University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215C QFT Spring 2025
Assignment 9

Due never

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1. **The Hohenberg-Mermin-Wagner-Coleman Fact.**

   (a) Consider a massless scalar X in 2d, with (Euclidean) action

   $$S[X] = \frac{1}{4\pi g} \int d^2\sigma \partial_a X \partial^a X. \tag{1}$$

   Show that the euclidean propagator

   $$G_2(z, z') \equiv \langle X(z) X(z') \rangle$$

   satisfies

   $$\nabla^2 G_2(z, z') = b\delta^2(z - z') \tag{2}$$

   where $z = \sigma_1^E + \mathbf{i}\sigma_2^E$, for some constant b; find b. Show that the solution is given by

   $$G_2(z, z') = a \ln |z - z'|,$$

   for some constant a (for example by Fourier transform); find a.

   (b) The long-distance behavior of G_2 has important implications for the spontaneous breaking of continuous symmetries in $D = 2$ – it can't happen. Argue that if a system with a continuous (say $\mathsf{U}(1)$, for definiteness) symmetry were to have an unsymmetric groundstate, the excitations about that state would include a field X with the action (1). Conclude from the form of G_2 that there is in fact no long-range order.

2. **Correlators of composite operators made of free bosons in 1+1 dimensions.**

   Consider a collection of n two-dimensional free bosons X^μ governed by the (Euclidean) action

   $$S = \frac{1}{4\pi g} \int d^2\sigma \partial_a X_\mu \partial^a X^\mu.$$

   Until further notice, we will assume that X takes values on the real line.

   [If $X \in \mathbb{R}$, the coupling g can be absorbed into the definition of X if we prefer, but it is useful to leave this coupling constant arbitrary for several reasons. First, different physicists use different conventions for the normalization and as you will see this affects the appearance of the final answer. But more importantly, in part 2d, g will become meaningful.]

(a) Compute the Euclidean generating functional

$$Z[J] = \left\langle e^{\int (d^2\sigma)_E J^\mu X_\mu} \right\rangle \equiv Z_0^{-1} \int [dX] e^{-S} e^{\int (d^2\sigma)_E J^\mu X_\mu}$$

(where $Z_0^{-1} \equiv Z[J=0]$ but please don't worry too much about the normalization of the path integral).

[Hint: use the Green function from the previous problem, and Wick's theorem. Or use our general formula for Gaussian integrals with sources.]

[Warning: In the problem at hand, even the euclidean kinetic operator has a kernel, namely the zero-momentum mode. You will need to do this integral separately.]

[Cultural remark 1: this field theory describes the propagation of featureless strings in n-dimensional flat space $\mathbb{R}^n$ – think of $X^\mu(\sigma)$ as the parametrizing the position in $\mathbb{R}^n$ to which the point σ is mapped.

Cultural remark 2: this is an example of a conformal field theory. In particular recall that massless scalars in $D = 2$ have engineering dimension zero.]

(b) Show that

$$\left\langle \prod_{i=1}^{N} : e^{-i\sqrt{2\alpha'} k_i \cdot X(\sigma^{(i)})} : \right\rangle = \delta^n \left(\sum_i k_i^\mu \right) \prod_{i,j=1}^{N} |z_i - z_j|^{-\alpha' g k_i \cdot k_j} \tag{3}$$

where $\sigma^{(i)}$ label points in 2d Euclidean space, $z_i \equiv \sigma_1^{(i)} + i\sigma_2^{(i)}$, α' is a parameter with dimensions of $[X^2/g]$ (called the 'Regge slope'), and k_i^μ are a set of arbitrary n-vectors in the target space. The : ... : indicate the following prescription for *defining* composite operators. The prescription is simply to leave out Wick contractions of objects within a pair of : ... :. Give a symmetry explanation of the delta function in k.

[Cultural remark: this calculation is the central ingredient in the *Veneziano amplitude* for scattering of bosonic strings at tree level.]

(c) Conclude that the composite operator $\mathcal{O}_a \equiv : e^{\mathbf{i}aX} :$ has *scaling dimension* $\Delta_a = \frac{ga^2}{2}$, in the sense that

$$\left\langle \mathcal{O}_a(z) \mathcal{O}_b^\dagger(0) \right\rangle = \delta(a-b) \frac{1}{|z|^{2\Delta_a}}.$$

Notice that the correlation functions of these operators do not describe the propagation of *particles* in any sense. The operator $\mathcal{O}$ produces some power-law excitation of the CFT soup.

(d) Suppose we have one field ($n = 1$) X which takes values on the circle, that is, we identify

$$X \simeq X + 2\pi R \ .$$

What values of a label single-valued operators $: e^{\mathbf{i}aX} :$? How should we modify (3)?

3. **The stress tensor is not a conformal primary if $c \neq 0$.**

(a) For any 2d CFT, use the general form of the TT OPE to show that the transformation of T under an infinitesimal conformal transformation $z \mapsto z + \xi(z)$ is

$$\delta_\xi T(w) = (\xi\partial + 2\partial\xi)T(w) + \frac{c}{12}\partial^3\xi. \quad (1)$$

(b) Consider the *finite* conformal transformation $z \mapsto f(z)$. Show that (1) is the infinitesimal version of the transformation law

$$T(z) \mapsto (\partial f)^2 T\left(f(z)\right) + \frac{c}{12}\{f, z\}$$

where

$$\{f, z\} \equiv \frac{\partial f \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2}$$

is called a *Schwarzian derivative.*

[Optional: verify that this extra term does the right thing when composing two maps $z \to f(z) \to g(f(z))$.]

(c) Given that the conformal map from the cylinder to the plane is $z = e^{-iw}$, show that (b) means that

$$T_{\text{cyl}}(w)(dw)^2 = \left(T_{\text{plane}}(z) - \frac{c}{24}\right)(dz)^2.$$

Use this relation to show that the Hamiltonian on the cylinder

$$H = \int \frac{d\sigma}{2\pi} T_{\tau\tau}$$

is

$$H = L_0 + \tilde{L}_0 - \frac{c + \bar{c}}{24}.$$

Comment: After all this complication, the result has a very simple physical interpretation: when putting a CFT on a cylinder, the scale invariance is spontaneously broken by the fact that the cylinder has a *radius*, *i.e.* the cylinder introduces a (worldsheet) length scale into the problem. The term in the energy extensive in the radius of the cylinder but not the length (and proportional to c) is actually experimentally observable sometimes.

4. $\mathsf{SU}(2)$ **current algebra from free scalar.**

Consider again a compact free boson $\phi \simeq \phi + 2\pi$ in $D = 1+1$ with action

$$S[\phi] = \frac{R^2}{8\pi} \int \mathrm{d}x\mathrm{d}t \partial_\mu \phi \partial^\mu \phi. \tag{4}$$

[Notice that if we redefine $\tilde{\phi} \equiv R\phi$ then we absorb the coupling R from the action $S[\tilde{\phi}] = \frac{1}{8\pi} \int \mathrm{d}x\mathrm{d}t \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi}$ but now $\tilde{\phi} \simeq \tilde{\phi} + 2\pi R$ has a different period – hence the name 'radius'.[1]]

So: there is a special radius (naturally called the $\mathsf{SU}(2)$ radius) where new operators of dimension $(1,0)$ and $(0,1)$ appear, and which are charged under the boson number current $\partial_\pm \phi$. Their dimensions tell us that they are (chiral) currents, and their charges indicate that they combine with the obvious currents $\partial_\pm \phi$ to form the (Kac-Moody-Bardakci-Halpern) algebra $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$.

Here you will verify that the model (4) does in fact host an $SU(2)_L \times SU(2)_R$ algebra involving *winding modes* – configurations of ϕ where the field winds around its target space circle as we go around the spatial circle. We'll focus on the holomorphic (R) part, $\phi(z) \equiv \phi_R(z)$; the antiholomorphic part will be identical, with bars on everything.

Define

$$J^\pm(z) \equiv : e^{\pm i\phi(z)} :, \qquad J^3 \equiv i\partial\phi(z).$$

The dots indicate a normal ordering prescription for defining the composite operator: no wick contractions between operators within a set of dots.

(a) Show that $J^3, J^\pm$ are single-valued under $\phi \to \phi + 2\pi$.

(b) Compute the scaling dimensions of $J^3, J^\pm$. Recall that the scaling dimension Δ of a holomorphic operator in 2d CFT can be extracted from its two-point correlation function:

$$\left\langle \mathcal{O}^\dagger(z)\mathcal{O}(0) \right\rangle \sim \frac{1}{z^{2\Delta}} \ .$$

For free bosons, all correlation functions of composite operators may be computed using Wick's theorem and

$$\langle \phi(z)\phi(0) \rangle = -\frac{1}{R^2} \log z.$$

Find the value of R where the vertex operators $J^\pm$ have dimension 1.

[1] Relative to the notation I used in lecture, I have set $\pi T \equiv R^2$. A note for the string theorists: I am using units where $\alpha' = 2$.

(c) Defining $J^{\pm} \equiv \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)$ show that the operator product algebra of these currents is

$$J^a(z)J^b(0) \sim \frac{k\delta^{ab}}{z^2} + i\epsilon^{abc}\frac{J^c(0)}{z} + ...$$

with $k = 1$. This is the level-$k = 1$ SU(2)Kac-Moody-Bardakci-Halpern algebra.

(d) [Bonus tedium] Defining a mode expansion for a dimension 1 operator,

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

show that

$$[J^a_m, J^b_n] = i\epsilon^{abc}J^c_{m+n} + mk\delta^{ab}\delta_{m+n}$$

with $k = 1$, which is an algebra called Affine $SU(2)$ at level $k = 1$. Note that the $m = 0$ modes satisfy the ordinary $SU(2)$ lie algebra.

For hints (and some applications in string theory) see problem 5 here.

5. **Constraints from Unitarity.** Show that in a unitary CFT, $c > 0$, and $h \geq 0$ for all primaries. Hint: consider $\langle\phi| [L_n, L_{-n}] |\phi\rangle$.