Simons theory and its relation to QHE.

1. **Another quantized coupling constant.** Consider the worldline theory of a charged particle. The degrees of freedom are the coordinates of the particle $x^i(t), i = 1..d$ as a function of time, a QFT in $0+1$ with d fields. We might include terms like
$$S_0[x] = \int dt \left(\frac{1}{2}m\left(\dot{x}\right)^2 - V(x)\right). \tag{1}$$
Suppose the particle is charged under a $\mathsf{U}(1)$ symmetry, and we would like to couple it to a background gauge field A_μ. This means that its worldline action contains a term of the form
$$S_c[A, x] = q\int A \equiv q\int dt\dot{x}^i A_i(x(t)). \tag{2}$$
This is an example of a Chern-Simons term in 0+1 dimensions.
The path integral measure $e^{\mathbf{i}S[x,A]}$ should be invariant under gauge transformations
$$A_\mu \to A_\mu + \mathbf{i}g^{-1}\partial_\mu g \tag{3}$$
where $g(t) = e^{\mathbf{i}\phi(t)}$ is an element of $\mathsf{U}(1)$. Show that this means that the charge q must be quantized. Hint: consider the case where the worldline is a circle.

2. **Quantization of the level.**
   (a) Show that the Chern-Simons action is gauge invariant under $a \to a + d\lambda$, as long as there is no boundary of spacetime Σ. Compute the variation of the action in the presence of a boundary of Σ.
   (b) Actually, the situation is a bit more subtle than the previous part suggests. The actual form of a $\mathsf{U}(1)$ gauge transformation is
$$a \to a - \mathbf{i}g^{-1}dg$$
where $g = e^{\mathbf{i}\lambda}$. This reduces to the previous expression for the gauge transformation when λ is small, but the latter ignores the global structure of the gauge group (*e.g.* in the abelian case, the fact that g is a periodic function).

Consider the case where spacetime is $\Sigma = S^1 \times S^2$. Find the variation of the $\mathsf{U}(1)$ Chern-Simons action

$$S_0[a] = \int_\Sigma \frac{k}{4\pi} a \wedge da$$

under a *large gauge transformation*, meaning that

$$g = e^{\mathbf{i}n\theta}$$

where θ is the coordinate on the circle. Conclude that in the absence of *other* interestingness (such as degenerate groundstates not coming from the dynamics of a), the level k must be an even integer.

Here is the logic: Since the action appears in the path integral in the form $e^{\mathbf{i}S}$, convince yourself that the path integrand is gauge invariant if
(1) $\int_\Gamma f \in 2\pi\mathbb{Z}$ for all closed 2-surfaces Γ in spacetime, and
(2) $k \in 2\mathbb{Z}$ – the Chern-Simons level is quantized as an *even* integer.

The first condition is called flux quantization, and is closely related to Dirac's condition.

The quantization of the level k, i.e. the Chern-Simons coupling, has a dramatic consequence: it means that this coupling constant cannot be renormalized by a little bit, only by an integer shift. This is an enormous constraint on the dynamics of the theory.

Let's write $\omega = d\phi = -\mathbf{i}g^{-1}dg$. This is a closed form, $d\omega = 0$, but it is not exact, since ϕ is not necessarily a globally well-defined function (it can jump by 2π anywhere).

The variation is $\delta S_0 = \frac{k}{4\pi}\int_M \omega \wedge da$. You might be tempted to integrate by parts and say $d\omega = 0$ and therefore this vanishes. But a is not globally well-defined, so it's not true that d of something involving a has to vanish on a closed manifold. A familiar example is $\int_{S^2} F = 2\pi$ for the sphere surrounding a magnetic monopole.

We can argue that $\frac{1}{2\pi}\int \omega \wedge da$ is 2π times an integer by following the logic: first show that it's topological, in the sense of independent of local variations of its arguments, then evaluate it on nice configurations where we can do the integral.

The first step follows because both ω and da are closed. For the second step, we can choose a nice 3-manifold, such as $S^1 \times S^2$, where the period of the circle is L and the coordinate is t ($t \equiv t + L$). Consider a field configuration where the gauge flux is constant in t. If we take $g = e^{\frac{2\pi \mathbf{i}t}{L}}$, then $\omega = \frac{2\pi}{L}dt$, we find

$$\delta S_0 = -\frac{k}{4\pi}\int_0^L \frac{2\pi}{L} dt \underbrace{\int_{S^2} f}_{\in 2\pi\mathbb{Z}} \in \pi k \mathbb{Z}.$$

Therefore, k must be an even integer, if there is nothing else around to make the amplitude gauge invariant. But, you say, we've been talking about the case $k = 1$ all the time as a description of the integer QHE! The answer is that the theory with odd k does make sense, but only if the system is fermionic. We'll come back to this later.

(c) [bonus] In the case where G is a non-abelian lie group, the argument for quantization of the level k is more straightforward. Show that the variation of the CS Lagrangian

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\mathrm{tr}\left(a \wedge da + \frac{2}{3} a \wedge a \wedge a\right)$$

under $a \to gag^{-1} - dgg^{-1}$ is

$$\mathcal{L}_{CS} \to \mathcal{L}_{CS} + \frac{k}{4\pi} d\mathrm{tr} dgg^{-1} \wedge a + \frac{k}{12\pi}\mathrm{tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right).$$

The integral of the second term over any closed surface is an integer. Conclude that $e^{\mathbf{i}S_{CS}}$ is gauge invariant if $k \in \mathbb{Z}$.

The first term integrates to zero on a closed manifold. The second term is the winding number of the map $g : \Sigma \to \mathsf{G}$

3. **Hall conductivity from Chern-Simons theory**

(a) For the abelian Chern-Simons theory with gauge group $\mathsf{U}(1)$ at level k,

$$S[a, \mathcal{A}] = \int \left(\frac{k}{4\pi} a \wedge da + \mathcal{A} \wedge \frac{da}{2\pi}\right).$$

do the (gaussian!) path integral over a to find the effective action for the background field $\mathcal{A}$. Find the Hall conductivity.

See the next problem.

(b) Now do it for the general K matrix and general charge vector t^I, with

$$S[a^I, \mathcal{A}] = \int \left(\frac{K_{IJ}}{4\pi} a^I \wedge da^J + \mathcal{A} \wedge t_I \frac{da^I}{2\pi}\right).$$

Let's just do it all at once. The path integral is

$$\int [Da] e^{\mathbf{i}S[a,\mathcal{A}]} = e^{\mathbf{i}S_{\text{eff}}[\mathcal{A}]}.$$

Since the Hall conductivity is a local quantity, let's just put the system on the plane or the sphere, where there is no opportunity for a to create any topological mischief, and we can just do the integral. Complete the square in the exponent:

$$\begin{aligned}&\mathbf{i}\int\left(\frac{K_{IJ}}{4\pi}a^I\wedge da^J+\mathcal{A}\wedge t_I\frac{da^I}{2\pi}\right)\\ &=\mathbf{i}\int\frac{K_{IJ}}{4\pi}\left(a^I+\left(K^{-1}\right)^{IK}t_K\mathcal{A}\right)d\left(a^J+\left(K^{-1}\right)^{JL}t_L\mathcal{A}\right)\\ &\quad-\frac{K_{IJ}}{4\pi}\left(\left(K^{-1}\right)^{IK}t_K\mathcal{A}\right)d\left(\left(K^{-1}\right)^{JL}t_L\mathcal{A}\right).\end{aligned} \tag{4}$$

Now change variables in the integral $a^I\to a^I+\left(K^{-1}\right)^{IK}t_K\mathcal{A}$. On the plane this is fine, and the integral is just a constant. All that is left is

$$S_{\text{eff}}=-t_I\left(K^{-1}\right)^{IJ}t_J\int\frac{\mathcal{A}\wedge d\mathcal{A}}{4\pi}.$$

We conclude that the Hall conductivity is

$$\sigma^{xy}=\frac{e^2}{h}t_I\left(K^{-1}\right)^{IJ}t_J.$$

I should make here a legal disclaimer that although the integral over a is gaussian and therefore it is irresistible to do the integral, it is not quite safe to integrate it out. You can see this from the fact that we get a CS theory for $\mathcal{A}$ with a level that is not an integer! The reason this is consistent with gauge invariance for the background $\mathsf{U}(1)$ gauge group is that a large gauge transformation takes one groundstate of a to a different one.

4. **Flux attachment.** Now consider

$$S_j[A]=\int\left(\frac{k}{4\pi}a\wedge da+a\wedge\star j\right).$$

Find the equations of motion. Show that the Chern-Simons term *attaches* k *units of flux* to the particles: $F_{12}\propto\rho$.

5. **Anyons.**

   (a) Show using the Bohm-Aharonov effect that the particles whose current density is j^μ have anyonic statistics with exchange angle $\frac{\pi}{k}$ (supposing they were bosons before we coupled them to A).

One way to do this is to consider a configuration of j which describes one particle adiabatically encircling another. Show that its wavefunction acquires a phase $e^{\mathbf{i}2\pi/k}$. This is twice the phase obtained by going halfway around, which (when followed by an innocuous translation) would exchange the particles.

See the next problem.

(b) Describe the statistics of the anyonic quasiparticles in the case with general K matrix.

The EoM are

$$\frac{K_{IJ}}{2\pi} da^J = \star j_I$$

which means $da^I = 2\pi \left(K^{-1}\right)^{IJ} \star j_J$. Bringing anyon one with charge l_1 all the way around anyon two with charge l_2 gives the phase

$$\Phi_{2\pi} = \left(l_1\right)_I \oint_C a^I = \left(l_1\right)_I \int_{R, \partial R = C} 2\pi \left(K^{-1}\right)^{IJ} \left(\rho_2\right)_J = 2\pi \left(l_1\right)_I \left(K^{-1}\right)^{IJ} \left(l_2\right)_J .$$

The exchange phase is half of this.