University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 230 Quantum Phases of Matter, Spr 2024
Assignment 8

Due 11pm Thursday, June 6, 2024

The following problems are all bonus problems.

---

1. **Helical majorana mode on a surface domain wall of a TI.**

   Consider a $D = 3+1$ fermion topological insulator protected by charge conservation and time-reversal symmetry. If we stick an s-wave superconductor on the surface, we can gap out the surface Dirac cone, but the vortices are interesting because they carry majorana zeromodes.

   What happens if we (somehow) produce on the surface a domain wall in the phase of the superconducting order? That is, suppose that on the surface ($z = 0$), when $y > 0$ the superconducting pairing field is $\Delta = \Delta_0$, but for $y < 0$, it is $\Delta = e^{\mathbf{i}\gamma}\Delta_0$ for some phase γ. If $\gamma = \pi$, what happens on the $D = 1+1$ dimensional locus at $y = 0$?

2. **Majorana chain.**

   Consider the majorana chain

   $$H = \mathbf{i}\sum_i \left(t_e \gamma_i \tilde{\gamma}_i + t_o \tilde{\gamma}_i \gamma_{i+1}\right)$$

   with $\{\gamma_i, \gamma_j\} = \delta_{ij} = \{\tilde{\gamma}_i, \tilde{\gamma}_j\}, \{\gamma_i, \tilde{\gamma}_i\} = 0$.

   (a) Show that the Hamiltonian can be rewritten as a p-wave superconductor of spinless electrons.

   (b) Find the energy spectrum. Check that when $t_e = t_o$, the gap closes.

3. **Majorana lasagna.**

   Consider a layer in the coupled-layer construction to be the critical limit of the Kitaev chain, that is, a massless 1+1d Majorana fermion field. Apply the coupled-layer construction to make a model in $D = 2+1$ dimensions with no symmetry. What 2+1d free-fermion SPT state do you make this way?

4. **All-fermion toric code.**

(a) Check that $\mathsf{U}(1)^4$ CS theory with K-matrix

$$K_{\mathsf{SO}(8)} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

has the same spectrum of anyons as the all-fermion toric code.

(b) Check that the statistics of the 16 anyons in two copies of the toric code is related to that of two copies of the all-fermion toric code by a relabelling. (At least check the self-statistics.)

(c) [Super bonus] A perhaps better way to address the previous part, using the result of the first part: show that the $\mathsf{U}(1)^8$ Chern-Simons theory with K-matrix

$$K_{\mathsf{SO}(8)} \oplus \left(-K_{\mathsf{SO}(8)}\right)$$

is equivalent to two copies of the toric code, plus trivial theories. (Note that we are not worried about preserving any symmetry here, so there is no notion of charge vector, and the theory with $K = \sigma^x$ is trivial.)

5. **Cluster state from group cohomology.**

(a) A projective representation of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by π rotations of a spin-half particle. Call the elements of $G = \{e, x, y, z\}$ (with multiplication table

$$\begin{pmatrix} e & x & y & z \\ x & e & z & y \\ y & z & e & x \\ z & y & x & e \end{pmatrix} .$$

Then we can take the representation matrices to be

$$U(e) = 1, U(x) = \mathbf{i}X, U(y) = \mathbf{i}Y, U(z) = \mathbf{i}Z.$$

Check that this is a projective representation of G

$$U(g)U(h) = \nu(e, g, gh)U(gh)$$

and find the 2-cocycle ν. Check that it satisfies the cocycle condition.

(b) Show that we can regard the single-site Hilbert space $\mathcal{H}_0 = \mathrm{span}\{|e\rangle, |x\rangle, |y\rangle, |z\rangle\}$ as a pair of qubits, and write the state

$$|1\rangle = \sum_{g\in G} |g\rangle$$

in terms of the Pauli operators on this qubit.

(c) Find the solvable Hamiltonian that results from the construction of Chen-Gu-Wen:

$$H = -U \sum_i |1_i\rangle\langle 1_i| U^\dagger$$

in terms of the Pauli operators.

Compare to the cluster hamiltonian

$$H = -\sum_i Z_{i-1} X_i Z_{i+1} = -U_{CZ} X_i U_{CZ}^\dagger,$$

with $U_{CZ} \equiv \prod_i \mathsf{CZ}_{i,i+1}$ where CZ is the control-Z operation.

6. **Haldane phase from the path integral.**

In this problem we will give a field theory description of a spin-s antiferromagnetic chain with $G = \mathrm{SO}(3)$ symmetry.

Consider the $D = 1+1$ nonlinear sigma model with target space S^2 at $\theta = 2\pi s$. The field variable is a 3-component unit vector $\hat{n} \in S^2$. The fact that $\pi_2(S^2) = \mathbb{Z}$ will play an important role. Think of this $\vec{n}$ as arising from coherent-state quantization of a spin chain. So take the (imaginary-time) action to be

$$S = \int \mathrm{d}\tau \mathrm{d}x \left(\frac{1}{g^2} \partial_\mu \hat{n} \cdot \partial^\mu \hat{n} + \mathbf{i} \frac{\theta}{4\pi} \epsilon_{abc} n^a \partial_\tau n^b \partial_x n^c \right).$$

We will focus on $\theta \in 2\pi\mathbb{Z}$, integer spin. In this case the model (flows to strong coupling $g \to \infty$ in the IR and) has a gap (we take this as an assumption). We would like to understand what is different between $\theta = 0$ and $\theta = 2\pi$.

Recall the role of the θ term: Because θ multiplies a quantity that evaluates to an integer on a closed spacetime manifold M_D,

$$Z_{M_D}(\theta) \equiv \int [Dn] e^{-S} = \sum_{n \in \pi_2(S^2)} e^{\mathbf{i}\theta n} Z_n$$

and $Z_{M_D}(\theta) = Z_{M_D}(\theta + 2\pi)$. In particular, we can take $M_D = S^1 \times N_{D-1}$ to compute the partition function on any spatial manifold N_{D-1}. This means the bulk spectrum is periodic in θ with period 2π.

The θ term is a total derivative in the action, so it can manifest itself when we study the path integral on a spacetime with boundary.

(a) Put this field theory on the half-line $x > 0$. Suppose that the boundary conditions respect the $\mathsf{SO}(3)$ symmetry, so that the boundary values $\vec{n}(\tau, x = 0)$ are free to fluctuate. By remembering that the θ-term is a total derivative, and considering the strong-coupling (IR) limit, $g \to \infty$, show that there is a spin-$\frac{1}{2}$ at the boundary. (Hint: Recall the coherent state path integral for a spin-$\frac{1}{2}$.)

(b) Now cut the path integral open at some fixed euclidean *time* $\tau = 0$. (Consider periodic boundary conditions in space.) Such a path integral computes the groundstate wavefunction, as a function of the boundary values of the fields, $\vec{S}(x, \tau = 0)$. Find the groundstate wavefunctional is $\Psi[\vec{n}(x, \tau = 0)]$ in the strong coupling limit $g \to \infty$ (where the gap is big).