University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2025
Assignment 1

Due 11:59pm Wednesday, April 9, 2025

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1. **An application of effective field theory in quantum mechanics.**

   Consider a model of two canonical quantum variables ($[\mathbf{x}, \mathbf{p}_x] = \mathbf{i} = [\mathbf{y}, \mathbf{p}_y], 0 = [\mathbf{x}, \mathbf{p}_y] = [\mathbf{x}, \mathbf{y}]$, etc) with Hamiltonian

   $$\mathbf{H} = \mathbf{p}_x^2 + \mathbf{p}_y^2 + \lambda \mathbf{x}^2 \mathbf{y}^2.$$

   (This is similar to the degenerate limit of the model we studied in lecture with two QM variables where both natural frequencies are taken to zero.)

   (a) Based on a semiclassical analysis, would you think that the spectrum is discrete or continuous?

   (b) Study large, fixed x near $y = 0$. We will treat x as the slow (= low-energy) variable, while y gets a large restoring force from the background x value. Solve the y dynamics, and find the groundstate energy as a function of x:

   $$V_{\text{eff}}(x) = E_{\text{g.s. of y}}(x).$$

   (c) Presumably you did the previous part using your knowledge of the spectrum of the harmonic oscillator Hamiltonian. Redo the previous part using path integral methods.

   (d) The result for $V_{\text{eff}}(x)$ is not analytic in x at $x = 0$. Why did that happen?

   (e) Is the spectrum of the resulting 1d model with

   $$\mathbf{H}_{\text{eff}} = \mathbf{p}_x^2 + V_{\text{eff}}(\mathbf{x})$$

   discrete? Is this description valid in the regime that matters for the semi-classical analysis?

   [Bonus: determine the spectrum of $\mathbf{H}_{\text{eff}}$.]

2. **Entanglement and EFT.** [Bonus problem]

   In a given state ρ_{AB}, one measure of the entanglement between two parts of a bipartite Hilbert space $\mathcal{H} = A \otimes B$ is the entanglement entropy, the von Neumann entropy $S_A \equiv -\text{tr}_A \rho_A \log \rho_A$ of the reduced density matrix of one of the parts,

$\rho_A = \text{tr}_B \rho_{AB}$. When the whole system is in a pure state $\rho_{AB} = |\psi\rangle\langle\psi|$, this is a useful measure of their entanglement.

Consider the problem of two oscillators we discussed in lecture as a parable about integrating out high-energy modes. Compute the entanglement entropy S_q of the light mode q in the groundstate of the combined system, by whatever means necessary.

If $S_q \neq 0$, it means that q does not have its own wavefunction, only a probability distribution on wavefunctions. Show that S_q is very small in the limit $\omega_0 \ll \Omega$, so that it is not a terrible idea to approximate the low-energy physics of q as being described by a pure state.

3. **Emergence of the Dirac equation.** Consider a chain of free fermions with

$$H = -t \sum_n c_n^\dagger c_{n+1} + h.c.$$

Show that the low-energy excitations at a generic value of the filling are described the the massless Dirac lagrangian in 1+1 dimensions. [Hint: the problem has (discrete) translation-invariance and is linear.]

Find an explicit choice of $1+1$-d gamma matrices that matches the answer from the lattice model. Show that the right-movers are right-handed $\gamma^5 \equiv \gamma^0\gamma^1 = 1$ and the left-movers are left-handed.