University of California at San Diego – Department of Physics – Prof. John McGreevy

# Physics 230 Quantum Phases of Matter, Spr 2024
# Assignment 3

**Due 11pm Thursday, April 25, 2024**

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**Problems about Abelian Chern-Simons theory and its relation to QHE.**

1. **Another quantized coupling constant.** Consider the worldline theory of a charged particle. The degrees of freedom are the coordinates of the particle $x^i(t), i = 1..d$ as a function of time, a QFT in $0+1$ with $d$ fields. We might include terms like
$$S_0[x] = \int dt \left( \frac{1}{2} m \left( \dot{x} \right)^2 - V(x) \right). \tag{1}$$
Suppose the particle is charged under a $\mathsf{U}(1)$ symmetry, and we would like to couple it to a background gauge field $A_\mu$. This means that its worldline action contains a term of the form
$$S_c[A, x] = q \int A \equiv q \int dt \dot{x}^i A_i(x(t)). \tag{2}$$
This is an example of a Chern-Simons term in 0+1 dimensions.
The path integral measure $e^{\mathbf{i}S[x,A]}$ should be invariant under gauge transformations
$$A_\mu \to A_\mu + \mathbf{i} g^{-1} \partial_\mu g \tag{3}$$
where $g(t) = e^{\mathbf{i}\phi(t)}$ is an element of $\mathsf{U}(1)$. Show that this means that the charge $q$ must be quantized. Hint: consider the case where the worldline is a circle.

2. **Quantization of the level.**
   (a) Show that the Chern-Simons action is gauge invariant under $a \to a + d\lambda$, as long as there is no boundary of spacetime $\Sigma$. Compute the variation of the action in the presence of a boundary of $\Sigma$.
   (b) Actually, the situation is a bit more subtle than the previous part suggests. The actual form of a $\mathsf{U}(1)$ gauge transformation is
   $$a \to a - \mathbf{i} g^{-1} dg$$
   where $g = e^{\mathbf{i}\lambda}$. This reduces to the previous expression for the gauge transformation when $\lambda$ is small, but the latter ignores the global structure of the gauge group (*e.g.* in the abelian case, the fact that $g$ is a periodic function).

Consider the case where spacetime is $\Sigma = S^1 \times S^2$. Find the variation of the $\mathsf{U}(1)$ Chern-Simons action

$$S_0[a] = \int_\Sigma \frac{k}{4\pi} a \wedge da$$

under a *large gauge transformation*, meaning that

$$g = e^{\mathbf{i}n\theta}$$

where $\theta$ is the coordinate on the circle. Conclude that in the absence of *other* interestingness (such as degenerate groundstates not coming from the dynamics of $a$), the level $k$ must be an even integer.

Here is the logic: Since the action appears in the path integral in the form $e^{\mathbf{i}S}$, convince yourself that the path integrand is gauge invariant if

(1) $\int_\Gamma f \in 2\pi\mathbb{Z}$ for all closed 2-surfaces $\Gamma$ in spacetime, and

(2) $k \in 2\mathbb{Z}$ – the Chern-Simons level is quantized as an *even* integer.

The first condition is called flux quantization, and is closely related to Dirac's condition.

The quantization of the level $k$, i.e. the Chern-Simons coupling, has a dramatic consequence: it means that this coupling constant cannot be renormalized by a little bit, only by an integer shift. This is an enormous constraint on the dynamics of the theory.

(c) [bonus] In the case where $\mathsf{G}$ is a non-abelian lie group, the argument for quantization of the level $k$ is more straightforward. Show that the variation of the CS Lagrangian

$$\mathcal{L}_{CS} = \frac{k}{4\pi} \mathrm{tr}\left( a \wedge da + \frac{2}{3} a \wedge a \wedge a \right)$$

under $a \to gag^{-1} - dgg^{-1}$ is

$$\mathcal{L}_{CS} \to \mathcal{L}_{CS} + \frac{k}{4\pi} d\mathrm{tr} dgg^{-1} \wedge a + \frac{k}{12\pi} \mathrm{tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right).$$

The integral of the second term over any closed surface is an integer. Conclude that $e^{\mathbf{i}S_{CS}}$ is gauge invariant if $k \in \mathbb{Z}$.

3. **Hall conductivity from Chern-Simons theory**

(a) For the abelian Chern-Simons theory with gauge group $\mathsf{U}(1)$ at level $k$,

$$S[a, \mathcal{A}] = \int \left( \frac{k}{4\pi} a \wedge da + \mathcal{A} \wedge \frac{da}{2\pi} \right).$$

do the (gaussian!) path integral over $a$ to find the effective action for the background field $\mathcal{A}$. Find the Hall conductivity.

(b) Now do it for the general $K$ matrix and general charge vector $t^I$, with

$$S[a^I, \mathcal{A}] = \int \left( \frac{K_{IJ}}{4\pi} a^I \wedge da^J + \mathcal{A} \wedge t_I \frac{da^I}{2\pi} \right).$$

4. **Flux attachment.** Now consider

$$S_j[A] = \int \left( \frac{k}{4\pi} a \wedge da + a \wedge \star j \right).$$

Find the equations of motion. Show that the Chern-Simons term *attaches $k$ units of flux* to the particles: $F_{12} \propto \rho$.

5. **Anyons.**

(a) Show using the Bohm-Aharonov effect that the particles whose current density is $j^\mu$ have anyonic statistics with exchange angle $\frac{\pi}{k}$ (supposing they were bosons before we coupled them to $A$).

One way to do this is to consider a configuration of $j$ which describes one particle adiabatically encircling another. Show that its wavefunction acquires a phase $e^{\mathbf{i}2\pi/k}$. This is twice the phase obtained by going halfway around, which (when followed by an innocuous translation) would exchange the particles.

(b) Describe the statistics of the anyonic quasiparticles in the case with general $K$ matrix.