University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 230 Quantum Phases of Matter, Spr 2024
Assignment 5 – Solutions

Due 11pm Thursday, May 9, 2024

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1. **Charges of quasiparticles in abelian CS EFT.**

   In an abelian CS theory with K-matrix K, show that a quasiparticle with charge ℓ^I under CS gauge field a^I has electric charge

   $$q_l = tK^{-1}l.$$

   The EFT for a charge at the origin is

   $$L = \frac{1}{4\pi}K_{IJ}a^I da^J + \frac{1}{2\pi}At_I da^I + \ell_I a_0^I \delta^2(x).$$

   The EOM for a_0^I is

   $$0 = \frac{\delta S}{\delta a_0} = \frac{1}{2\pi}Kda + \ell\delta^2$$

   so

   $$da = 2\pi K^{-1}\ell\delta.$$

   The source for A_0 is then

   $$\frac{1}{2\pi}t_I da^I = tK^{-1}\ell\delta^2(x).$$

2. **Quasiparticle wavefunctions.**

   (a) Use the flux-threading argument starting from the Laughlin $\nu = \frac{1}{m}$ state to construct wavefunctions for the quasihole and quasiparticle. That is, write down a wavefunction of N electrons with the property that it acquires a phase $e^{\pm \mathbf{i}\theta}$ when the coordinate z_i of *any* electron is taken around the point $z_i = w$ by an angle θ: $z_i - w \to e^{\mathbf{i}\theta}(z_i - w)$, $\forall i = 1..N$.

   In the latter case, don't forget to project onto the lowest Landau level.

   Alternatively, you can try to use the parton construction, *i.e.* add or remove a single parton.

   A simple (parton-independent) way to motivate the quasihole wavefunction is to find the wavefunction that results by threading 2π flux at the point w

in the complex plane. (We saw earlier that on general grounds, if the state is gapped, this produces an excitation with statistics $\pi\sigma^{xy}$.) Threading 2π flux at w means that the wavefunction should acquire the phase $e^{\mathbf{i}\theta}$ when we move *any* of the electrons around the point w: $z_i - w \to e^{\mathbf{i}\theta}(z_i - w)$, $\forall i = 1..N$. A very easy way to accomplish this is to multiply the wavefunction by the factor

$$\prod_{i=1}^{N}(z_i - w) \ .$$

That's it. No need for an LLL projection, since it's still holomorphic. The full wavefunction for a quasihole at w is then

$$\tilde{\Psi}_w(z) = \prod_{i=1}^{N}(z_i - w)\prod_{i<j}^{N}(z_i - z_j)^m e^{-\sum_i \frac{|z_i|^2}{4\ell_B^2}} .$$

Notice that this is still a wavefunction for N electrons.

The quasiparticle wavefunction should acquire the opposite phase, so we'd like to multiply by

$$\prod_{i=1}^{N}(\bar{z}_i - \bar{w})$$

but that's not a LLL wavefunction. The projection of this to the LLL is the quasiparticle wavefunction.

$$\tilde{\Psi}_{\bar{w}}(z) = \mathcal{P}_{LLL}\prod_{i=1}^{N}(\bar{z}_i - \bar{w})\prod_{i<j}^{N}(z_i - z_j)^m = \prod_i \left(2\ell_B^2\partial_{z_i} - \bar{w}\right)\prod_{i<j}(z_i - z_j)^m.$$

Let's try acting on the parton groundstate with a single parton annihilation operator, $f_\alpha(w)$, where α is a species label on the partons: $c = \prod_\alpha f_\alpha$. The problem with this idea is that it removes a parton. But the projection to the gauge invariant Hilbert space requires that there be the same number of electrons as each type of parton, so that the state has a nonzero overlap $\Psi(r) = \langle 0|\prod_i c(r_i)\,|\text{parton state}\rangle$.

If we act with a single creation operator $f^\dagger(\eta)$, we must start occupying the second parton Landau level, so that's a good sign that we'll need a LLL projection.

But I conclude that at the moment I don't know how to motivate the Laughlin quasihole and quasiparticle wavefunctions from partons. Please let me know if you do.

(b) Using the plasma analogy, show that your quasihole wavefunction produces a localized charge deficit of charge $1/m$.

See the footnote in the answer to the next part.

(c) Construct a wavefunction with *two* quasiholes and use it to verify their statistics (by adiabatically moving them around each other and computing the resulting Berry phase).

This calculation was first done here.

The state is

$$\tilde{\Psi}_{12}(z) = \prod_{i=1}(z_i - w_1) \prod_{i=1}(z_i - w_2) \prod_{i<j}(z_i - z_j)^m.$$

Let's compute the Berry connection for varying w_1:

$$\mathcal{A}_{w_1} = \langle \Psi_{12} | \mathbf{i}\partial_{w_1} | \Psi_{12} \rangle \ .$$

First we have to make sure the state is normalized: the normalization factor is

$$Z(w) = \int \prod_{i=1}^{N} d^2 z_i \prod_i |w_1 - z_i|^2 |w_2 - z_i|^2 \prod_{i<j} |z_{ij}|^{2m} e^{-\sum |z|^2/(2\ell_B^2)}, \tag{1}$$

and I'll write $\Psi = Z^{-1/2}\Psi_0$. So

$$\begin{aligned} \mathcal{A}_{w_1} &= \mathbf{i}(Z^{-1} \langle \Psi_0 | \partial_{w_1} | \Psi_0 \rangle - \frac{1}{2} \langle \Psi_0 | Z^{-1} \partial_{w_1} \log Z | \Psi_0 \rangle & (2) \\ &= \mathbf{i} \left(\left\langle \sum_i \frac{1}{w_1 - z_i} \right\rangle - \frac{1}{2} \partial_{w_1} \log Z \right) & (3) \\ &= \frac{1}{2} \langle \Psi_{12} | \sum_i \frac{\mathbf{i}}{w_1 - \hat{z}_i} | \Psi_{12} \rangle . & (4) \end{aligned}$$

The Berry phase accumulated by moving w_1 in a circle (of radius, say R) around w_2 is then

$$\begin{aligned} \gamma_{12} &\equiv \oint_{C_{w_2}} dw_1 \mathcal{A}_{w_1} + h.c. & (5) \\ &= \frac{1}{2} \langle \Psi_{12} | \mathbf{i} \oint_{C_{w_2}} dw_1 \sum_i \frac{1}{w_1 - \hat{z}_i} | \Psi_{12} \rangle + h.c. & (6) \\ &= \frac{1}{2} \langle \Psi_{12} | (-2\pi) \sum_i \Theta(\hat{z}_i \in C_{w_2}) | \Psi_{12} \rangle + h.c. & (7) \end{aligned}$$

where we used Cauchy's theorem, and

$$\Theta(s) \equiv \begin{cases} 1, & \text{if the statement } s \text{ is true} \\ 0, & \text{else} \end{cases}.$$

This last expression is the average number of electrons inside the circle of radius R about w_2 (times -2π). If there were no quasihole at w_2, this would be (for large enough R) just $-2\pi\nu\frac{\Phi}{\Phi_0}$, where $\Phi = \int_{C_{w_2}} \vec{B} \cdot d\vec{a}$ is the flux through the circle. This contribution is not necessarily 2π times an integer, and would be there even if the path C did not go around another quasihole. It represents the Aharonov-Bohm phase acquired by the particle, which you can see therefore has charge $1/m$.

The presence of the quasihole at $z = w_2$ decreases the electron density. It decreases the expected number of electrons in the neighboring region by $\frac{1}{m}$[1], and therefore the contribution from w_2 to the Berry phase is $\gamma_{12} = -2\pi\frac{1}{m}$. The quasihole exchange phase is then

$$\theta_{12} = \frac{\gamma_{12}}{2} = \frac{\pi}{m} = \pi\nu.$$

See David Tong's notes pp 93-96 for some more discussion of this calculation. I think the trick I used above is a successful simplification.

[1]Here I am appealing to a result from the plasma analogy. The charge density

$$\rho(z,\bar{z}) = \int \prod_{i=2}^{N} d^2 z_i |\Psi_w(z, z_2 \cdots z_N)|^2 = \int \prod_{i=2}^{N} d^2 z_i e^{\sum_{1<i<j} \log|z_i - z_j|^2 + \sum_{1<i} \log|z-w|^2 - \sum_i \frac{|z_i|^2}{2\ell_B^2}}$$

in the quasihole wavefunction is the density of a one-component plasma of charge-m objects (with logarithmic mutual interactions) that see a neutralizing background (that's the quadratic term) plus an extra potential from a fixed impurity of positive unit charge at $z = w$. As Girvin and Yang say (page 447), 'the chief desire of the plasma is to maintain charge neutrality'. This is accomplished by forming a screening cloud near $z = w$ to screen the impurity. Screening the cloud requires a deficit of $1/m$th of a charge-m particle. Those particles sit at the electron positions, so this is $1/m$th of an electron missing.