University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2025
Assignment 4

Due 11:59pm Wednesday, April 30, 2025

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1. **All possible terms.**

   Perturb the gaussian fixed point of the XY model in $D \geq 3$ by a term

   $$\delta S_6 = \int d^D x \; g_6 (\phi^\star \phi)^3 \; .$$

   This term is a marginal perturbation of the gaussian fixed point in $D = 3$. Should we worry that we left it out of our study of the Wilson-Fisher fixed point? At leading order in $\epsilon = 4 - D$, what does it do to the Wilson-Fisher fixed point?

2. **An example of the power of the RG logic.**

   Consider quantum mechanics of a single particle in d dimensions, with Hamiltonian

   $$H = \frac{p^2}{2m} + V(q), \quad [q, p] = i.$$

   Consider the (say, euclidean) path integral for this problem,

   $$Z = \int [dq] e^{-S[q]}, \quad S[q] = \int dt \left(\frac{m}{2} \dot{q}^2 - V(q) \right) .$$

   To be more precise, with periodic boundary conditions, $Z(\beta) = \int_{q(t+\beta)=q(t)} [dq] e^{-S[q]} = \text{tr} e^{-\beta H}$ is the thermal partition function. Alternatively, instead of Z, we could consider the Green's function $G(q_1, t_1; q_2, t_2) = \int_{q(t_1)=q_1}^{q(t_2)=q_2} [dq] e^{-S[q]}$.

   Working by analogy with our treatment of field theory, show that any smooth[1] potential V is a *relevant* perturbation of the free particle, *i.e.* the Gaussian fixed point with $H = \frac{p^2}{2m}$.

   Hint: change variables to $\phi(t) \equiv \sqrt{m} q(t)$.

   Use this to explain in words why the high energy asymptotics of the density of states

   $$N(E) \equiv \{\# \text{ of eigenvalues of } H \text{ less than } E\}$$

[1] Some singular potentials are also relevant perturbations. If $V(q) \sim q^{-\alpha}$, how big can α be for my statement to remain true? Thanks to Brian Vermilyea for reminding me that a singular enough potential will cause trouble.

is given by the *Weyl formula* (even for $V(q) \neq 0$):

$$N(E) = E^{d/2} K_d L^d + \ldots$$

where $K_d = \frac{\Omega_{d-1}}{(2\pi)^d}$ as usual, and L is the linear size of the box in which we put the particle (an IR cutoff).

Hint: we can represent the density of states by a path integral using an inverse Laplace transform:

$$\mathrm{tr}\frac{1}{\omega - H} = \int d\beta\, e^{\beta\omega} Z(\beta)$$

and the relation

$$\mathrm{Im}\frac{1}{\omega + \mathbf{i}\epsilon - H} = \pi\delta(\omega - H).$$

In the next two problems, we will study free massless bosons in two dimensions. This system is solvable and has many physical applications – *e.g.* in string theory, and at the edge of quantum Hall systems. It is an example of a *conformal field theory.* It is a field theory where the excitations are *not particles.* And it's an example where we can understand all the composite operators exactly.

3. **There are no Goldstone bosons in two dimensions.**

(a) Consider a massless scalar X in 2d, with action

$$S[X] = -\frac{1}{4\pi}\int d^2\sigma \partial_a X \partial^a X. \tag{1}$$

Show that the euclidean Green function G_2 satisfies

$$\nabla^2 G_2(z, z') = -2\pi\delta^2(z - z') \tag{2}$$

($z = \sigma_1^E + \mathbf{i}\sigma_2^E$)[2] and is given by

$$G_2(z, z') = -\ln|z - z'|,$$

for example by Fourier transform.

(b) [Perhaps this part is more of a diatribe than a problem.] The long-distance behavior of G_2 has important implications for the physics of massless scalars in two dimensions. Thinking of G_2 as the two point function of the massless scalar

$$G_2(z, z') = \langle X(z) X(z') \rangle$$

let's ask the following question:

[2] Recall Schwinger-Dyson equations from the previous problem set.

There is no potential energy for the field X in (1). Someone used to doing physics in 3+1 dimensions might think that this means that there is a vacuum for every value of X. Let's try to fix the expectation value of the scalar $\langle X \rangle = x$ and see what happens. Perturb the putative vacuum $|x\rangle$ a little bit at the position z by inserting the operator X there. To measure what happens, insert the operator X at z'. The correlator G_2 can thus be interpreted as a measurement of how the effects of our perturbation fall off with distance. What happens? Contrast this with the behavior you would see for a scalar field with a flat potential in more than two dimensions. Note that the case of (0+1) dimensional QFT (*i.e.* quantum mechanics) is even more problematic in the infrared.

One way to arrive at an action like (1) is if the field X arises as a Goldstone boson associated with a symmetry $X \to X + a$, which would be broken by fixing the vacuum $|x\rangle$. Then it is guaranteed by Goldstone's theorem that the action can only depend on derivatives of X. We will say more about Goldstone's theorem later, but note that the Goldstone-ness of the massless bosons (*i.e.* whether they are massless because of a broken symmetry) is not crucial for this discussion. One might expect massless bosons whose masslessness is not protected by a symmetry to be lifted (to acquire a mass) quantumly, but there are special cases (such as in supersymmetric theories) where different points on the space of minima of the potential need not be related by a symmetry.

This result is called the Coleman-Mermin-Wagner (sometimes Hohenberg, too) Theorem. Coleman's paper on the subject is S. Coleman, "There are no Goldstone bosons in two-dimensions," *Commun. Math. Phys.* **31**:259-264 (1973).

4. **Correlators of composite operators made of free bosons in 1+1 dimensions.**

Consider a collection of n two-dimensional free bosons X^μ governed by the action

$$S = -\frac{1}{4\pi g} \int d^2\sigma \partial_a X_\mu \partial^a X^\mu.$$

[The coupling g can be absorbed into the definition of X if we prefer, but it is useful to leave this coupling constant arbitrary since different physicists use different conventions for the normalization and as you will see this affects the appearance of the final answer.]

Until further notice, we will assume that X takes values on the real line.

(a) Rotate $e^{\mathbf{i}S}$ to Euclidean space ($d^2\sigma = -\mathbf{i}(d^2\sigma)_E$) and compute the Euclidean generating functional

$$Z[J] = \left\langle e^{\int (d^2\sigma)_E J^\mu X_\mu} \right\rangle \equiv Z_0^{-1} \int [dX] e^{\mathbf{i}S} e^{\int (d^2\sigma)_E J^\mu X_\mu}$$

(where $Z_0^{-1} \equiv Z[J=0]$ but please don't worry too much about the normalization of the path integral).

[Hint: use the Green function from the previous problem, and Wick's theorem. Or use our general formula for Gaussian integrals with sources.]

[Warning: In the problem at hand, even the euclidean kinetic operator has a kernel, namely the zero-momentum mode. You will need to do this integral separately.]

[Cultural remark 1: this field theory describes the propagation of featureless strings in n-dimensional flat space $\mathbb{R}^n$ – think of $X^\mu(\sigma)$ as the parametrizing the position in $\mathbb{R}^n$ to which the point σ is mapped.

Cultural remark 2: this is an example of a *conformal field theory*. In particular recall that massless scalars in $D=2$ have engineering dimension zero.]

(b) Show that

$$\left\langle \prod_{i=1}^{N} : e^{-i\sqrt{2\alpha'}k_i\cdot X(\sigma^{(i)})} : \right\rangle = \delta^n\left(\sum_i k_i^\mu\right) \prod_{i,j=1}^{N} |z_i - z_j|^{+\alpha' g k_i\cdot k_j} \tag{3}$$

where $\sigma^{(i)}$ label points in 2d Euclidean space, $z_i \equiv \sigma_1^{(i)} + i\sigma_2^{(i)}$, α' is a parameter with dimensions of $[X^2/g]$ (called the 'Regge slope'), and k_i^μ are a set of arbitrary n-vectors in the target space. The : ... : indicate the following prescription for *defining* composite operators. The prescription is simply to leave out Wick contractions of objects within a pair of : ... :. Give a symmetry explanation of the delta function in k.

[Cultural remark: this calculation is the central ingredient in the *Veneziano amplitude* for scattering of bosonic strings at tree level.]

(c) Conclude that the composite operator $\mathcal{O}_a \equiv: e^{\mathbf{i}aX}$: has *scaling dimension* $\Delta_a = \frac{ga^2}{2}$, in the sense that

$$\left\langle \mathcal{O}_a(z)\mathcal{O}_b^\dagger(0) \right\rangle = \delta(a-b)\frac{1}{|z|^{2\Delta_a}}.$$

Notice that the correlation functions of these operators do not describe the propagation of *particles* in any sense. The operator $\mathcal{O}$ produces some power-law excitation of the CFT soup.

(d) Suppose we have one field ($n=1$) X which takes values on the circle, that is, we identify

$$X \simeq X + 2\pi R\ .$$

What values of a label single-valued operators : $e^{\mathbf{i}aX}$: ? How should we modify (3)?