University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 230 Quantum Phases of Matter, Spr 2024
Assignment 6 – Solutions

Due 11pm Thursday, May 16, 2024

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1. **Hall plateaux as a crazy manifestation of quantum oscillations.** Check the claim that the hierarchy states at fillings $\nu = \frac{\nu^\star}{2\nu^\star \pm 1}$ for $\nu^\star \in \mathbb{Z}$ can be regarded as an extreme version of quantum oscillations in the HLR state at $\nu = \frac{1}{2}$.

   We work at fixed electron density ρ throughout, so B and ν are related by $\rho = \nu B/\Phi_0$. Write $B = B_{\nu=\frac{1}{2}} + \delta B = 2\Phi_0\rho + \delta B$, so

   $$\delta B = \Phi_0 \rho \left(\frac{1}{\nu} - 2\right).$$

   If $\frac{1}{\delta B} = \pm \nu^\star \frac{1}{\rho \Phi_0}$, we find

   $$\pm \frac{1}{\nu^\star} = \frac{1}{\nu} - 2$$

   which indeed gives the relation for the states at the first level of the hierarchy.

2. **Quantum Hall states of quasiparticles.** In lecture we explained how to find incompressible states with filling fraction $\nu = \frac{1}{k - \frac{1}{\tilde{k}}}$ by placing the quasiparticle excitations of a $\nu = 1/k$ FQH state in a $\tilde{\nu} = 1/\tilde{k}$ FQH state. Check this relation. When $\tilde{k} = 2$, this reproduced one branch of the composite fermion states we found previously. Explain how to get the other branch.

   The other branch can be obtained from $\tilde{k} = -2$. One way is with the K matrix

   $$K = \begin{pmatrix} \tilde{k} & 1 \\ 1 & k \end{pmatrix}, \quad t^I = (0, 1)^I \tag{1}$$

   which for $k = 2$ gives $\tilde{\nu} = 1, 2/3, 3/5, 4/7 \cdots$ for $\tilde{k} = 1, 2, 3, 4 \cdots$.

3. **Excitations of hierarchy states.** Find the torus groundstate degeneracy, and the charges and statistics of the quasiparticle excitations of the abelian incompressible FQH state at $\nu = \frac{2}{5}$ (for example, using the description in terms of the K-matrix CS theory).

   This state is described by the hierarchy construction with $m = 3$ and $\nu^\star = 2$. The EFT is

   $$4\pi L = 3ada + 2Ada + 2ad\tilde{a} + 2\tilde{a}d\tilde{a},$$

that is, the K-matrix is

$$K = \begin{pmatrix} k & 1 \\ 1 & \tilde{k} \end{pmatrix}$$

and the charge vector is $t = (1, 0)$. You can check that indeed the Hall conductivity is $tK^{-1}t = \frac{2}{5}$.

$$\det K = 5$$

so the torus GSD is 5-fold.

A single qp with charge ℓ^I under $a^I = (a, \tilde{a})$ has electric charge

$$tK^{-1}\ell.$$

For $\ell = (1, 0)$, this gives $q = \frac{2}{5}$ and for $\ell = (0, 1)$, this gives $q = -\frac{1}{5}$.

4. **Boson Integer Quantum Hall State from Partons.** Consider a system made from two species of bosons, $b_\uparrow, b_\downarrow$. They could be distinguished by living in two layers. We'll assume that only the total boson number, acting by $(b_\uparrow, b_\downarrow) \to e^{\mathbf{i}\alpha}(b_\uparrow, b_\downarrow)$ is conserved (so that if the label is a layer label, the particles are able to tunnel between layers), and couple to a background field $\mathcal{A}$ for that symmetry.

   (a) Consider the parton ansatz:

   $$b_\uparrow = f_0 f_\uparrow, \quad b_\downarrow = f_0 f_\downarrow f_1 f_2$$

   where all the fs are fermionic partons. There are three $\mathsf{U}(1)$ gauge fields that glue these partons back together, and the charge assignments are as follows:

	a_1	a_2	a_3	$\mathcal{A}$	Chern # in Phase 1	Chern # in Phase 2	Chern # in Phase 3
$f_\uparrow$	1	0	0	1	1	1	1
$f_\downarrow$	1	1	0	1	1	1	1
f_0	-1	0	0	0	-1	-1	-1
f_1	0	-1	1	0	-1	-1	-1
f_2	0	0	-1	0	-1	0	1

   Also in the table are the Chern numbers of the bands filled by each of the partons in three distinct phases. (Only the Chern number of f_2 changes.) Identify the three phases, and describe the critical theories separating them.

   Hint: I recommend describing the parton currents in terms of dynamical gauge fields $j_\mu^{(\alpha)} = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial_\nu b_\rho^{(\alpha)}$, where $\alpha = \uparrow, \downarrow, 0, 1, 2$.

Following the advice of the hint, the effective action is

$$4\pi L = 2\left((b_\uparrow + b_\downarrow - b_0)da_1 + (b_\downarrow - b_1)da_2 + (b_1 - b_2)da_3\right) \tag{2}$$

$$+ b_\uparrow db_\uparrow + b_\downarrow db_\downarrow - b_0 db_0 - b_1 db_1 + cb_2 db_2 + 2\mathcal{A}d(b_\uparrow + b_\downarrow). \tag{3}$$

The $a_{1,2,3}$ are lagrange multipliers gluing the partons back together. Solving the constraints as $b_1 = b_2 = b_\downarrow, b_0 = b_\uparrow + b_\downarrow$, we have

$$4\pi L = b_\uparrow db_\uparrow + cb_\downarrow db_\downarrow - (b_\uparrow + b_\downarrow)d(b_\uparrow + b_\downarrow) + 2(b_\uparrow + b_\downarrow)d\mathcal{A}.$$

Here $c = -1, 0, 1$ in Phases 1-3. In phase 3, we find

$$4\pi L = Kbdb + 2tbd\mathcal{A}, \quad K = \sigma^x, t = (1, 1)$$

which is our K-matrix description of the boson IQHE. In phase 1 where $c = -1$, we find $K = \begin{pmatrix} 0 & 1 \\ 1 & 1-c \end{pmatrix}$ which when $c = -1$ gives $\sigma^{xy} = 0$.

Finally, phase 2 is trickier. It is a superfluid. To see this, integrate out all the b_α leaving the a_i instead.

This result, and the tex for the table, are from this paper. This paper came out on the same day (from the same department) and is very similar.

(b) For this part of the problem, let's retreat to the continuum. Consider the simpler parton ansatz:

$$b_\uparrow = f_0 f_\uparrow, \quad b_\downarrow = f_0 f_\downarrow$$

where all the fs are fermionic partons. Choose the $\mathsf{U}(1)_\mathcal{A}$ to be charges $q_0 = 2, q_\uparrow = -1, q_\downarrow = -1$.

Consider an equal number N of $b_\uparrow$ and $b_\downarrow$ particles, so that the total filling fraction is $\nu = 2$. How many f_0 particles are there, and how many $f_\downarrow, f_\uparrow$ particles are there?

There are $2N$ f_0s and N of each of the other two species.

Write a candidate groundstate wavefunction $\Psi(r_i^\uparrow, r_i^\downarrow)$ for the bosons.

There are twice as many f_0 particles, but they see a magnetic field that's twice as big, so they can fill their landau level: $\psi_0 = \prod_{i<j} z_{ij}^\uparrow z_{ij}^\downarrow \prod_{ij}(z_i^\uparrow - z_j^\downarrow)$. Similarly the $f_\uparrow$ and $f_\downarrow$ fill their Landau levels, but they see a magnetic field of the opposite sign, so $\psi_{\uparrow/\downarrow} = \prod_{i<j} \bar{z}_{ij}^{\uparrow/\downarrow}$. The gauge projection glues the coordinates of the f_0 to both the coordinates of $f_\downarrow$ and $f_\uparrow$. Therefore

$$\Psi(r_i^\uparrow, r_i^\downarrow) = \prod_{i<j} |z_{ij}^\uparrow|^2 |z_{ij}^\downarrow|^2 \prod_{ij} (z_i^\uparrow - z_j^\downarrow).$$

(c) Bonus question: why does the simpler ansatz of the previous part produce a wavefunction in the same phase as one of the phases of the first part?

The change in the wavefunction from including f_1, f_2 with the assigned Chern numbers is just a factor of a real number $\prod |z_{ij}^{\downarrow}|^2$. This doesn't change the statistics of the quasiparticles. The factors $|z_{ij}^{\uparrow/\downarrow}|^2$ could similarly be argued not to affect things. However, they matter for producing a uniform particle density in each layer and make a big difference for the quality of the variational ansatz.

(d) Actually, here is a simpler description of the same phase diagram, closer to what I said in lecture. Consider a single species of boson, with the simple parton ansatz with $b = d_1 d_2$ in terms of two fermions. Let d_1 and d_2 fill Chern bands with total Chern number c_1 and c_2. Fix $c_1 = -1$. Consider what happens when $c_2 = 2$.

Describe the effective field theory of d_2 filling two bands with chern number 1 by introducing two gauge fields each with CS term $\frac{1}{4\pi} b_a db_a$.

See Appendix B of this paper.