University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2025
Assignment 8

Due 11:59pm Wednesday, May 22, 2025

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1. **Diagrammatic understanding of BCS instability of Fermi liquid theory.**

   (a) Recall that only the four-fermion interactions with special kinematics are marginal. Keeping only these interactions, show that cactus diagrams (like this: ) dominate.

   (b) To sum the cacti, we can make bubbles with a corrected propagator. Argue that this correction to the propagator is innocuous and can be ignored.

   (c) Armed with these results, compute diagrammatically the Cooper-channel susceptibility (two-particle Green's function),

   $$\chi(\omega_0) \equiv \left\langle \mathcal{T} \psi^\dagger_{\vec{k},\omega_3,\downarrow} \psi^\dagger_{-\vec{k},\omega_4,\uparrow} \psi_{\vec{p},\omega_1,\downarrow} \psi_{-\vec{p},\omega_2,\uparrow} \right\rangle$$

   as a function of $\omega_0 \equiv \omega_1 + \omega_2$, the frequencies of the incoming particles. Think of χ as a two point function of the Cooper pair field $\Phi = \epsilon_{\alpha\beta}\psi_\alpha\psi_\beta$ at zero momentum.

   Sum the geometric series in terms of a (one-loop) integral kernel.

   (d) Do the integrals. In the loops, restrict the range of momenta to $|\epsilon(k)| < E_D$, the Debye energy, since it is electrons with these energies that experience attractive interactions.

   Consider for simplicity a round Fermi surface. For doing integrals of functions singular near a round Fermi surface, approximate the dispersion relation as $\epsilon(k) \simeq v_F(|k| - k_F)$, so that $d^dk \simeq k_F^{d-1} \frac{d\xi}{v_F} d\Omega_{d-1}$. I recommend doing to the frequency integral first (by residues).

   (e) Show that when $V < 0$ is attractive, $\chi(\omega_0)$ has a pole. Does it represent a bound-state? Interpret this pole in the two-particle Green's function as indicating an instability of the Fermi liquid to superconductivity. Compare the location of the pole to the energy E_{BCS} where the Cooper-channel interaction becomes strong.

   (f) **Cooper problem.** [optional] We can compare this result to Cooper's influential analysis of the problem of two electrons interacting with each other in the presence of an inert Fermi sea. Consider a state with two electrons with antipodal momenta and opposite spin

$$|\psi\rangle = \sum_k a_k \psi^\dagger_{k,\uparrow}\psi^\dagger_{-k,\downarrow} |F\rangle$$

where $|F\rangle = \prod_{k<k_F} \psi^\dagger_{k,\uparrow}\psi^\dagger_{k,\downarrow} |0\rangle$ is a filled Fermi sea. Consider the Hamiltonian

$$H = \sum_k \epsilon_k \psi^\dagger_{k,\sigma}\psi_{k,\sigma} + \sum_{k,k'} V_{k,k'} \psi^\dagger_{k,\sigma}\psi_{k,\sigma}\psi^\dagger_{k',\sigma'}\psi_{k',\sigma'}.$$

Write the Schrödinger equation as

$$(\omega - 2\epsilon_k) a_k = \sum_{k'} V_{k,k'} a_{k'}.$$

Now assume (following Cooper) that the potential has the following form:

$$V_{k,k'} = V w^\star_{k'} w_k, \qquad w_k = \begin{cases} 1, & 0 < \epsilon_k < E_D \\ 0, & \text{else} \end{cases}.$$

Defining $C \equiv \sum_k \omega^\star_k a_k$, show that the Schrödinger equation requires

$$1 = V \sum_k \frac{|w_k|^2}{\omega - 2\epsilon_k}. \tag{1}$$

Assuming V is attractive, find a bound state. Compare (14) to the condition for a pole found from the bubble chains above.

2. **Fermion propagator in a metal.** [bonus problem]

Starting from

$$G(p,t) = -\frac{1}{2\pi\mathbf{i}} \langle \text{gs}| \mathcal{T} c_p(t) c^\dagger_p(0) |\text{gs}\rangle \tag{2}$$

and using the free fermion time evolution operator, and the fact that the groundstate has all levels filled up to the Fermi level:

$$\langle \text{gs}| c^\dagger_p c_p |gs\rangle = \begin{cases} 1, & \epsilon_p < 0 \\ 0, & \epsilon_p > 0 \end{cases} \tag{3}$$

show that the free fermion propagator can be written as

$$G(p,\omega) = \frac{a}{\omega - \epsilon_p - \mathbf{i}\eta b \text{sgn}(\epsilon_p)} \tag{4}$$

or

$$G(p,\omega) = \frac{a'}{\omega(1 + \mathbf{i}b'\eta) - \epsilon_p} \tag{5}$$

where $\eta = 0^+$ is an infinitesimal for some constants a, b, a', b' to be determined.