University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 230 Quantum Phases of Matter, Spr 2024
Assignment 4 – Solutions

Due 11pm Thursday, May 2, 2024

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1. **A simple avatar of the Lieb-Schulz-Mattis theorem.** Consider the effective theory describing a system living in the continuum that spontaneously forms a solid, say a cubic lattice in d dimensions. Since translation symmetry is spontaneously broken, the degrees of freedom must include a collection of Goldstone bosons θ^I, where $I = 1..d$ runs over the spatial dimensions. $\theta^I(x)$ is the shift of the atom at location x in the I direction relative to its equilibrium position. These fields live on a circle, because if I shift all the atoms by the lattice spacing, I get back the original lattice.

   (a) Convince yourself that the effective action takes the form

   $$S_{\text{elastic}}[\theta^I] = \int d^{d+1}x \; \kappa^{ijKL} \partial_i \theta^K \partial_j \theta^L + \text{terms with more derivatives}, \quad (1)$$

   where the coupling constant κ^{ijKL} is the elasticity tensor. With various symmetries imposed, it can be decomposed further into various tensors with names from the 19th century. These tensors describe things like bending moduli – the rigidity of the solid to various kinds of strain.

   Because the θ^I are Goldstone bosons, they can only appear in terms with derivatives. Rotation invariance forbids terms with a single derivative.

   (b) Now suppose that the number of atoms is a conserved quantity. That is, consider a situation where there is also a $\mathsf{U}(1)$ symmetry. So we can couple the system to a background gauge field A_μ for this $\mathsf{U}(1)$ symmetry. We'll assume this $\mathsf{U}(1)$ symmetry is not spontaneously broken. What are the leading terms in the (local!) effective action $S_{\text{eff}}[\theta^I, A_\mu]$ that preserve gauge invariance and translation symmetry?

   I wrote the most interesting ones below. There can also be terms involving dA.

   (c) Consider the case of $d = 1$. In addition to the terms involving dA, one interesting term is

   $$S_\nu[\theta, A] \equiv \frac{\nu}{2\pi} \int A \wedge d\theta = \frac{\nu}{2\pi} \int dx dt A_\mu \partial_\nu \theta \epsilon^{\mu\nu}. \quad (2)$$

One point to notice about it is that it is not obviously gauge invariant, because it depends explicitly on A and not just the gauge-invariant object F. Show that $e^{\mathbf{i}S_\nu}$ is gauge invariant if ν is an integer.

Under a gauge transformation, it changes by

$$\delta S_\nu = \frac{\nu}{2\pi}\int d\theta \wedge g^{-1}dg. \tag{3}$$

This is not obviously zero. But we don't actually need the variation of the action to be zero, we just need it to be an integer multiple of $2\pi\mathbf{i}$, since it only ever appears exponentiated in the path integral. And in fact, if θ and g are continuous functions and spacetime has no boundaries, (3) is always $2\pi\mathbf{i}\nu$ times an integer. (To see this, first show that it is invariant under small changes of g or θ:

$$\frac{\delta\left(\delta S_\nu\right)}{\delta g} = \frac{\delta\left(\delta S_\nu\right)}{\delta\theta} = 0.$$

So it is topological. Then we can compute it for some representative configuration. If, for definiteness, we periodically identify the spacetime coordinates, (3) is an expression for ($2\pi\mathbf{i}$ times) the winding number of the map $T^2 \to T^2, (x,t) \to (g(x,t), \theta(x,t))$. Note that maps g : spacetime $\to G$ that are not continuously connected to the map to the identity are called 'large gauge transformations'.) Therefore, if $\nu \in \mathbb{Z}$, then (2) is gauge invariant[1].

(d) What does the new term (2) do? Well, the first question we should ask about an effective action for a background gauge field is: what is the resulting charge density:

$$\rho(x) = \frac{\delta S}{\delta A_0(x)} \text{ ?}$$

Interpret your result.

$$\rho(x) = \frac{\delta S}{\delta A_0(x)} = \frac{\nu}{2\pi}\partial_x\theta + \cdots .$$

This equation correctly expresses the fact that deforming the lattice away from a uniform configuration will make the density vary.

[1] Alternatively, if spacetime is a manifold without boundary, we can integrate by parts and write

$$S_\nu = -\frac{\nu}{2\pi}\int \theta \wedge F.$$

This is manifestly gauge invariant, but it is not manifestly single-valued under $\theta \to \theta + 2\pi$, as it must be to be well-defined. Fortunately, $\int_S F/2\pi \in \mathbb{Z}$ is an integer if A is a background $\mathsf{U}(1)$ gauge field on a manifold S without boundary (this is called flux quantization), and so again we conclude that $e^{\mathbf{i}S_\nu}$ is well-defined if $\nu \in \mathbb{Z}$.

The $\cdots$ is contributions from other terms in the action, such as a term like $\int A_0 \rho_0$ that adds a background density. If ρ_0 is constant in time and integrates to an integer, this is also gauge invariant. More generally, we could add $\int A_\mu j^\mu$ which you can show is gauge invariant (even under large gauge transformations) as long as $\partial_\mu j^\mu = 0$.

(e) What is the analog of (2) in d dimensions? (That is, find a term in d spatial dimensions involving a single power of A and derivatives of the θ^I that can be written without using the metric.) Show that its coefficient ν is quantized to be an integer. What contribution does it make to the density?

(f) We can identify the goldstone field θ with the phase field describing the displacements of the atoms from their equilibrium positions:

$$u^i(x,t) = \frac{1}{2\pi} a^i_I \theta^I(x,t) - x^i$$

where $\vec{a}_I$ are generators of the lattice Γ. Then the equilibrium configuration is actually $\theta^I(x,t) = K^I_i x^i$ where $K^I_i \left(\frac{a}{2\pi}\right)^j_I = \delta^j_i$, so K^I_i is the matrix whose columns are the reciprocal lattice generators.

The generalization of (2) in d spatial dimensions is

$$\frac{\nu}{(2\pi)^d} \int A \wedge d\theta^1 \wedge d\theta^2 \cdots \wedge d\theta^d \ . \tag{4}$$

Again $\nu \in \mathbb{Z}$ is required by gauge invariance. This gives the density

$$\rho(x) = \frac{\delta S}{\delta A_0(x)} = \frac{\nu}{(2\pi)^d} \frac{1}{d!} \epsilon_{I_1 \cdots I_d} \epsilon^{i_1 \cdots i_d} \partial_{x_{i_1}} \theta^{I_1} \cdots \partial_{x_{i_d}} \theta^{I_d}.$$

Plugging in the equilibrium configuration gives

$$\rho_0(x) = \nu \frac{\det K}{(2\pi)^d} = \frac{\nu}{V}$$

where $V \equiv \det a$ is the volume of the unit cell. This says that ν is the (integer!) number of atoms per unit cell.

(g) The conclusion you should find by the gauge invariance argument above, under the present assumptions, is that ν, and hence the equilibrium number of particles per unit cell, must be an integer. This is an avatar of the Lieb-Schulz-Mattis-Oshikawa-Hastings (LSMOH) theorem. Now, you may say to yourself, why can't I make a system at some filling which is not an integer? Indeed, I can take 20007 particles and place them in a volume with 20004 unit cells, and the system must have some groundstate. What gives?

2. **Edge modes of CS theory.** Now we return to abelian Chern-Simons theory (for an extra challenge, redo this part in the non-Abelian case). If there is a boundary of spacetime, something must be done to fix up the fact that the action is not invariant under would-be gauge transformations that are nontrivial at the boundary. Consider the case where $\Sigma = \mathbb{R} \times \text{UHP}$ where $\mathbb{R}$ is the time direction, and UHP is the upper half-plane $y > 0$. One way to fix the problem is simply to declare that the would-be gauge transformations which do not vanish at $y = 0$ are not redundancies. This means that they represent physical degrees of freedom.

   (a) First consider the simplest case of $\mathsf{U}(1)$ CS theory at level k. Choose $a_0 = 0$ gauge, and plug the solution of the bulk equations of motion $a = \tilde{d}\phi$ (where $\phi(x, y \to 0) \equiv \phi(x)$ is a scalar field, and $\tilde{d}$ is the exterior derivative on the spatial manifold) into the Chern-Simons action to find the resulting action for ϕ.

   The exterior derivative on this spacetime decomposes into $d = \partial_t dt + \tilde{d}$ where $\tilde{d}$ is just the spatial part, and similarly the gauge field is $a = a_0 dt + \tilde{a}$. Let us choose the gauge $a_0 = 0$. We must still impose the equations of motion for a_0 (in the path integral it is a Lagrange multiplier) which says $\tilde{d}\tilde{a} = 0$ (just the spatial part). This equation is solved by $\tilde{a} = \tilde{d}\phi$ (or rather $\tilde{a} = g^{-1}dg$ where g is a $\mathsf{U}(1)$-valued function). This is pure gauge except at the boundary. Plugging this into the CS term gives

$$S = \frac{k}{4\pi}\int_{\mathbb{R}\times D} \tilde{a} \wedge \left(dt\partial_t + \tilde{d}\right)\tilde{a} \tag{5}$$
$$= \frac{k}{4\pi}\int_{\mathbb{R}\times D} \tilde{d}\phi \wedge dt\partial_t \tilde{d}\phi \tag{6}$$
$$= \frac{k}{4\pi}\int_{\mathbb{R}\times D} \tilde{d}\left(\phi \wedge dt\partial_t \tilde{d}\phi\right) \tag{7}$$
$$\overset{\text{Stokes}}{=} \frac{k}{4\pi}\int_{\mathbb{R}\times \partial D} \phi dt \partial_t \tilde{d}\phi \tag{8}$$
$$= \frac{k}{4\pi}\int_{\mathbb{R}\times \partial D} dx dt \phi \partial_t \partial_x \phi \tag{9}$$
$$\overset{\text{IBP}}{=} -\frac{k}{4\pi}\int_{\mathbb{R}\times \partial D} dx dt \partial_x\phi \partial_t \phi. \tag{10}$$

   (b) We can also add local terms at the boundary to the action. Consider adding $\Delta S = g\int_{\partial\Sigma} a_x^2$ (for some coupling constant g). Find the equations of motion for ϕ.

This term evaluates to $\Delta S = \int_{\partial\Sigma} v\,(\partial_x \phi)^2$. Altogether we now have

$$S_{\text{edge}}[\phi] = \int_{y=0} dx dt \partial_x \phi \left(\frac{k}{4\pi} \partial_t \phi + g \partial_x \phi \right).$$

The EoM is then

$$\frac{\delta}{\delta \phi(x)} S_{\text{edge}}[\phi] = \partial_t \left(\frac{k}{4\pi} \partial_t \phi + g \partial_x \phi \right)$$

which is solved if $\frac{k}{4\pi}\partial_t \phi + g \partial_x \phi = 0$. This describes a dispersionless wave which moves only in the sign$k$ direction – a chiral bosonic edge mode.

For more, I recommend the textbook by Xiao-Gang Wen.

Interpretation: the Chern-Simons theory on a space with boundary necessarily produces a chiral edge mode.

(c) If you feel like it, redo the previous parts for the general K-matrix theory.