University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2025
Assignment 5

Due 11:59pm Wednesday, May 7, 2025

---

1. **Order parameter exponent at the Wilson-Fisher fixed point.** [bonus problem] In lecture we outlined the computation of η using position space diagrams. Find the coefficient c in $\eta = c\epsilon^2 + \mathcal{O}(\epsilon^3)$ as a function of n at the $\mathsf{O}(n)$ Wilson-Fisher fixed point. Check that the factors of r_2 drop out.

   [Hint: The answer for the Ising model ($n = 1$) is $\eta = \frac{\epsilon^2}{54}$.]

2. **OPE.** [bonus problem] Consider the Gaussian fixed point with $\mathsf{O}(n)$ symmetry. Compute the OPE coefficients for the operators $\mathcal{O}_2 \equiv: \phi_a\phi_a :$, $\mathcal{O}_4 \equiv: (\phi_a\phi_a)^2 :$, and the identity operator (here $a = 1..n$ and the repeated index is summed). Use this information to compute the beta function, find the Wilson-Fisher fixed point and the correlation length critical exponent ν there.

3. **RG analysis of less-symmetric spin systems.**

   Suppose that we break the rotation symmetry of the $\mathsf{O}(n)$ model to the subgroup of $\pi/2$ rotations, *i.e.* the cubic symmetry, (for example, for $n = 2$, $(s_1, s_2) \to (s_2, -s_1)$.) If the spins live on the cubic lattice, a spin-orbit coupling could do this.

   (a) Don't look at the next part of the problem yet! What functions of an n-vector ϕ_a are invariant under $\pi/2$ rotations, but not general rotations?

⋮

⋮

⋮

⋮

⋮

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

(b) Show that in addition to the usual $\mathsf{O}(n)$-symmetric interaction

$$\int d^d x \sum_{a,b=1}^{n} u\phi_a^2\phi_b^2,$$

the LG free energy should include a term of the form

$$\int d^d x \sum_{a=1}^{n} v\phi_a^4.$$

Argue that this is the only new term (preserving cubic symmetry but not the full $\mathsf{O}(n)$ symmetry) which can be a relevant perturbation of the Gaussian fixed point near $d = 4$.

(c) Treating $\mathcal{O}(u) = \mathcal{O}(v) = \mathcal{O}(\epsilon)$, redo the analysis of the running couplings in $d = 4 - \epsilon$ dimensions to derive beta functions for u and v up to corrections of order $\mathcal{O}(u^3) = \mathcal{O}(v^2 u) = ... = \mathcal{O}(\epsilon^3)$.

(d) Your answer to the previous part will be of the form

$$\begin{aligned} -s\partial_s u &= -\epsilon u + A_1 u^2 + A_2 uv + A_3 v^2 + \mathcal{O}(u^3) \\ -s\partial_s v &= -\epsilon v + B_1 u^2 + B_2 uv + B_3 v^2 + \mathcal{O}(u^3). \end{aligned} \tag{1}$$

You should find that $A_3 = B_1 = 0$. Find four fixed points:

- The gaussian fixed point.
- A fixed point where only $u \neq 0$.
- A fixed point where only $v \neq 0$. Describe the physics of this fixed point. (Hint: the action is a sum of n terms.)
- A fixed point where both (u, v) are nonzero.

(In every case, the assumption of $u \sim v \sim \epsilon$ is self-consistent.)

(e) Analyze the stability of these fixed points (by computing the matrix of derivatives of the beta functions at each fixed point). Draw the phase diagram. Which fixed point dominates the critical behavior? You will want to consider different cases depending on whether $n > 4$ or $n < 4$.

(f) When $n > 4$ you may find that v wants to become negative. This means that the effective potential for m becomes unbounded, within our approximation. What have we left out that will restore sanity? What does this mean for the order of the phase transition? (Notice that mean field theory predicts a continuous transition, so any change in this conclusion is a dramatic effect of the fluctuations, more dramatic than just changing the values of critical exponents by a little.)