University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2025
Assignment 2

Due 11:59pm Wednesday, April 16, 2025

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1. **Brain-warmer.** [optional] Find the constant a in

$$a \prod_{\mu=0}^{D-1} \gamma^\mu = \gamma^5$$

so that $(\gamma^5)^2 = 1$.

2. **Chiral anomaly in two dimensions.**

Consider a massive relativistic Dirac fermion in 1+1 dimensions, with

$$S = \int \mathrm{d}x\mathrm{d}t \bar{\psi} \left(\mathbf{i}\gamma^\mu \left(\partial_\mu + eA_\mu \right) - m \right) \psi.$$

By heat-kernel regularization of its expectation value, show that the divergence of the axial current $j^5_\mu \equiv \mathbf{i}\bar{\psi}\gamma_\mu\gamma^5\psi$ is

$$\partial_\mu j^5_\mu = 2\mathbf{i}m\bar{\psi}\gamma^5\psi + \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$

3. **Chiral anomaly in six dimensions.** [optional] Find the divergence of the axial current in QED in $D = 5 + 1$.

4. **An application of the anomaly to a theory without gauge fields.**

Consider a 1+1d theory of Dirac fermions coupled to a background scalar field θ as follows:

$$\mathcal{L} = \bar{\Psi} \left(\mathbf{i}\partial\!\!\!/ + m e^{\mathbf{i}\theta\gamma^5} \right) \Psi.$$

We wish to ask: if we subject the fermion to various configurations of $\theta(x)$ (such as a domain wall where $\theta(x > 0) = \pi + \theta(x < 0)$) what does the fermion number do in the groundstate?

   (a) Convince yourself that when θ is constant

$$\langle j^\mu \rangle = 0$$

   where $j^\mu = \bar{\Psi}\gamma^\mu\Psi$ is the fermion number current.

(b) Minimally couple the fermion to a *background* gauge field A_μ. Let $e^{\mathbf{i}\Gamma[A,\theta]} = \int [d\Psi] e^{\mathbf{i}S}$. Convince yourself that the term linear in A in $\Gamma[A,\theta] = \text{const} + \int A_\mu J^\mu + \mathcal{O}(A^2)$ is the vacuum expectation value of the current $\langle j^\mu \rangle = J^\mu$ (at $A = 0$).

(c) Show that by a local chiral transformation $\Psi \to e^{-\mathbf{i}\theta(x)\gamma^5/2}\Psi$ we can remove the dependence on θ from the mass term.

(d) Where does the theta-dependence go? Use the 2d chiral anomaly to relate $\langle j^\mu \rangle$ to $\partial\theta$. Notice that the result is independent of m. [This relation was found by Goldstone and Wilczek. The associated physics is realized in Polyacetylene.]

(e) Show that a domain wall where θ jumps from 0 to π localizes *fractional* fermion number.

(f) Consider the Dirac hamiltonian in the presence of such a domain wall. Show that there is an exponentially-localized mode of zero energy.