University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2025
Assignment 6

Due 11:59pm Wednesday, May 15, 2025

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1. $\mathsf{O}(N)$ **model at large** N**.** In lecture we studied the $\mathsf{O}(N)$ model in an expansion in $\epsilon = 4 - D$. When N is large, there is another small parameter in which to expand. We'll see that the results are consistent with the ϵ expansion. It is also an example that illustrates the manipulations we'll do in describing the BCS phenomenon.

   Consider the (Euclidean) partition function for an N-vector of scalar fields in D dimensions:

$$Z = \int [d\phi] e^{\mathbf{i}S[\phi]}, \quad S[\vec{\phi}] = \int \mathrm{d}^D x \left(\partial_\mu \phi^a \partial^\mu \phi^a - r\phi^a\phi^a - \frac{g}{N} (\phi^a\phi^a)^2 \right).$$

   (a) At the free fixed point, what is the dimension of the coupling g as a function of the number of spacetime dimensions D? Show that it is classically marginal in $D = 4$, so that this action is (classically) scale invariant.

   (b) [optional] Show that the definition above of $u = g/N$ is a good idea if we want to take $N \to \infty$, at fixed g. Do this by considering the N-dependence of diagrams that contribute to, say, the free energy, and demanding that in the large-N limit the interaction terms contribute with the same power of N as the leading term.

   (c) Analyze, at large N, the critical behavior of the model as r is varied. You'll need to consider separately the regimes $D > 4, D = 4, 2 < D < 4, D \leq 2$. Here are the steps: first use the Hubbard-Stratonovich trick to replace ϕ^4 by $\sigma\phi^2 + \sigma^2$ (up to factors) in the action, where σ is a new scalar field[1]. Then integrate out the ϕ fields. Find the saddle point equation for σ; argue that the saddle point dominates the integral for large N. Regulate the integrals in a convenient way. Find a translation invariant saddle point (*i.e.* where σ is constant). Plug the saddle point configuration of σ back into the action for ϕ and describe the resulting dynamics.

[1] To be more explicit, use (a path-integral version of) the identity

$$e^{u\phi^4} = \frac{1}{\sqrt{\pi u}} \int_{-\infty}^{\infty} d\sigma \; e^{-\sigma^2/u - 2\phi^2\sigma}.$$

(d) [bonus] Compute the correlation-length critical exponent ν at leading order in large N. Compare with the epsilon expansion results.

2. **Right-handed neutrinos.**

Consider adding a right-handed singlet (under all gauge groups) neutrino N_R to the Standard Model. It may have a majorana mass M; and it may have a coupling g_ν to leptons, so that all the dimension ≤ 4 operators are

$$\mathcal{L}_N = \bar{N}_R \mathbf{i} \partial\!\!\!/ N_R - \frac{M}{2} \bar{N}_R^c N_R - \frac{M}{2} \bar{N}_R N_R^c + \left(g_\nu \bar{N}_R H_i^T L_j \epsilon^{ij} + h.c. \right)$$

where $N_R^c = C \left(\bar{N}_R \right)^T$ is the the charge conjugate field, $C = \mathbf{i}\gamma_2\gamma_0$ (in the Dirac representation), H is the Higgs doublet, L is the left-handed lepton doublet, containing ν_L and e_L. Take the mass M to be large compared to the electroweak scale. Integrate out the right-handed neutrinos at tree level. [Hint: you may find it useful to work in terms of the Majorana field

$$N \equiv N_R + N_R^c$$

which satisfies $N = N^c$.]

Show that the leading term in the expansion in $1/M$ is a dimension-5 operator made of Standard Model fields. Explain the consequences of this operator for neutrino physics, assuming a vacuum expectation value for the Higgs field.

Place a bound on M assuming that the observed neutrinos have masses $m_\nu < 0.5$ eV.