Physics 215C: Quantum Field Theory, Part 3
Spring 2025

Lecturer: McGreevy
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0.1 Introductory remarks

Welcome to the third quarter of the field theory sequence. My main goal for this quarter is to undo the high-energy bias that I introduced by talking about so much particle physics last quarter. Specifically, from our discussion of perturbative renormalization last quarter, you could have reasonably gotten the idea that renormalization is about divergences and sweeping them under the rug. A much more physical point of view on renormalization (in QFT and in general) is due to Kadanoff and Wilson and I want to tell you about it this quarter.

Quantum field theory is very useful in particle physics, but its domain is much broader. In fact, while in particle physics we only get to study the one example of the Standard Model (and speculate about its extensions), in condensed matter physics, essentially every system is a different universe, with its own low-energy QFT, governing its own laws of physics. Some of them are like particle physics, in the sense that the elementary excitations above the groundstate (the analog of the vacuum) are well-defined, maybe weakly-coupled particles. These are usually called *quasiparticles* to emphasize the fact that their properties need have nothing to do with the particles out of which the system is made. (Recall the example of the balls and springs from last quarter – the quasiparticles there are massless phonons, while the microscopic particles are massive.) An exciting fact is that some of these universes have low-energy physics very different from our own. In some of them (for example when the low-energy physics is governed by a conformal field theory (CFT)), the low-energy excitations cannot even be organized as particles. (In discussions of this phenomenon in the context of speculations about physics beyond the Standard Model, this is sometimes called Unparticle Physics.)

All of that was just to say that, if one is interested in understanding QFT, as an object of study in its own right, it's a good idea to broaden our perspective beyond particle physics. Besides quantum condensed matter, another large area of physics that has a lot to teach us is statistical physics. One basic connection here (in case I forget to do it later, I recommend Peskin chapter 8 for this purpose) is that the euclidean path integral for a QFT in $d+1$ dimensions,

$$Z \equiv \int D\phi e^{-S[\phi]} = \mathrm{tr} e^{-\beta H} \tag{0.1}$$

has the form of the equilibrium (canonical ensemble) partition function of some (classical) statistical system in $d+1$ space dimensions, with (classical) Hamiltonian $S[\phi]$ at inverse temperature (coolness) $\beta = 1$ (or we can factor some coupling constant out of S and regard this as determining the temperature). Here ϕ could be a continuum variable, or (more obviously well-defined) it could live on a lattice. In the former point

of view, the fluctuations are due to quantum mechanics; in the latter point of view, the fluctuations are thermal. More generally, *non-equilibrium* statistical physics problems can often be studied using path integrals, and therefore using field theory methods. (I plan to say more about this next quarter in 210B.)

All of the field theories that arise in condensed matter physics are *effective field theories* (EFT). This term just means that we don't worry about whether the field theory describes physics down to arbitrarily short distances (since we know already that it doesn't). EFT comes with a strategy for thinking about physics, which I hope to convey this quarter. If you want to start an interesting argument, try telling people that "every QFT is an EFT". The alternative point of view, that there is an important platonic notion of continuum QFT, valid to arbitrarily short distances, can be defended. But in doing physics (e.g. in trying to describe a particular real physical system) it is good to keep in mind that we only ever have access to a finite range of length scales.

An important goal for the course is demonstrating that many fancy phenomena precious to particle physicists can emerge from humble origins in the kinds of (completely well-defined) local quantum lattice models we will study. Here I have in mind: fermions, gauge theory, photons, anyons, strings, topological solitons, CFT, and many other sources of wonder I'm forgetting right now.

The crux of many problems in physics is the correct choice of variables with which to label the degrees of freedom. Often the best choice is very different from the obvious choice; a name for this phenomenon is 'duality'. There are many examples of it and we will study some of them. This word is dangerous because it is *about* ambiguities in our (physics) language. I would like to reclaim it.

An important bias in deciding what is meant by 'correct' or 'best' in the previous paragraph is: we will be interested in low-energy and long-wavelength physics, near the groundstate. For one thing, this is the aspect of the present subject which is like 'elementary particle physics'; the high-energy physics of these systems is of a very different nature and bears little resemblance to the field often called 'high-energy physics' (for example, there is volume-law entanglement).

Topics that I hope to discuss this quarter include:

- Wilsonian theory of renormalization (things can look different depending on how closely you look; this is how we should organize our understanding of extensive quantum systems)
- illustrations of effective field theory (perhaps cleverly mixed in with the other subjects) in diverse areas of physics
- effects of topology in QFT (this includes anomalies, topological solitons and

defects, topological terms in the action)

- the uses and limitations of path integrals of various kinds
- large-N expansions
- more deep mysteries of gauge theory and its emergence in physical systems.
- duality.

Some other modern topics in QFT, which we could consider discussing, include: CFT, entanglement, generalized symmetries, various bootstrap methods, scattering amplitudes, QFT in curved spacetime. I welcome your suggestions regarding which subjects in QFT we should study.

0.2 Sources and acknowledgement

The material in these notes is collected from many places, among which I should mention in particular the following:

Peskin and Schroeder, *An introduction to quantum field theory*

Zee, *Quantum Field Theory* (2d Edition)

Banks, *Modern Quantum Field Theory: A Concise Introduction*

Schwartz, *Quantum field theory and the standard model*

David Tong's lectures on gauge theory

Many other bits of wisdom come from the Berkeley QFT courses of Prof. L. Hall and Prof. M. Halpern.

Some other books that might be useful to us are:

Xiao-Gang Wen, *Quantum Field Theory of Many-Body Systems*

Sidney Coleman, *Aspects of Symmetry*

Alexander Polyakov, *Gauge Fields and Strings*

Eduardo Fradkin, *Field Theories of Condensed Matter Systems*

Eduardo Fradkin, *Quantum Field Theory, an Integrated Approach*

R. Shankar, *Quantum Field Theory and Condensed Matter*

0.3 Conventions

Following most QFT books, I am going to use the $+---$ signature convention for the Minkowski metric. I am (somehow, still) used to the other convention, where time is the weird one, so I'll need your help checking my signs. More explicitly, denoting a small spacetime displacement as $dx^\mu \equiv (dt, d\vec{x})^\mu$, the Lorentz-invariant distance is:

$$ds^2 = +dt^2 - d\vec{x}\cdot d\vec{x} = \eta_{\mu\nu}dx^\mu dx^\nu \quad \text{with} \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} .$$

(spacelike is negative). We will also write $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\partial_t, \vec{\nabla}_x\right)^\mu$, and $\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$. I'll use $\mu, \nu, ...$ for Lorentz indices, and $i, j, k, ...$ for spatial indices.

The convention that repeated indices are summed is always in effect unless otherwise indicated. d is the number of space dimensions, D is the number of spacetime dimensions (it's bigger!).

$\equiv$ means 'equals by definition'. $A \stackrel{!}{=} B$ means we are demanding that $A = B$. $A \stackrel{?}{=} B$ means A probably doesn't equal B.

A consequence of the fact that english and math are written from left to right is that time goes to the left.

A useful generalization of the shorthand $\hbar \equiv \frac{h}{2\pi}$ is $đk \equiv \frac{dk}{2\pi}$. I will also write $\not{\delta}^d(q) \equiv (2\pi)^d\delta^{(d)}(q)$. I will try to be consistent about writing Fourier transforms as

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{ikx}\tilde{f}(k) \equiv \int đ^d k\; e^{ikx}\tilde{f}(k) \equiv f(x).$$

IFF $\equiv$ if and only if.
RHS $\equiv$ right-hand side. LHS $\equiv$ left-hand side. BHS $\equiv$ both-hand side.
IBP $\equiv$ integration by parts. WLOG $\equiv$ without loss of generality.
$+\mathcal{O}(x^n) \equiv$ plus terms which go like x^n (and higher powers) when x is small.
$+h.c. \equiv$ plus hermitian conjugate.
$\mathcal{L} \ni \mathcal{O}$ means the object $\mathcal{L}$ contains the term $\mathcal{O}$.

We work in units where $\hbar$ and the speed of light, c, are equal to one unless otherwise noted. When I say 'Peskin' I usually mean 'Peskin & Schroeder'.

Please tell me if you find typos or errors or violations of the rules above.

1 A parable on integrating out degrees of freedom

Here's another parable from QM which gives some useful perspective on renormalization in QFT and on the notion of effective field theory.

First let's ask a question about classical physics. Suppose we have a single variable q with action

$$S[q] = \int dt \frac{1}{2}\left(\dot{q}^2 + \omega_0^2 q^2 + \frac{\ddot{q}^2}{\Omega^2}\right) \tag{1.1}$$

where $\omega_0 \ll \Omega$ are two frequency scales. In this regime, that last term is a small perturbation of our harmonic oscillator. But this higher-derivative term has a big singular effect, because the equations of motion involve $\partial_t^4 q$ and hence require *four* integration constants! What do we do? How can we regard it as a small perturbation if it totally changes the form of the initial value problem?

[Banks p. 138] Now consider a system of two coupled harmonic oscillators. We will assume one of the springs is much stiffer than the other: let's call their natural frequencies ω_0, Ω, with $\omega_0 \ll \Omega$. The euclidean-time action is

$$S[Q,q] = \int dt \left[\frac{1}{2}\left(\dot{q}^2 + \omega_0^2 q^2\right) + \frac{1}{2}\left(\dot{Q}^2 + \Omega^2 Q^2\right) + gQq^2\right] \equiv S_{\omega_0}[q] + S_\Omega[Q] + S_{\rm int}[Q,q].$$

(The particular form of the q^2Q coupling is chosen for convenience. Don't take too seriously the physics at large negative Q.) We can construct physical observables in this model by studying the path integral:

$$Z = \int [dQdq] e^{-S[Q,q]}.$$

Since I put a minus sign rather than an i in the exponent (and the potential terms in the action have + signs), this is a euclidean path integral.

Let's consider what happens if we do the path integral over the heavy mode Q, and postpone doing the path integral over q. This step, naturally, is called *integrating out* Q, and we will see below why this is a good idea. The result just depends on q; we can think of it as an *effective action* for q:

$$\begin{aligned} e^{-S_{\rm eff}[q]} &:= \int [dQ] e^{-S[q,Q]} \\ &= e^{-S_{\omega_0}[q]} \left\langle e^{-S_{\rm int}[Q,q]} \right\rangle_Q \end{aligned}$$

Here $\langle ... \rangle_Q$ indicates the expectation value of ... in the (free) theory of Q, with the action $S_\Omega[Q]$. It is a gaussian integral (because of our choice of $S_{\rm int}$):

$$\left\langle e^{-S_{\rm int}[Q,q]} \right\rangle_Q = \int [dQ] e^{-S_\Omega[Q] - \int ds J(s)Q(s)} = \mathcal{N} e^{\frac{1}{4}\int ds dt J(s)G(s,t)J(t)} \ .$$

This last equality is an application of the 'fundamental theorem of path integrals,' *i.e.* the gaussian integral. Here $J(s) \equiv gq(s)^2$. The normalization factor $\mathcal{N}$ is independent of J and hence of q. And $G(s,t)$ is the inverse of the linear operator appearing in S_Ω, the euclidean Green's function:

$$S_\Omega[Q] = \int dsdt Q(s) G^{-1}(s,t) Q(t).$$

More usefully, G satisfies

$$\left(-\partial_s^2 + \Omega^2\right) G(s,t) = \delta(s-t) \ .$$

The fact that our system is time-translation invariant means $G(s,t) = G(s-t)$. We can solve this equation in fourier space: $G(s) = \int đ\omega e^{-i\omega s} G_\omega$ makes it algebraic:

$$G_\omega = \frac{1}{\omega^2 + \Omega^2}$$

and we have

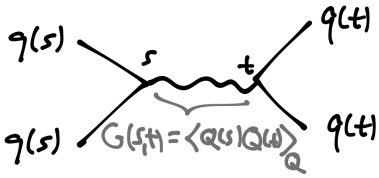
$$G(s) = \int đ\omega e^{-\mathbf{i}\omega s} \frac{1}{\omega^2 + \Omega^2}. \tag{1.2}$$

So we have:

$$e^{-S_{\text{eff}}[q]} = e^{-S_{\omega_0}[q]} e^{-\int dtds \frac{g^2}{4} q(s)^2 G(s,t) q(t)^2}$$

or taking logs

$$S_{\text{eff}}[q] = S_{\omega_0}[q] + \int dtds \frac{g^2}{4} q(s)^2 G(s,t) q(t)^2 \ . \tag{1.3}$$



Q mediates an interaction of four qs, an anharmonic term, a self-interaction of q. In Feynman diagrams in the full theory, the leading interaction between q's mediated by Q comes from the diagram at left.

And the whole thing comes from exponentiating disconnected copies of this diagram. There are no other diagrams: once we make a Q from two qs what can it do besides turn back into two qs? Nothing. And no internal q lines are allowed, they are just sources, for the purposes of the Q integral.

But it is *non-local*: we have two integrals over the time in the new quartic term. This is unfamiliar, and also *bad*: *e.g.* classically we don't know how to pose an initial value problem using this action.

But now suppose we are interested in times much longer than $1/\Omega$, say times comparable to the period of oscillation of the less-stiff spring $2\pi/\omega_0$. We can accomplish

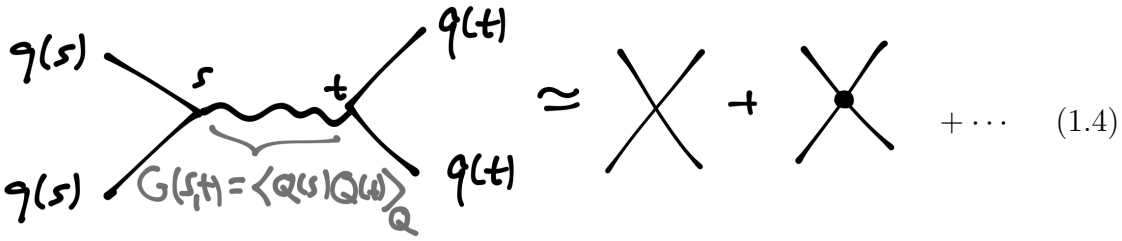
this by Taylor expanding under the integrand in (1.2):

$$G(s) = \int \text{đ}\omega e^{-\text{i}\omega s} \frac{1}{\Omega^2} \underbrace{\frac{1}{1+\frac{\omega^2}{\Omega^2}}}_{=\sum_n (-1)^n \left(\frac{\omega^2}{\Omega^2}\right)^n} \overset{s \gg 1/\Omega}{\simeq} \frac{1}{\Omega^2}\delta(s) + \frac{1}{\Omega^4}\partial_s^2\delta(s) + \ldots$$

Plug this back into (1.3):

$$S_{\text{eff}}[q] = S_{\omega_0}[q] + \int dt \frac{g^2}{4\Omega^2} q(t)^4 + \int dt \frac{g^2}{4\Omega^4} \dot{q}^2 q^2 + \ldots$$

The effects of the heavy mode Q are now organized in a *derivative expansion*, with terms involving more derivatives suppressed by more powers of the high energy scale Ω.



(1.4)

A useful mnemonic for integrating out the effects of the heavy field in terms of Feynman diagrams: to picture Q as propagating for only a short time (compared to the external time $t-s$), we can contract its propagator to a point. The first term on the RHS shifts the q^4 term, the second shifts the kinetic term (by something that depends on q^2), the third involves four time derivatives ...

On the RHS of this equation, we have various interactions involving four qs, which involve increasingly many derivatives. The first term is a quartic potential term for q: $\Delta V = \frac{g}{\Omega^2} q^4$; the leading effect of the fluctuations of Q is to *shift* the quartic self-coupling of q by a finite amount (note that we could have included a bare $\lambda_0 q^4$ potential term).

Notice that if we keep going in this expansion, we get terms with *more than two derivatives* of q. This is OK. We've just derived the right way to think about such terms: we treat them as a perturbation, and they are part of a never-ending series of terms, which become less and less important for low-energy questions. If we want to ask questions about q at energies of order ω, we can get answers that are correct up to effects of order $\left(\frac{\omega}{\Omega}\right)^{2n}$ by keeping the nth term in this expansion.

Conversely if we are doing an experiment with precision Δ at energy ω, we can measure the effects of up to the nth term, with

$$\left(\frac{\omega}{\Omega}\right)^{2n} \sim \Delta.$$

Another important lesson: $S_{\text{eff}}[q]$ contains couplings with negative dimensions of energy

$$\sum_n c_n \left(\partial_t^n q\right)^2 q^2, \quad \text{with } c_n \sim \frac{1}{\Omega^{2n}},$$

exactly the situation where the S-matrix grows too fast at high energies that we discussed last quarter. In this case we know exactly where the probability is going: if we have enough energy to see the problem ($E \sim \Omega$), we have enough energy to kick the heavy mode Q out of its groundstate.

1.0.1 Attempt to consolidate understanding

We've just done some coarse graining: focusing on the dofs we care about (q), and actively ignoring the dofs we don't care about (Q), except to the extent that they affect those we do (e.g. the self-interactions of q).

Above, we did a calculation in a QM model with two SHOs. This is a microcosm of QFT in many ways. For one thing, free quantum fields are bunches of harmonic oscillators with natural frequency depending on k, $\Omega = \sqrt{\vec{k}^2 + m^2}$. Here we kept just two of these modes (one with large k, one with small k) for clarity. Perhaps more importantly, QM is just QFT in 0+1d. The more general QFT path integral just involves more integration variables. The idea of the Wilsonian RG (for continuum field theory) is essentially to do the integrals over the modes in descending order of wavenumber, and at each stage make the expansion described above to get a local action. And notice that basically all possible terms are generated, consistent with the symmetries (here for example, there is a $\mathbb{Z}_2$ symmetry under which $q \to -q$, so there are no odd powers of q).

The result of that calculation was that fluctuations of Q mediate various q^4 interactions. It adds to the action for q the following: $\Delta S_{\text{eff}}[q] \sim \int dtdsq^2(t)G(t-s)q^2(s)$, as in Fig. 1.4.

If we have the hubris to care about the exact answer, it's nonlocal in time. But if we want exact answers then we'll have to do the integral over q, too. On the other hand, the hierarchy of scales $\omega_0 \ll \Omega$ is useful if we ask questions about energies of order ω_0, e.g.

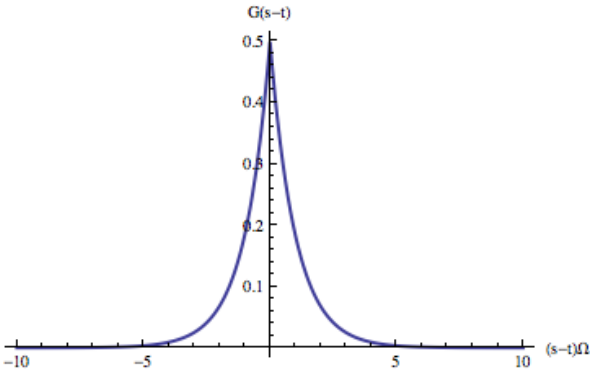
$$\langle q(t)q(0)\rangle \text{ with } t \sim \frac{1}{\omega_0} \gg \frac{1}{\Omega}.$$

Then we can Taylor expand the function $G(t-s)$, and we find a series of corrections in powers of $\frac{1}{t\Omega}$ (or more accurately, powers of $\frac{\partial_t}{\Omega}$).

Notice an important asymmetry: Why do we do the integral over the heavy mode first? It's not so useful to integrate out light degrees of freedom to get an action for the

heavy degrees of freedom; that would necessarily be nonlocal and stay nonlocal and we wouldn't be able to treat it using ordinary techniques.

The crucial point is that the scary nonlocality of the effective action that we saw only extends a distance of order $\frac{1}{\Omega}$; the kernel $G(s-t)$ looks like this:



The mechanism we've just discussed is an essential ingredient in getting *any* physics done at all. *Why* can we do physics despite the fact that we do not understand the theory of quantum gravity that governs Planckian distances? We happily do lots of physics without worrying about this! This is because the effect of those Planckian quantum gravity fluctuations – whatever they are, call them Q – on the degrees of freedom we do care about (*e.g.* the Standard Model, or an atom, or the sandwich you made this morning, call them collectively q) are encoded in terms in the effective action of q that are suppressed by powers of the high energy scale $M_{\rm Planck}$, whose role in the toy model is played by Ω. And the natural energy scale of your sandwich is much less than $M_{\rm Planck}$. I picked the Planck scale as the scale to ignore here for rhetorical drama, and because we really are ignorant of what physics goes on there. But this idea is equally relevant for *e.g.* being able to describe water waves by hydrodynamics (a classical field theory) without worrying about atomic physics, or to understand the physics of atoms without needing to understand nuclear physics, or to understand the nuclear interactions without knowing about the Higgs boson, and so on deeper into the onion of physics.

This wonderful situation, which makes physics possible, has a price: since physics at low energies is so insensitive to high energy physics, it makes it hard to learn about high energy physics! People have been very clever and have learned a lot in spite of this vexing property of the RG called *decoupling*. We can hope that will continue. (Cosmological inflation plays a similar role in hiding the physics of the early universe. It's like whoever designed this game is *trying* to hide this stuff from us.)

The explicit functional form of $G(s)$ (the inverse of the (euclidean) kinetic operator for Q) is:

$$G(s) = \int \text{đ}\omega \frac{e^{-\text{i}\omega s}}{\omega^2 + \Omega^2} = e^{-\Omega|s|}\frac{1}{2\Omega}. \tag{1.5}$$

Do it by residues: the integrand has poles at $\omega = \pm \mathbf{i}\Omega$. The absolute value of $|s|$ is crucial, and comes from the fact that the contour at infinity converges in the upper (lower) half plane for $s < 0$ ($s > 0$).

[End of Lecture 1]

Some comments:

- Please don't be confused by the formal similarity of the above manipulations with the construction of the generating functional of correlation functions of Q:

$$Z[J] \equiv \left\langle e^{\int dt Q(t) J(t)} \right\rangle_Q \quad , \quad \langle Q(t_1)Q(t_2)... \rangle_Q^{\text{connected}} = \frac{\delta}{\delta J(t_1)} \frac{\delta}{\delta J(t_2)} ... \log Z[J]$$

  It's true that what we did above amounts precisely to constructing $Z[J]$, and plugging in $J = g_0 q^2$. But the motivation is different: in the above q is also a dynamical variable, so we don't get to pick q and differentiate with respect to it; we are merely postponing doing the path integral over q until later.

- Having said that, what is this quantity $G(s)$ above? It is the (euclidean) two-point function of Q:

$$G(s,t) = \langle Q(s)Q(t) \rangle_Q^{\text{connected}} = \frac{\delta}{\delta J(t)} \frac{\delta}{\delta J(s)} \log Z[J]|_{J=0}.$$

  The middle expression makes it clearer that $G(s,t) = G(s-t)$ since nobody has chosen the origin of the time axis in this problem. This euclidean Green's function, the inverse of $-\partial_\tau^2 + \Omega^2$, is unique (unlike the real-time Green's function).

- **Adding more labels.** Quantum mechanics is quantum field theory in 0+1 dimensions. Except for our ability to do all the integrals, everything we are doing here generalizes to quantum field theory in more dimensions: quantum field theory *is* quantum mechanics (with infinitely many degrees of freedom). With more spatial dimensions, we'll want to use the variable x for the spatial coordinates (which are just labels on the fields!) and it was in anticipation of this step that I used q instead of x for my oscillator position variables.

  The Kadanoff-Wilson perspective on the renormalization group is quite similar to what we've done above. It starts with not just two oscillator modes, but many (one for each momentum), and does the integrals in a certain order, starting with the stiffest (fastest, heaviest) modes, to derive an effective action for the remaining (slow, light) modes. The integrals are more complicated, but the idea is the same.

- **Entanglement.** When we study the effective theory of q governed by the effective action in a derivative expansion, we use the ordinary laws of quantum mechanics for a closed, isolated system. That is, we treat it as a pure state. However, in the full system, the two oscillator modes q and Q interact, and in the true groundstate of the full system, the two modes will be *entangled*. This means that neither has its own pure-state wavefunction, rather, the q system by itself only has a distribution of pure state wavefunctions, *i.e.* a density matrix.

  This entanglement between q and Q is visible in the exact path integral description in the nonlocality of S_{eff}[1]. (Maybe later we will talk about path integrals that prepare mixed states.) At the step where we expand under the integral to get a local action, we approximate the theory of the light mode q as closed, *i.e.* its groundstate is a pure state. The fact that cross-sections mediated by higher-derivative interactions grow with energy too quickly to be consistent with unitarity is a sign that nevertheless such a system is not closed.

[1] I learned this from Daniel Harlow.

2 Anomalies

[Zee §IV.7; Polyakov, *Gauge Fields and Strings*, §6.3; K. Fujikawa, *Phys. Rev. Lett.* **42** (1979) 1195; Argyres, 1996 lectures on supersymmetry §14.3; Peskin, chapter 19; highly recommended: David Tong, Gauge Theory, chapter 3]

Topology means the study of quantities that can't vary smoothly, but can only vary by jumping. Good examples are quantities that must be integers. *Anomalies* provide an example of a topological phenomenon in QFT, which is therefore robust against any change in the QFT which can be made continuously (like varying masses or couplings, or the cutoff or the resolution of our description, *i.e.* a renormalization group transformation).

Here is the historical origin of anomalies, which at least motivates the name. Suppose we have in our hands a classical field theory in the continuum that has some symmetry. Is there a well-defined QFT whose classical limit produces this classical field theory and preserves that symmetry? The path integral construction of QFT offers some insight here. The path integral involves two ingredients: (1) an action, which is shared with the classical field theory, and (2) a path integral measure. It is possible that the action is invariant but the measure is not. This is called an anomaly. It means that the symmetry is (explicitly) broken, and its current conservation is violated by a known amount, and this often has many other consequences that can be understood by humans. It means that the theory *cannot* be regulated in a way that preserves the symmetry.

Notice that here I am speaking about actual, global symmetries. I am not talking about gauge redundancies. If you think that two field configurations are equivalent but the path integral tells you that they would give different contributions, you are doing something wrong. Such a 'gauge anomaly' means that the system has more degrees of freedom than you thought. (It does not mean that the world is inconsistent. For a clear discussion of this, please see Preskill, 1990.)

You could say that we have already seen a dramatic example of an anomaly: the violation of classical scale invariance (*e.g.* in massless ϕ^4 theory, or in massless QED) by quantum effects. A regulator necessarily introduces a length scale into the problem and explicitly breaks scale invariance.

Notice that the name 'anomaly' betrays the bias that we imagine constructing a QFT by starting with a continuum action for a classical field theory; you would never imagine that *e.g.* scale invariance was an exact symmetry if you instead started from a well-defined quantum lattice model. Partly for this reason, the concept of 'anomaly' is not native to the condensed matter literature (but has recently been flourishing there).

2.1 Anomaly parable

[N. Seiberg's 2023 TASI lectures (lecture 1)] The chiral anomaly is one (important) example of an anomaly. It involves fermions, and it involves infinitely many degrees of freedom and the regularization of the UV. I want to convey the fact that anomalies can happen even without either of these ingredients. I will explain the simplest possible example. First let me give a better definition of anomaly, that makes no reference to any classical limit (which may not exist).

A physical system can be specified by its Hilbert space $\mathcal{H}$ and its Hamiltonian H. For $\mathcal{H}$, consider a single qubit, that is a two-dimensional Hilbert space. And for H, consider the simplest possible example, $H = 0$. What is the symmetry of this system? An argument can be made that the symmetry group is $\mathsf{SO}(3)$. This is because a basis of hermitian operators on $\mathcal{H}$ is $\{\mathbb{1}, X, Y, Z\}$ (the Pauli matrices), and the most general possible transformation acts by

$$\mathcal{O} \to U\mathcal{O}U^\dagger \tag{2.1}$$

where $U = e^{\mathbf{i}\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}}e^{\mathbf{i}\varphi}$ is an arbitrary unitary on $\mathcal{H}$ ($\hat{n}$ is a unit 3-vector). Clearly this is a symmetry, in the sense that $[U, H] = 0$ (since $H = 0$). U is actually an element of the unitary group $\mathsf{U}(2)$, but in its action on the operators, clearly φ does not matter, and furthermore, the operator $-\mathbb{1} = e^{\mathbf{i}\pi\hat{n}\cdot\vec{\sigma}} \in \mathsf{SU}(2)$ (where $\theta = 2\pi$) does not act on $\mathcal{O}$. The quotient of $\mathsf{SU}(2)$ by this element is the group $\mathsf{SO}(3)$.

Now I claim that the single qubit is an *anomalous* action of $\mathsf{SO}(3)$. There are two ways to think about this. A sort of silly one is that the representations of the group $\mathsf{SO}(3)$ all have integer spin. In constrast, the qubit transform in the spin-half representation, which is a projective representation, meaning that the group law is only satisfied by the representation matrices up to phases. In this case, the group element $e^{\mathbf{i}\pi\hat{n}\cdot\vec{\sigma}}$ (with $\theta = 2\pi$) is realized as $-\mathbb{1}$, whereas in the group $\mathsf{SO}(3)$ it should be the identity operator.

A perspective that generalizes better is the following. Whenever we have a symmetry of a physical system, we automatically have a collection of nice probes of the system. That is, we can couple to background gauge fields for the symmetry. In the simplest case of a $\mathsf{U}(1)$ symmetry where there is a conserved current j_μ, this means adding to the Lagrangian a term of the form $\int d^D x j^\mu A_\mu$ (and possibly some more terms to make the action gauge invariant). I emphasize that $A_\mu(x)$ here is a collection of coupling constants, not a dynamical field; we are *not* gauging the symmetry, the symmetry still acts on the Hilbert space. Then we can think about the partition function

$$Z[A] = \int D\phi e^{-S[\phi,A]} = \mathrm{tr}_{\mathcal{H}} e^{-\beta H[A]} \tag{2.2}$$

as a functional of these background fields. In the example of continuous symmetries,

it is a generating functional for correlation functions of the currents. More generally, I claim that turning on background fields is the same as *inserting symmetry operators* into the path integral. (I'll explain this in the example below.)

With this in mind, here's a good, general definition of anomaly: the symmetry is anomalous if $Z[A]$ is not gauge invariant[2]. In the case of a $\mathsf{U}(1)$ symmetry (which is easy to write), we have

$$Z[A + g^{-1}dg] = e^{\mathrm{i}\mathcal{A}(g)} Z[A], \tag{2.3}$$

and the phase $\mathcal{A}(g)$ is the anomaly. Two simple but very important observations:

- The anomaly $\mathcal{A}$ is an obstruction to gauging the symmetry. Clearly if the partition function is not gauge-invariant, we cannot regard the transformation as an equivalence relation. Sometimes it is useful to think of "obstruction to gauging" as the defining property of anomaly.

- The renormalization group is accomplished by doing the path integral in a certain order and redefining variables in such a way that, by construction, Z is invariant. That is, we have some UV description, which can be very different from the IR description, but what they have in common is that they produce the same Z. This immediately implies that the transformation rule (2.3) is *an invariant of the RG*. The RG is a complicated thing that we generally don't understand well at all. Identifying a quantity that doesn't change under this operation, and must match between UV and IR descriptions is precious.

I glossed over one important thing: sometimes it is possible to add some terms to the Lagrangian to remove the anomalous transformation. That is, the prescription for coupling to background fields is a little ambiguous. Sometimes $\int j_\mu A^\mu$ is not gauge invariant by itself. Consider for example the case of a free scalar field φ, with $j_\mu = \partial_\mu \varphi$ the current for the shift symmetry. Then under the gauge transformation

$$\varphi \to \varphi + \alpha(x), A_\mu \to A_\mu - \partial_\mu \alpha, \tag{2.4}$$

$j_\mu A^\mu \mapsto j_\mu A^\mu - \partial_\mu \alpha A^\mu + j_\mu \partial^\mu \alpha$, which is not invariant. But we can fix this by adding some local terms, *i.e.* instead choosing the Lagrangian

$$(\partial_\mu \varphi + A_\mu)^2 \tag{2.5}$$

which is manifestly gauge invariant. This is not an anomaly. So actually the anomaly is the variation of the phase of the partition function $\mathcal{A}$ modulo the variation of possible local terms in the action.

[2]This is an oversimplification that I will correct in a minute.

In our example above, what are the possible 'background fields'? The most general thing is to insert the general symmetry operator:

$$Z[\theta, \hat{n}] \equiv \mathrm{tr}_{\mathcal{H}} e^{\mathbf{i}\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}}. \tag{2.6}$$

In the path integral description, we have a QFT on a (euclidean time) circle of radius β. The most general background gauge field is of the form $A^a_\mu dx^\mu = A^a_0 d\tau$, where $a = x, y, z$ is an adjoint-of-$\mathsf{SO}(3)$ index, and it only has a time component. The partition function (2.6) can be written as

$$Z[A] = Z[\theta, \hat{n}] = \int D\phi e^{-S_0} e^{\mathbf{i}\oint A} \tag{2.7}$$

that is, turning on background fields is the same as inserting a Wilson line around the thermal circle. Its only gauge-invariant effect is to specify the boundary conditions around the circle: when we go around the circle we can do a $\mathsf{SO}(3)$ rotation of our qubit. So this Wilson line is specified by a choice of $\hat{n}$ and θ. So, using the fact that $(\hat{n}\cdot\vec{\sigma})^2 = \mathbb{1}$,

$$Z[A] \equiv Z(\theta) = \mathrm{tr}_{\mathcal{H}} e^{\mathbf{i}\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}} = 2\cos\theta/2. \tag{2.8}$$

Now here's the crucial question for our example: Is $Z[A]$ gauge invariant? In $\mathsf{SO}(3)$ (but not in $\mathsf{SU}(2)$) there is a gauge redundancy under $\theta \mapsto \theta + 2\pi$. (Let's pick $\hat{n} = \hat{z}$. Take $A \to A + \mathbf{i}g^{-1}dg$ with $g = e^{\mathbf{i}nZ\tau/\beta}$ with n odd.) Under this transformation,

$$Z(\theta + 2\pi) = -Z(\theta)\ , \tag{2.9}$$

the partition function changes sign. This is a failure of gauge invariance.

Actually, if we just think about a single $\mathsf{U}(1) \subset \mathsf{SU}(2)$, it is possible to remove the problem, by redefining $A_\tau = Z\theta/2 \to Z\theta/2 + \mathbb{1}\theta/2$. But then something funny will happen under an operation that takes $\hat{z} \to -\hat{z}$. That is, with the above definition,

$$Z(\theta + 2\pi) = -Z(\theta), Z(-\theta) = Z(\theta). \tag{2.10}$$

With the modified definition, we have instead

$$Z'(\theta + 2\pi) = Z'(\theta), Z'(-\theta) = e^{-\mathbf{i}\theta} Z'(\theta). \tag{2.11}$$

Now this violates invariance under the (non-abelian) gauge transformation $A \to g^{-1}(A + \mathbf{i}d)g$ with a choice of g that anticommutes with Z. In fact, to see a violation of gauge invariance that can't be removed by a local redefinition of the action (an anomaly), it's enough to consider just a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of $\mathsf{SO}(3)$ (*i.e.* , choose $\theta = \pi$). Notice that the qubit is a projective representation of this $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the π rotations about orthogonal axes, *e.g.* X and Z.

2.2 Chiral anomaly

The example we will focus on here is the *chiral anomaly*. This is encapsulated by an equation for the violation of the chiral (aka axial) current for fermions coupled to a background gauge field. The chiral anomaly was first discovered in perturbation theory, by computing a certain Feynman diagram with a triangle; the calculation was motivated by the experimental observation of the process $\pi^0 \to \gamma\gamma$, which would not happen if the chiral current were conserved. (I'll explain the relationship between the chiral current and the pion later.)

I will outline a derivation of this effect (using the fermionic path integral) which is more illuminating than the triangle diagram. It shows that the one-loop result is exact – there are no other corrections. It shows that the quantity on the right hand side of the continuity equation for the would-be current integrates to an integer. It gives a (physics) proof of the *index theorem*, relating numbers of solutions of the Dirac equation in a background field configuration to a certain integral of field strengths. It butters your toast. [End of Lecture 2]

Chiral symmetries. In even-dimensional spacetimes, the Dirac representation of $\mathrm{SO}(D-1,1)$ is reducible. This is because

$$\gamma^5 \equiv a \prod_{\mu=0}^{D-1} \gamma^\mu \neq 1, \quad \text{satisfies} \quad \{\gamma^5, \gamma^\mu\} = 0, \forall \mu \ .$$

(In odd D, $\prod \gamma^\mu$ instead commutes with each of the γ^μ, and is in fact proportional to the identity.) This means that γ^5 commutes with the Lorentz generators

$$[\gamma^5, \Sigma^{\mu\nu}] = 0, \qquad \Sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu].$$

We can choose a so that $(\gamma^5)^2 = 1$ so that $\frac{1}{2}(1 \pm \gamma^5)$ are projectors. A left- or right-handed Weyl spinor is an irreducible representation of $\mathrm{SO}(D-1,1)$, $\psi_{L/R} \equiv \frac{1}{2}(1 \pm \gamma^5)\psi$. This allows the possibility that the L and R spinors can transform differently under a symmetry; such a symmetry is called a chiral symmetry. (Note that when $D = 2$, the Dirac equation says that a left-(right-) handed spinor really only moves to the left (right) – see the homework. In higher dimensions, the name just comes from the fact that L and R are interchanged by the parity operation.)

In $D = 4k$ dimensions, if ψ_L is a left-handed spinor in representation $\mathbf{r}$ of some group G, then its image under CPT, $\psi_L^{CPT}(t,\vec{x}) \equiv \mathbf{i}\gamma^0 (\psi_L(-t,-\vec{x}))^\star$, is right-handed and transforms in representation $\bar{\mathbf{r}}$ of G. Therefore chiral symmetries arise in $D = 4k$ only when the Weyl fermions transform in *complex representations* of the symmetry group, where $\bar{\mathbf{r}} \neq \mathbf{r}$. (In $D = 4k+2$, CPT maps left-handed fields to left-handed fields. For more detail on discrete symmetries and Dirac fields, see Peskin §3.6.)

Some more explicit words (of review) about chiral fermions in $D = 3 + 1$, mostly notation. Recall Peskin's *Weyl* basis of gamma matrices in 3+1 dimensions, in which γ^5 is diagonal:

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\boldsymbol{\sigma}}^\mu \\ \boldsymbol{\sigma}^\mu & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^\mu \equiv (\mathbb{1}, \vec{\boldsymbol{\sigma}})^\mu, \quad \bar{\boldsymbol{\sigma}}^\mu \equiv (\mathbb{1}, -\vec{\boldsymbol{\sigma}})^\mu, \quad \gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

This makes the reducibility of the Dirac representation of $\mathrm{SO}(3,1)$ manifest, since the Lorentz generators are $\propto [\gamma^\mu, \gamma^\nu]$ block diagonal in this basis. The gammas are a map from the $(1, \mathbf{2_R})$ representation to the $(\mathbf{2_L}, \mathbf{1})$ representation. It is sometimes useful to denote the $\mathbf{2_R}$ indices by $\alpha, \beta = 1, 2$ and the $\mathbf{2_L}$ indices by $\dot{\alpha}, \dot{\beta} = 1, 2$. Then we can define two-component Weyl spinors $\psi_{L/R} = P_{L/R}\psi \equiv \frac{1}{2}(1 \pm \gamma^5)\psi$ by simply forgetting about the other two components. The conjugate of a L spinor $\chi = \psi_L$ (L means $\gamma^5\chi = \chi$) is right-handed:

$$\bar{\chi} = \chi^\dagger\gamma^0, \quad \bar{\chi}\gamma^5 = \chi^\dagger\gamma^0\gamma^5 = -\chi^\dagger\gamma^5\gamma^0 = -\chi^\dagger\gamma^0 = -\bar{\chi}.$$

We can represent any system of Dirac fermions in terms of a collection of twice as many Weyl fermions.

Let me be more explicit about the meaning of a complex representation of a continuous symmetry G. The statement that ψ is in representation $\mathbf{r}$ means that its transformation law is

$$\delta\psi_a = \mathbf{i}\epsilon^A \left(t^A_{\mathbf{r}}\right)_{ab} \psi_b$$

where $t^A_{\mathbf{r}}$, $A = 1..\dim G$ are generators of G in representation $\mathbf{r}$; for a compact Lie group G, we may take the t^A to be Hermitian. The conjugate representation, by definition, is one with which you can make a singlet of G – it's the way $\psi^{\star T} = \psi^\dagger$ transforms:

$$\delta\psi^{\star T}_a = -\mathbf{i}\epsilon^A\psi^{\star T}_b \left(t^A_{\mathbf{r}}\right)_{ba} = -\mathbf{i}\epsilon^A \left(t^A_{\mathbf{r}}\right)^T_{ab} \psi^{\star T}_b \ .$$

So:

$$t^A_{\bar{\mathbf{r}}} = -\left(t^A_{\mathbf{r}}\right)^T.$$

The condition for a complex representation is that this is different from $t^A_{\mathbf{r}}$ (actually we have to allow for relabelling of the generators and the basis – two representations $\mathbf{r}_{1,2}$ are equivalent, $\mathbf{r}_1 \cong \mathbf{r}_2$ if there is a change of basis (the same for all A) that relates the generators: $t^A_{\mathbf{r}_1} = U^\dagger t^A_{\mathbf{r}_1} U$. $\mathbf{r}$ is complex if $\mathbf{r} \ncong \mathbf{r}_2$. The simplest case is $G = \mathrm{U}(1)$, where t is just a number indicating the charge. In that case, any nonzero charge gives a complex representation.

We're going to think about the case where the set of left-moving and right-moving fields is the same, but we'll think about a symmetry that acts in a chiral way. (A

similar analysis can be done more generally.) Consider the effective action produced by integrating out Dirac fermions coupled to a *background* gauge field (the gauge field is just going to sit there for this whole calculation):

$$e^{\mathbf{i}S_{\text{eff}}[A]} \equiv \int [D\psi D\bar{\psi}]\; e^{\mathbf{i}S[\psi,\bar{\psi},A]} \ .$$

We must specify how the fermions are coupled to the gauge field. The simplest example is if A is a $U(1)$ gauge field and ψ is minimally coupled:

$$S[\psi,\bar{\psi},A] = \int \mathrm{d}^D x \bar{\psi}\mathbf{i}\not{D}\psi, \qquad \not{D}\psi \equiv \gamma^\mu \left(\partial_\mu + \mathbf{i}A_\mu\right)\psi.$$

We will focus on this abelian example, but you could imagine instead that A_μ is a non-Abelian gauge field for the group G, and ψ is in a representation $\mathbf{r}$, with gauge generators $t^A_{\mathbf{r}}$ ($A = 1...\dim G$), so the coupling would be

$$\bar{\psi}\not{D}\psi = \bar{\psi}_a\gamma^\mu\left(\partial_\mu\delta_{ab} + \mathbf{i}A^A_\mu\left(t^A_{\mathbf{r}}\right)_{ab}\right)\psi_b \ . \tag{2.12}$$

Much of the discussion below applies for any even D.

Notice that we are turning on a background gauge field for the vector-like (*i.e.* non-chiral) symmetry that acts by $\psi \to e^{\mathbf{i}\theta^A t^A_{\mathbf{r}}}\psi$ (with no γ^5).

In the absence of a mass term, the action (in the Weyl basis) involves no coupling between L and R:

$$S[\psi,\bar{\psi},A] = \int \mathrm{d}^D x \left(\psi_L^\dagger \mathbf{i}\sigma^\mu D_\mu\psi_L + \psi_R^\dagger \mathbf{i}\bar{\sigma}^\mu D_\mu\psi_R\right)$$

and therefore is invariant under the global chiral rotation

$$\psi \to e^{\mathbf{i}\alpha\gamma^5}\psi, \quad \psi^\dagger \to \psi^\dagger e^{-\mathbf{i}\alpha\gamma^5}, \quad \bar{\psi} \to \bar{\psi}e^{+\mathbf{i}\alpha\gamma^5}. \quad \text{That is:} \ \ \psi_L \to e^{\mathbf{i}\alpha}\psi_L, \quad \psi_R \to e^{-\mathbf{i}\alpha}\psi_R.$$

(The mass term couples the two components

$$L_m = \bar{\psi}\left(\mathrm{Re}m + \mathrm{Im}m\gamma^5\right)\psi = m\psi_L^\dagger\psi_R + h.c.;$$

notice that the mass parameter is complex.) The associated Noether current is $j^5_\mu = \bar{\psi}\gamma^5\gamma_\mu\psi$, and it seems like we should have $\partial^\mu j^5_\mu \stackrel{?}{=} 0$ if $m = 0$. This follows from the massless (classical) Dirac equation $0 = \gamma^\mu\partial_\mu\psi$. (With the mass term, we would have instead $\partial^\mu j^5_\mu \stackrel{?}{=} 2\mathbf{i}\bar{\psi}\left(\mathrm{Re}m\gamma^5 + \mathrm{Im}m\right)\psi$.)

Notice that there is another current $j^\mu = \bar\psi\gamma^\mu\psi$. j^μ is the current that is coupled to the gauge field, $L \ni A_\mu j^\mu$. The conservation of this current is required for gauge invariance of the effective action

$$S_{\text{eff}}[A_\mu] \stackrel{!}{=} S_{\text{eff}}[A_\mu + \partial_\mu\lambda] = -\mathbf{i}\log\left\langle e^{\mathbf{i}\int\lambda(x)\partial_\mu j^\mu}\right\rangle + S_{\text{eff}}[A_\mu].$$

No matter what happens we can't find an anomaly in j^μ, since this is the symmetry that's gauged in QED. The anomalous one is the other one, the *axial current*.

To derive the conservation law for the axial current we can use the Noether method. This amounts to substituting $\psi'(x) \equiv e^{\mathbf{i}\alpha(x)\gamma^5}\psi(x)$ into the action:

$$S_F[\psi'] = \int \mathrm{d}^D x\bar\psi e^{+\mathbf{i}\alpha\gamma^5}\mathbf{i}\not{D}e^{\mathbf{i}\alpha\gamma^5}\psi = \int \mathrm{d}^D x\left(\bar\psi\mathbf{i}\not{D}\psi + \bar\psi\mathbf{i}\gamma^5\left(\not{\partial}\alpha\right)\psi\right) \stackrel{\text{IBP}}{=} S_F[\psi] - \mathbf{i}\int\alpha(x)\partial^\mu\mathrm{tr}\bar\psi\gamma^5\gamma_\mu\psi$$

up to terms of $\mathcal{O}(\alpha^2)$. Then we can completely get rid of $\alpha(x)$ if the change of integration variables in the path integral, *i.e.* if $[D\psi'] \stackrel{?}{=} [D\psi]$. Usually this is true, but here we pick up an interesting Jacobian.

Claim:

$$e^{\mathbf{i}S_{\text{eff}}[A]} = \int[D\psi' D\bar\psi']e^{\mathbf{i}S_F[\psi']} = \int[D\psi D\bar\psi]e^{\mathbf{i}S_F[\psi]+\mathbf{i}\int\mathrm{d}^D x\alpha(x)\left(\partial_\mu j_5^\mu - \mathcal{A}(x)\right)}$$

where $\mathcal{A}$ comes from the variation of the measure. That is,

$$[D\psi' D\bar\psi'] = [D\psi D\bar\psi]\det{}^{-2}\left(e^{\mathbf{i}\alpha\gamma^5}\right) \equiv [D\psi D\bar\psi]e^{-\mathbf{i}\int\alpha\mathcal{A}}$$

so

$$e^{-\mathbf{i}\int\alpha\mathcal{A}} = e^{-2\mathrm{Tr}\log e^{\mathbf{i}\alpha\gamma^5}} = e^{-2\mathrm{Tr}\left(\mathbf{i}\alpha\gamma^5\right)}$$

or more explicitly but very formally we can write the anomaly as:

$$\mathcal{A}(x) = 2\sum_n \mathrm{tr}\bar\xi_n(x)\gamma^5\xi_n(x) = 2\mathrm{tr}\left\langle x\right|\gamma^5\left|x\right\rangle \tag{2.13}$$

where ξ_n are a basis of eigenspinors of the Dirac operator. This big Tr is the trace over the space of functions on which $\not{D}$ acts, including both spinor indices and function labels. I'll use tr or tr_γ for the spinor trace. So we've derived the equation $\int d^D x\mathcal{A}(x) = 2\mathrm{Tr}\ \gamma^5$. This is a very formal expression since it is a sum over infinitely many terms each of which is ±1. Note that the trace over the spinor space is $\mathrm{tr}\gamma^5 = 0$, but here we are summing over the whole single-particle Hilbert space.

The expression above for S_{eff} is actually independent of α, since the path integral is invariant under a change of variables. For a conserved current, α would multiply the

divergence of the current and this demand would imply current conservation. Here this implies that instead of current conservation we have a specific violation of the current:

$$\partial^\mu j^5_\mu = \mathcal{A}(x).$$

What is the anomaly $\mathcal{A}$? [Polyakov §6.3, Peskin §19.2] An alternative useful (perhaps more efficient) perspective is that the anomaly arises from trying to define the axial current operator, which after all is a composite operator. Thus we should try to compute

$$\partial_\mu \langle j^\mu_5 \rangle = \partial_\mu \left\langle \bar\psi(x)\gamma^\mu\gamma^5\psi(x)\right\rangle$$

– the coincident operators on the RHS need to be regulated.

The classical (massless) Dirac equation immediately implies that the axial current is conserved (up to contact terms, meaning collisions with other operators in the expectation value)

$$\partial_\mu \left(\bar\psi\gamma^\mu\gamma^5\psi\right) \stackrel{?}{=} 0.$$

Consider, on the other hand, the (vacuum) expectation value

$$\begin{aligned} J^5_\mu &\equiv \left\langle \bar\psi(x)\gamma_\mu\gamma^5\psi(x)\right\rangle \equiv Z^{-1}[A]\int [D\psi D\bar\psi] e^{\mathbf{i}S_F[\psi]} j^5_\mu(x) \\ &= \quad = \\ &= -\mathrm{tr}_\gamma \gamma_\mu\gamma^5 G^{[A]}(x,x) \end{aligned} \tag{2.14}$$

where the blob represents $G^{[A]}$, the Green's function of the Dirac operator in the gauge field background (and the figure is from Polyakov's book (which works in Euclidean signature)). The $\mathbf{x}$ is the insertion of the current $j^5_\mu = \bar\psi\gamma^5\gamma_\mu\psi$. The minus sign in the last line is from the fermion loop.

We can construct it out of eigenfunctions of $\mathbf{i}\not{D}$:

$$\mathbf{i}\not{D}\xi_n(x) = \epsilon_n\xi_n(x), \qquad \bar\xi_n(x)\mathbf{i}\gamma^\mu\left(-\overleftarrow{\partial}_\mu + \mathbf{i}A_\mu\right) = \epsilon_n\bar\xi_n \tag{2.15}$$

in terms of which[3]

$$G^{[A]}(x,x') = \mathbf{i}\sum_n \frac{1}{\epsilon_n}\xi_n(x)\bar\xi_n(x'). \tag{2.17}$$

[3]Actually, this step is full of danger, but I promise it works out. See §2.3 below for the full story.

Also, the factor of $\mathbf{i}$ in front of this expression for the real-time Green's function is a bit confusing, and would be absent in Euclidean spacetime (*e.g.* in Polyakov's treatment). Recall that

$$\langle\theta_i\bar\theta_j\rangle = Z^{-1}\int d\theta d\bar\theta\theta_i\bar\theta_j e^{-\bar\theta A\theta} = A^{-1}_{ij}\ , Z \equiv \int d\theta d\bar\theta e^{-\bar\theta A\theta}. \tag{2.16}$$

(To check the sign, do the case with just one θ and one $\bar\theta$.) Relative to this expression, the real-time action is missing the minus sign and has an extra $\mathbf{i}$.

(I am suppressing spinor indices all over the place, note that here we are taking the outer product of the spinors to make a matrix.)

We want to define the coincidence limit, as $x' \to x$. The problem with this limit arises from the large $|\epsilon_n|$ eigenvalues; the contributions of such short-wavelength modes are local and most of them can be absorbed in renormalization of couplings. It should not (and does not) matter how we regulate them, but we must pick a regulator. A convenient choice here is heat-kernel regulator:

$$G_s^{[A]}(x,x') \equiv \mathbf{i}\sum_n e^{-s\epsilon_n^2}\frac{1}{\epsilon_n}\xi_n(x)\bar{\xi}_n(x')$$

and

$$J_\mu^5(x) = -\mathbf{i}\sum_n e^{-s\epsilon_n^2}\frac{1}{\epsilon_n}\bar{\xi}_n(x)\gamma_\mu\gamma^5\xi_n(x)\ .$$

The anomaly is

$$\partial^\mu J_\mu^5 = \partial^\mu\left\langle j_\mu^5\right\rangle = \sum_n \mathbf{i}\partial^\mu\left(\bar{\xi}_n\gamma_\mu\gamma^5\xi_n\right)\frac{e^{-s\epsilon_n^2}}{\epsilon_n}.$$

The definition (2.15) says

$$\mathbf{i}\partial^\mu\left(\bar{\xi}_n\gamma_\mu\gamma^5\xi_n\right) = -2\epsilon_n\bar{\xi}_n\gamma^5\xi_n$$

using $\{\gamma^5,\gamma^\mu\}=0$. (Notice that the story would deviate dramatically here if we were studying the vector current which lacks the γ^5, and would give $\mathbf{i}\partial^\mu\left(\bar{\xi}_n\gamma_\mu\xi_n\right)=0$.) This gives

$$\partial^\mu J_\mu^5(x) = 2\mathrm{tr}_\gamma\left\langle x\right|\gamma^5 e^{-s\left(\mathbf{i}\not{D}\right)^2}\left|x\right\rangle \tag{2.18}$$

with

$$\left(\mathbf{i}\not{D}\right)^2 = -\left(\gamma_\mu\left(\partial_\mu+\mathbf{i}A_\mu\right)\right)^2 = -\left(\partial_\mu+\mathbf{i}A_\mu\right)^2 - \frac{\mathbf{i}}{2}\Sigma_{\mu\nu}F^{\mu\nu}$$

where $\Sigma_{\mu\nu}\equiv\frac{1}{2}[\gamma_\mu,\gamma_\nu]$ is the spin Lorentz generator. We used $\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}+\frac{1}{2}[\gamma^\mu,\gamma^\nu] = \eta^{\mu\nu}+\Sigma_{\mu\nu}$. (2.18) is the equation we got from the variation of the measure, (2.13), but now better defined by the heat kernel regulator.

We've shown that in any even dimension,

$$\partial^\mu\left\langle j_\mu^5(x)\right\rangle = 2\mathrm{tr}_\gamma\left\langle x\right|\gamma^5 e^{s\not{D}^2}\left|x\right\rangle \tag{2.19}$$

This can now be expanded in small s, which amounts to an expansion in powers of A, F. If there is no background field, $A=0$, we get

$$\left\langle x\right| e^{-s\left(\mathbf{i}\not{\partial}\right)^2}\left|x\right\rangle \overset{\text{Wick}}{=} \mathbf{i}\int \mathrm{d}\!\!\!^{\,D}p_E\ e^{-sp_E^2} = \mathbf{i}\underbrace{K_D}_{=\frac{\Omega_{D-1}}{(2\pi)^D}}\frac{1}{s^{D/2}} \overset{D=4}{=} \frac{\mathbf{i}}{16\pi^2 s^2}. \tag{2.20}$$

This term will renormalize the charge density

$$\rho(x) = \left\langle \psi^\dagger \psi(x) \right\rangle = \mathrm{tr}\gamma^0 G(x,x),$$

for which we must add a counterterm (in fact, it is accounted for by the counterterm for the gauge field kinetic term, *i.e.* the running of the gauge coupling). But it will not affect the axial current violation, which is proportional to

$$\mathrm{tr}\left(\gamma^5 G(x,x)\right)|_{A=0} \propto \mathrm{tr}\gamma^5 = 0.$$

Similarly, bringing down more powers of $(\partial + A)^2$ doesn't give something nonzero since the γ^5 remains.

In $D = 4$, the first term from expanding $\Sigma_{\mu\nu} F^{\mu\nu}$ is still zero from the spinor trace. (Not so in $D = 2$.) The first nonzero term comes from the next term:

$$\mathrm{tr}\left(\gamma_5 e^{-s\left(\mathbf{i}\not{D}\right)^2}\right)_{xx} = \underbrace{\left\langle x|e^{-s(\mathbf{i}D)^2}|x\right\rangle}_{\overset{(2.20)}{=}\frac{\mathbf{i}}{16\pi^2 s^2}+\mathcal{O}(s^{-1})} \cdot\frac{s^2}{8}\cdot(\mathbf{i}^2)\underbrace{\mathrm{tr}\left(\gamma^5\Sigma^{\mu\nu}\Sigma^{\rho\lambda}\right)}_{=4\mathbf{i}\epsilon^{\mu\nu\rho\lambda}}\cdot\underbrace{\mathrm{tr}_c}_{\text{color}}(F_{\mu\nu}F_{\rho\lambda}) + \mathcal{O}(s^1)\ .$$

In the abelian case, just ignore the trace over color indices, tr_c. The terms that go like positive powers of s go away in the continuum limit. Therefore

$$\partial_\mu J_5^\mu = -2\cdot\frac{1}{16\pi^2 s^2}\cdot\frac{s^2}{8}\cdot 4\epsilon^{\mu\nu\rho\lambda}\mathrm{tr}_c F_{\mu\nu}F_{\rho\lambda} + \mathcal{O}(s^1) = -\frac{1}{8\pi^2}\mathrm{tr}F_{\mu\nu}\left(\star F\right)^{\mu\nu}. \qquad (2.21)$$

(Here $(\star F)^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}$.) This is the chiral anomaly formula. It can also be usefully written as:

$$\partial_\mu J_5^\mu = -\frac{1}{8\pi^2}\mathrm{tr}F\wedge F = -\frac{1}{32\pi^2}\vec{E}\cdot\vec{B}.$$

- This object on the RHS is a total derivative. In the abelian case it is

  $$F\wedge F = \mathrm{d}\left(A\wedge F\right)\ .$$

  Its integral over spacetime is a topological invariant (in fact $16\pi^2$ times an integer) characterizing the gauge field configuration. How do I know it is an integer? The anomaly formula! The change in the number of left-handed fermions minus the number of right-handed fermions during some time interval is:

  $$\Delta Q_A \equiv \Delta\left(N_L - N_R\right) = \int \mathrm{d}t\partial_t J_0^5 = \int_{M_4}\partial^\mu J_\mu^5 = \int_{M_4}\frac{F\wedge F}{8\pi^2}$$

  where M_4 is the spacetime region under consideration. If nothing is going on at the boundaries of this spacetime region (*i.e.* the fields go to the vacuum, or there

is no boundary, so that no fermions are entering or leaving), we can conclude that the RHS is an integer.

More generally, $\frac{\mathrm{tr}F\wedge F}{8\pi^2}$ integrates to an integer on any closed 4-manifold on which we can put fermions (*i.e.* which admits a *spin structure*).

- Look back at the diagrams in (2.14). Which term in that expansion gave the nonzero contribution to the axial current violation? In $D = 4$ it is the diagram with three current insertions, the ABJ triangle diagram. So in fact we did end up computing the triangle diagram. But this calculation also shows that nothing else contributes, even non-perturbatively.

- We chose a particular regulator above. The answer we got did not depend on the cutoff; in fact, whatever regulator we used (as long as it preserves the chiral symmetry!) we would get this answer. I am not proving this, but it must be true if the theory makes any sense. We will see strong evidence for it below.

- Consider what happens if we redo this calculation in other dimensions. We only consider even dimensions because in odd dimensions there is no analog of γ^5 – the Dirac spinor representation is irreducible, and there is no notion of chirality. In $2n$ dimensions, we need n powers of $\Sigma^{\mu\nu}F_{\mu\nu}$ to soak up the indices on the epsilon tensor.

  Actually there is an analogous phenomenon in odd dimensions (sometimes called parity anomaly) of an effect that is independent of the masses of the fields, where the spinor trace produces an $\epsilon_{\mu\nu\rho}$, which you already studied on the homework last quarter. Instead of F^n, the thing that appears is the Chern-Simons term.

[End of Lecture 3]

- If we had kept the non-abelian structure in (2.12) through the whole calculation, the only difference is that the trace in (2.21) would have included a trace over representations of the gauge group. With multiple fermion flavors ($I = 1..N_f$), we could have considered also a non-abelian flavor transformation in the chiral symmetry

$$\psi_I \to \left(e^{\mathbf{i}\gamma^5 g^a \tau^a}\right)_{IJ} \psi_J$$

  for some $\mathsf{su}(N_f)$ flavor rotation generator τ^a. This is a symmetry of $\bar{\psi}_I \mathbf{i}\not{D}\psi_I$ (with no mass terms), and the current is $j^{\mu 5a} = \bar{\psi}\gamma^\mu\gamma^5\tau^a\psi$. Then we would have found (recall that $F = F^A T^A$):

$$\partial^\mu j^{5a}_\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\lambda}F^A_{\mu\nu}F^B_{\rho\lambda}\mathrm{tr}_{c,f}\left(T^A T^B \tau^a\right). \tag{2.22}$$

Note that in this expression, the matrix in the trace is more explicitly

$$(T^A \otimes \mathbb{1}_f)(T^B \otimes \mathbb{1}_f)(\mathbb{1}_c \otimes \tau^a) = T^A T^B \otimes \tau^a \ . \tag{2.23}$$

A special case of this is if we have multiple species of fermion fields but consider the diagonal chiral symmetry ($\tau^a = \mathbb{1}$): their contributions to the anomaly add. Sometimes they can cancel; the Electroweak gauge interactions are an example of this.

- Most generally, consider a collection of fermions transforming under symmetry group $\mathsf{G}_1 \times \mathsf{G}_2 \times \mathsf{G}_3$ and couple to background gauge fields $A^{1,2,3}$ for all three groups. We'll call a "$\mathsf{G}_1\mathsf{G}_2\mathsf{G}_3$ anomaly" the diagram with insertions of currents for $\mathsf{G}_1, \mathsf{G}_2$ and G_3.

  Above we computed the contribution from whole Dirac fermions. We can compute separately the contributions of the L and R Weyl components: there is a factor of half and a relative sign. The result for the anomaly of the current for G^1 coming from the background gauge fields for $\mathsf{G}^{2,3}$ is

$$\partial_\mu j_1^{A\mu} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^{2B} F_{\rho\sigma}^{3C} \sum_f (-1)^f \mathrm{tr}_{R(f)}\{T_1^A, T_2^B\} T_3^C . \tag{2.24}$$

  The sum is over each Weyl fermion, $R(f)$ is its representation under the combined group $\mathsf{G}_1 \times \mathsf{G}_2 \times \mathsf{G}_3$, and T_1^A are a basis of generators of the Lie algebra of G_1 etc. in the representation of the field f. By $(-1)^f$ I mean $\pm$ for left- and right-handed fermions respectively. Here $\{A, B\} = AB + BA$ means anticommutator; the second term comes from reversing the arrows of the fermion lines in the triangle diagram. Using this formula you can check that the Standard Model gauge group is anomaly-free.

2.3 Zeromodes of the Dirac operator

Do you see why I said that the step involving the fermion Green's function was full of danger? The danger arises because the Dirac operator (whose inverse is the Green's function) can have *zeromodes*, eigenspinors with eigenvalue $\epsilon_n = 0$. In that case, $\mathbf{i}\not{D}$ is *not* invertible, and the expression (2.17) for G is ambiguous. This factor of ϵ_n is about to be cancelled when we compute the divergence of the current and arrive at (2.13). Usually this kind of thing is not a problem because we can lift the zeromodes a little and put them back at the end. But here it is actually hiding something important. The zeromodes cannot just be lifted. This is true because nonzero modes of $\mathbf{i}\not{D}$ must come in left-right pairs: this is because $\{\gamma^5, \mathbf{i}\not{D}\} = 0$, so $\mathbf{i}\not{D}$ and γ^5 cannot be simultaneously

diagonalized in general. That is: if $\mathbf{i}\not{D}\xi = \epsilon\xi$ then $(\gamma^5\xi)$ is also an eigenvector of $\mathbf{i}\not{D}$, with eigenvalue $-\epsilon$. By taking linear combinations

$$\chi_n^{L/R} = \frac{1}{2}\left(1 \pm \gamma^5\right)\xi_n$$

these two partners can be arranged into a pair of simultaneous eigenvectors of $(\mathbf{i}\not{D})^2$ (with eigenvalue ϵ_n^2) and of γ^5 with $\gamma^5 = \pm$ respectively.

Only for $\epsilon = 0$ does this fail, so zeromodes can come by themselves. So you can't just smoothly change the eigenvalue of some ξ_0 from zero unless it has a partner with whom to pair up.

This leads us to a deep fact, called the (Atiyah-Singer) *index theorem*: *only* zeromodes can contribute to the anomaly. Any mode ξ_n with nonzero eigenvalue has a partner (with the same eigenvalue of $(\mathbf{i}\not{D})^2$) with the opposite sign of γ^5; hence they cancel exactly in

$$\mathrm{tr}\gamma^5 e^{-s(\mathbf{i}\not{D})^2} = \sum_{n,\epsilon_n\neq 0} \underbrace{\sum_{L/R} \bar{\chi}_n^{L/R}\gamma^5\chi_n^{L/R} e^{-s\epsilon_n^2}}_{=0} + \text{ zeromodes } .$$

We can choose our eigenfunctions to be normalized $\int d^D x \bar{\chi}_i \chi_j = \delta_{ij}$ and of definite chirality $\gamma^5\chi = \pm\chi$. So the anomaly equation tells us that the number of zeromodes of the Dirac operator $\mathbf{i}\not{D}^{[A]}$ in some configuration of the background field, weighted by handedness (*i.e.* with a + for L and - for R) is equal to

$$n_L - n_R = \int d^D x \mathcal{A}(x) = -\int \frac{1}{8\pi^2} F \wedge F.$$

A practical consequence for us is that it makes manifest that the result is independent of the regulator s.

Another consequence is that in the background of a gauge field configuration with nonzero $n_L - n_R = q$, the vacuum to vacuum amplitude

$$Z[A^q] = \int [D\psi D\bar{\psi}] e^{\mathbf{i}\int d^D x \bar{\psi}\mathbf{i}\not{D}\psi} = \det \mathbf{i}\not{D} = 0$$

vanishes, since it is the determinant of an operator with a kernel. Rather, only amplitudes for transitions that change the chiral charge by q are allowed. A (localized) gauge field configuration with $\int F \wedge F \neq 0$ is called an *instanton*.

2.4 The physics of the anomaly

Emergence of the Dirac equation. Consider free fermions hopping on a chain of

sites:

$$H = -t \sum_n c_n^\dagger c_{n+1} + h.c. \tag{2.25}$$

(The particular choice of nearest-neighbor hopping is not special for what I'm about to say.) Since this is translation-invariant, the single-particle Hamiltonian is diagonalized by Fourier modes: $c_n = \oint đk e^{\mathbf{i}kna} c_k$ (where a is the lattice spacing):

$$H = \oint đk c_k^\dagger c_k \epsilon(k) \tag{2.26}$$

and for the particular choice of nearest-neighbor hopping in one dimension we get

$$\epsilon(k) = -2t(\cos ka - 1) \ . \tag{2.27}$$

It is sometimes convenient to add a chemical potential term to account for the number of electrons:

$$H - \mu N = \oint đk c_k^\dagger c_k \left(\epsilon(k) - \mu\right) \ . \tag{2.28}$$

Introduce an infrared regulator so that the levels are discrete – put them in a box of length L, so that $k_\ell = \frac{2\pi\ell}{L}$, $\ell \in \mathbb{Z}$. The groundstate of N such fermions (N determines the chemical potential) is described by filling the N lowest-energy single particle levels, up to the Fermi momentum: $|k| \leq k_F$ are filled. The energy of the last filled level is $\epsilon(k_F) = \mu$, the Fermi energy. (In Figure 1, the red circles are possible 1-particle states, and the green ones are the occupied ones.)

Starting in the groundstate of N electrons, the lowest-energy state available in which to add a fermion is the one just above the Fermi level. Adding an electron in this level costs energy

$$\epsilon(k) - \mu = \epsilon(k_F) + (k - k_F)\partial_k \epsilon|_{k=k_F} + \mathcal{O}(k - k_F)^2 - \mu = v_F \delta k + \mathcal{O}(\delta k)^2. \tag{2.29}$$

The fields near these Fermi points $k = \pm k_F$ in k-space therefore satisfy the Dirac equation

$$(\omega - v_F \delta k)\, \psi_L = 0, \quad (\omega + v_F \delta k)\, \psi_R = 0 \tag{2.30}$$

where $\delta k \equiv k - k_F$ and $v_F \equiv |\partial_k \epsilon|_{k=k_F}$. I call this the Dirac equation because it is the equations of motion for the action

$$S[\psi, \bar\psi] = \int d^2x \bar\psi \mathbf{i} \not\partial \psi \tag{2.31}$$

where γ^μ are 2×2 and the upper/lower component of ψ creates fermions near the left/right Fermi point: $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ The basis of gammas that gives (2.30) is $\gamma^0 =$

$\sigma^1, \gamma^1 = \mathbf{i}\sigma^2$. I chose units of length where $v_F = 1$ (rather than the actual speed of light).

Thus the Dirac equation emerges from a very generic and simple lattice model. The left-(right-)handed fermion is left-(right-)moving, very convenient. The Dirac antiparticle is a *hole*: the lowest energy state with one fewer particle is obtained by removing an electron just below the Fermi level. The relativistic approximation breaks down when the $\mathcal{O}(k - k_F)^2$ terms are appreciable, *i.e.* if we put in enough energy to see the curvature of the band.

The chiral anomaly in the lattice model in finite volume. [Polyakov, page 102; Kaplan 0912.2560 §2.1; Alvarez-Gaumé]

This action is preserved by a chiral transformation and would therefore seem to imply a conserved axial current, whose conserved charge is the number of left moving fermions minus the number of right moving fermions. But the fields ψ_L and ψ_R are not independent; with high-enough energy excitations, you reach the bottom of the band (near $k = 0$ here) and you can't tell the difference. This means that the numbers are *not* separately conserved.

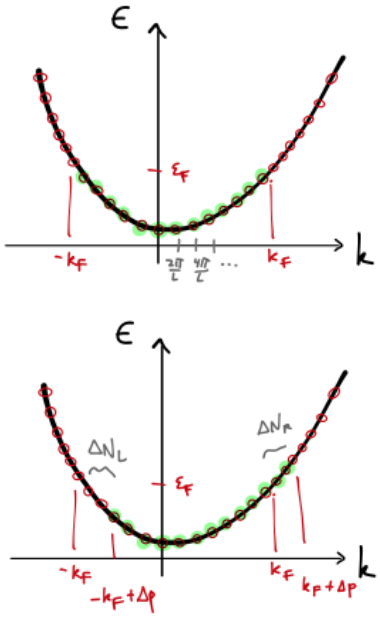


Figure 1: Green dots represent occupied 1-particle states. Top: In the groundstate. Bottom: After applying $E_x(t)$. Here I am just drawing the bottom of the band, where $\epsilon(k)$ can be approximated by $\frac{k^2}{2m}$.

We can do better in this 1+1d example and show that the amount by which the axial current is violated is given by the anomaly formula. Consider subjecting our poor 1+1d free fermions to an electric field $E_x(t)$ which is constant in space and slowly varies in time[4]. Suppose we gradually turn it on and then turn it off; here gradually means slowly enough that the process is adiabatic. Then each particle experiences a force $\partial_t p = eE_x$ and its net change in momentum is

$$\Delta p = e \int \mathrm{d}t E_x(t).$$

This means that the electric field puts the fermions in a state where the Fermi surface

[4]To do this in the lattice model, modify the Hamiltonian by

$$H = -t \sum_n c_n^\dagger e^{\mathbf{i}eA_x(t)} c_{n+1} + h.c. \tag{2.32}$$

$k = k_F$ has shifted to the right by Δp, as in the figure. Notice that the total number of fermions is of course the same – charge is conserved.

Now consider the point of view of the low-energy theory at the Fermi points. This theory has the action

$$S[\psi] = \int \mathrm{d}x\mathrm{d}t\bar{\psi}\mathbf{i}\gamma^\mu D_\mu\psi \ .$$

In the process above, we have added $N_R = \frac{\Delta p}{2\pi/L}$ right-moving particles and taken away $|N_L|$ left-moving particles (with $N_L = -N_R$), that is *added* N_L left-moving holes (aka anti-particles). The axial charge of the state has changed by

$$\Delta Q_A = \Delta(N_R - N_L) = 2\frac{\Delta p}{2\pi/L} = \frac{L}{\pi}\Delta p = \frac{L}{\pi}e\int \mathrm{d}tE_x(t) = \frac{e}{\pi}\int \mathrm{d}t\mathrm{d}xE_x = \frac{e}{2\pi}\int \epsilon_{\mu\nu}F^{\mu\nu}$$

On the other hand, the LHS is $\Delta Q_A = \int \partial^\mu J^A_\mu$. We can infer a local version of this equation by letting E vary slowly in space as well, and we conclude that

$$\partial_\mu J^\mu_A = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}.$$

This agrees exactly with the anomaly equation in $D = 1{+}1$ produced by the calculation above in (2.19) (see the homework).

2.5 't Hooft anomaly matching

The most important fact about anomalies is that they are RG invariants. The existence of the anomaly means that the partition function varies by some particular phase under the anomalous symmetry,

$$Z \to e^{\mathbf{i}\int \alpha\mathcal{A}}Z \ . \tag{2.33}$$

But an RG transformation (doing the integrals in a certain order and relabelling rulers) must preserve the partition function.

Why is this a big deal? Much of physics is about trying to match microscopic (UV) and long-wavelength (IR) descriptions. That is, we are often faced with questions of the form "what could be a microscopic Hamiltonian that produces these phenomena?" and "what does this microscopic Hamiltonian do at long wavelengths?". Anomalies are precious to us, because they are RG-invariant information: any anomaly in the UV description must be realized somehow in the IR description.

This tool has been used to great effect in the last few decades to study strongly-coupled and otherwise intractable theories. The first application (by 't Hooft) was to constrain the possibility that quarks themselves can be composite. He looked for gauge theories where there are gauge-invariant particles (like baryons) with the quantum

numbers of the quarks, that might be massless. Anomaly matching provides crucial evidence for the correctness of Seiberg duality.

It can also be used to help decide whether a gauge theory must spontaneously break chiral symmetry. The idea is: there may be no way for massless fermionic degrees of freedom in a candidate chiral-symmetry-preserving low-energy theory to saturate the anomaly, but there is another possibility. There is a way for bosons to contribute to the anomaly: if they transform non-linearly under the symmetry, *i.e.* if the symmetry is spontaneously broken, they can appear in Wess-Zumino-Witten terms, which can produce the required anomalous variation of the action. More on this in the section about pions.

Further comments:

- Another useful perspective on anomaly is as an obstruction to gauging the symmetry. Gauging a symmetry means creating a new system where the symmetry is a redundancy of the description, by coupling to gauge fields. If the symmetry is not conserved in the presence of background gauge fields, the resulting theory would be inconsistent.

- Above I've described an example of an anomaly of a continuous symmetry. Discrete symmetries can also be anomalous. One way to arrive at this is to start with a continuous symmetry with an anomaly and explicitly break it to a discrete subgroup.

- Anomaly is actually a more basic notion even than phase of matter: The anomaly is a property of the degrees of freedom (*i.e.* of the Hilbert space) and how the symmetry acts on them, independent of a choice of Hamiltonian or action. Multiple phases of matter can carry the same anomaly.

- There is a long story about anomalies and Symmetry Protected Topological (SPT) phases of matter. If you want to read about this, §3 here might be a place to start.

2.6 Some other anomalies

There are some other examples of anomalies whose existence is worth mentioning.

One is that there can be anomalies where the role of the field strength F is played by the curvature of spacetime R. The chiral anomaly gets such a contribution. Since gravity exists (space can be curved), this constrains the hypercharge assignments in the Standard Model.

Spinors in curved spacetime. To couple integer-spin fields to curved space is not such a big deal: just replace every $\eta_{\mu\nu}$ by $g_{\mu\nu}$, use covariant derivatives, and use the covariant volume form. For example:

$$\int d^D x \left(\partial_\mu \phi \partial_\nu \phi \eta^{\mu\nu} - V(\phi)\right) \rightsquigarrow \int d^D x \sqrt{g} \left(\partial_\mu \phi \partial_\nu \phi g^{\mu\nu}(x) - V(\phi)\right). \tag{2.34}$$

Coupling spinors to curved space is a little more involved, and requires the introduction of the spin connection. This is a useful device in other contexts, for example, for computing curvatures by hand (see *e.g.* §9 of these notes).

The first step is to introduce the vierbeins ('vier' means ''four' in German; in general dimension, they are called vielbeins instead, since 'viel' means 'many')

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab} \ . \tag{2.35}$$

Here $a, b = 0...D-1$ are tangent space indices, which are contracted with the ordinary Minkowski metric. The es are an orthonormal frame on the tangent space that varies from point to point. This description in terms of e introduces a redundancy under local Lorentz transformations $\mathsf{SO}(3,1)$ (that preserve $g_{\mu\nu}$). The required connection is the spin connection ω^{ab}_μ. It can be defined by demanding that the vielbeins are covariantly constant:

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega^a_{\mu b} e^b_\nu \stackrel{!}{=} 0 \tag{2.36}$$

where Γ is the usual Christoffel symbol. The field strength of the spin connection is

$$(R_{\mu\nu})^a{}_b = \partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} - [\omega_\mu, \omega_\nu]^a{}_b = R_{\mu\nu}{}^\rho{}_\sigma e^a_\rho e^\sigma_b \tag{2.37}$$

where the last object is the ordinary Riemann curvature.

In terms of these ingredients, the spinor covariant derivative is

$$\mathcal{D}_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{1}{4} \omega^{ab}_\mu \left(\Sigma_{ab}\right)^\beta{}_\gamma \psi_\beta \tag{2.38}$$

where $\Sigma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$ are the local Lorentz generators. The curved-space Dirac action is then

$$S[\Psi, g] = \int d^D x \sqrt{g} \bar{\Psi} \mathbf{i} \gamma^\mu \mathcal{D}_\mu \Psi \tag{2.39}$$

where $\gamma^\mu = \gamma^\mu(x) \equiv \gamma^a e^\mu_a(x)$. The γ^a are the ordinary flat-spacetime gammas.

Redoing our calculation above including this extra ω term in the Dirac action gives

(in $D = 3 + 1$)

$$\mathcal{D}_\mu j_A^\mu = -\frac{1}{384\pi^2} \sum_{\text{Weyl},f} Q_f (-1)^f \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\tau} R_{\rho\sigma}{}^{\lambda\tau} \tag{2.40}$$

where Q_f is the charge of Weyl fermion f under the current in question. This means that any $\mathsf{U}(1)$ factor of the gauge group must have $\sum_f Q_f (-1)^f = 0$. (This property is automatic for $\mathsf{U}(1)$ factors of a compact Lie group (since it is true for $\mathsf{SU}(2)$). This is a small hint toward Grand Unification.)

An extreme example of such an anomaly is an anomaly in diffeomorphism invariance – a purely gravitational anomaly. This only happens in $D = 8k + 2$ dimensions.

Back in $D = 3 + 1$, $\mathsf{SU}(2)$ gauge theory with an odd number of Weyl fermions in a half-integer spin representation is anomalous. The case of spin-half is called the Witten anomaly or $\mathsf{SU}(2)$ anomaly.

The $\mathsf{SU}(2)$ anomaly. There are no perturbative anomalies in $D = 3+1$ (meaning, ones coming from triangle diagrams) for the case of $\mathsf{SU}(2)$, since the **2** is pseudo-real (isomorphic to its conjugate representation). But there can be a more subtle way for the fermion measure to vary under a gauge transformation – a non-perturbative anomaly.

The path integral over a Dirac fermion Ψ in some representation of a gauge group G is

$$\int D\Psi D\bar{\Psi} e^{-\int d^4x \bar{\Psi} \mathbf{i} \not{D} \Psi} = \det\left(\mathbf{i}\not{D}\right) = \prod_n \epsilon_n. \tag{2.41}$$

Here $\mathbf{i}\not{D}$ is the Dirac operator with background fields for G. We've discussed above how to regulate such things in a gauge-invariant way, and it is gauge invariant.

Now consider a single Weyl fermion $\psi_L = \frac{1}{2}(1 + \gamma^5)\Psi$ in some representation of G:

$$\int D\psi_L D\bar{\psi}_L e^{-\int d^4x \bar{\psi}_L \mathbf{i}\sigma^\mu D_\mu \psi_L} = \pm \det\left(\mathbf{i}\not{D} \frac{1+\gamma^5}{2}\right) = \sqrt{\det \mathbf{i}\not{D}}. \tag{2.42}$$

Recall that the eigenvalues of $\mathbf{i}\not{D}$, when nonzero, come in left-right pairs. (Let's assume that no eigenvalues are zero, or else the whole thing is zero.) So the square root just means taking one of each pair. The problem is picking the sign.

To try to define the sign, pick a reference gauge field configuration $A^\star_\mu$, and define the square root for this configuration

$$\sqrt{\det \mathbf{i}\not{D}_{A^\star}} = \prod_{\epsilon > 0} \epsilon_n \tag{2.43}$$

to be just the positive eigenvalues. To define the square root for any other configuration A, find a path from $A^\star$ to A, and follow the eigenvalues. If an odd number of the positive eigenvalues at $A^\star$ go through zero, then the sign of the square root flips.

Now we must ask: is this choice of sign gauge invariant? That is, do we get the same sign for A_μ and for its gauge image

$$A^\Omega_\mu = \Omega(x) A_\mu \Omega(x)^{-1} + \mathbf{i}\Omega(x)\partial_\mu \Omega(x)^{-1} \quad ? \tag{2.44}$$

If not, then clearly this gauge transformation $\Omega(x)$ cannot be regarded as an equivalence, since A and A^Ω would have different weight in

$$Z = \int DA e^{-S[A]} \int D\psi_L D\bar{\psi}_L e^{-\int d^4x \bar{\psi}_L \mathbf{i}\sigma^\mu D_\mu \psi_L}. \tag{2.45}$$

Now, there are many Ω we could consider. Only Ω that approach the identity map at $x \to \infty$ in $\mathbb{R}^4$ are equivalences. For such Ω, we can identify all the points at infinity and $\mathbb{R}^4 \cup \infty \simeq S^4$. So such gauge transformations are maps $\Omega : S^4 \to \mathsf{G}$. In the case of $\mathsf{G} = \mathsf{SU}(2)$, it's a nontrivial fact that $\pi_4(\mathsf{SU}(2) \simeq S^3) = \mathbb{Z}_2$ – there are two classes of such gauge transformations.

The Witten anomaly happens because, with the definition of sign above,

$$\sqrt{\det \mathbf{i}\not{D}_{A^\Omega}} = -\sqrt{\det \mathbf{i}\not{D}_A} \tag{2.46}$$

if Ω is in the nontrivial homotopy class.

For more, see David Tong's notes on anomalies.

[End of Lecture 4]

3 Wilsonian Renormalization Group

[Fradkin, 2d edition, chapter 4; Cardy; Zee §VI; Álvarez-Gaumé and Vázquez-Mozo, *An Invitation to QFT*, chapter 8.4-5 ($\simeq$ §7.3-4 of hep-th/0510040)] What I want to explain next is an important piece of metaphysics, in the sense that it is an idea about how to go about doing physics. The following discussion describes a perspective that can (and should) be applied to any system of extensive degrees of freedom. This includes many statistical-mechanics systems, condensed-matter systems and also QFTs in high energy physics. The great insight of Kadanoff and Wilson about such systems is that we should organize our thinking about them by length scale. We should think about a *family* of descriptions, labelled by the resolution of our microscope.

3.1 Where do field theories come from?

A model with finitely many degrees of freedom per unit volume.

Consider the following system of extensive degrees of freedom – it is an example of a very well-regulated (euclidean) QFT. At each site i of a lattice we place a two-valued (classical) degree of freedom $s_i = \pm 1$, so that the path 'integral' measure is

$$\int [ds] ... \equiv \sum_{\{s_i\}} ... = \prod_{\text{sites, } i} \sum_{s_i = \pm 1} ... \ .$$

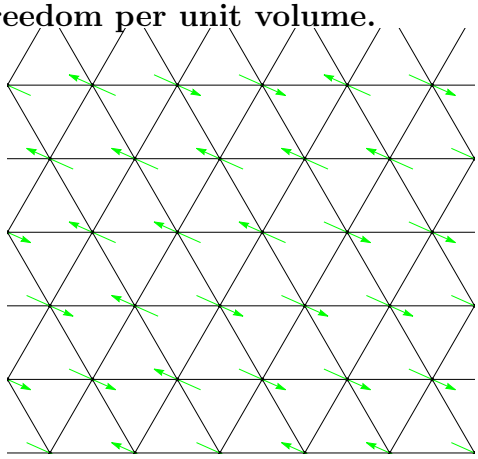


Figure 2: A configuration of classical Ising spins on the 2d triangular lattice.

Let's choose the euclidean action to be

$$S[s] = -\beta J \sum_{\langle i,j \rangle} s_i s_j \ .$$

Here βJ is some coupling; the notation $\langle i, j \rangle$ means 'sites i and j that are nearest neighbors'. The partition function is

$$Z = \int [ds] e^{-S[s]} = \sum_{\{s_i\}} e^{+\beta J \sum_{\langle i,j \rangle} s_i s_j}. \tag{3.1}$$

(I can't hide the fact that this is the thermal partition function $Z = \mathrm{tr} e^{-\beta H}$ for the *classical Ising model*, with $H = -J \sum_{\langle i,j \rangle} s_i s_j$, and $\beta \equiv 1/T$ is the *coolness*[5], *i.e.* the inverse temperature.)

[5]This nomenclature, due to the condensed matter physicist Miles Stoudenmire, does a great job of reminding us that at lower temperatures, quantum mechanics has more dramatic consequences.

In the thermodynamic limit (the number of sites goes to infinity), this model has a special value of $\beta J > 0$ above which there is spontaneous breaking of the $\mathbb{Z}_2$ symmetry $s_i \to -s_i$ by a nonzero *magnetization*, $\langle s_i \rangle \neq 0$.

The Ising model defined by (3.1) is a model of a magnet (more specifically, when $\beta J > 0$ which makes neighboring spins want to align, a *ferro*magnet). Some basic phenomenology: just below the Curie temperature T_c, the magnetization (average magnetic moment per unit volume) behaves like

$$|M| \sim (T_c - T)^\beta$$

where β is a pure number (it depends on the number of spatial dimensions)[6]. In terms of the Ising model, the magnetization is[7]

$$\langle M \rangle = \frac{1}{Z} \sum_{\{s_i\}} e^{-H(s)/T} \frac{\sum_i s_i}{\mathcal{V}} \quad . \tag{3.2}$$

($\mathcal{V}$ is the number of sites of the lattice, the volume of space.) How can you get such a *non-analytic* (at $T = T_c \neq 0$) function of T by adding a bunch of terms of the form $e^{-E/T}$? It is clearly impossible if there is only a finite number of terms in the sum, each of which is analytic near $T_c \neq 0$. It *is* actually possible if the number of terms is infinite – finite-temperature phase transitions only happen in the thermodynamic limit.

Landau and Ginzburg guess the answer. Starting from Z, even with clever tricks like Kramers-Wannier duality, and even for Onsager, it is pretty hard to figure out what the answer is for the magnetization. But the answer is actually largely determined on general grounds, as follows.

Let's ask what is the free energy G *at fixed magnetization*, $G[M]$. How would we do this in an experiment? We'd apply a uniform magnetic field, and find just the right field to get the desired M, and then measure the free energy (with our trusty free-energy-ometer, of course). In more formal terms, we should add a source for the magnetization and compute

$$e^{-\beta F[J]} = \mathrm{tr} e^{-\beta(H + \sum MJ)}.$$

Pick some magnetization M_c, and *choose* $J^{[M_c]}$ so that

$$\langle M \rangle = -\frac{\partial F}{\partial J} = M_c.$$

[6] The name is conventional; don't confuse it with the inverse temperature.

[7] In many real magnets, the magnetization can point in any direction in three-space – it's a vector $\vec{M}$. We are simplifying our lives.

Then $G[M_c] \equiv F[J^{[M_c]}] - \sum M_c J^{[M_c]}$. This is a Legendre transform of the usual F in $Z = e^{-\beta F}$. In this context, the source J is (minus) an external magnetic (Zeeman) field. This $G[M]$ is just the same idea as an object you may encounter called the euclidean effective action $\Gamma[\phi_c]$ (up to factors of β), where the analog of M is called the 'classical field' ϕ_c. G is the thing we should minimize to find the magnetization in the groundstate.

LG Effective Potential. We can even consider a model where the magnetization is a vector. If $\vec{M}$ is independent of position $\vec{x}$ [8] then spin rotation invariance (or even just $M \to -M$ symmetry) demands that

$$G(M) = V\left(r\vec{M}^2 + u\left(\vec{M}^2\right)^2 + ...\right)$$

where V is the volume of space, r, u are some functions of T that we don't know, and the dots are terms with a larger (even) number of Ms. These functions $a(T)$ and $b(T)$ have no reason not to be smooth functions of T. Now suppose there is a value of T for which $a(T)$ vanishes:

$$r(T) = r_1(T - T_c) + ...$$

with $r_1 > 0$ a pure constant. For $T > T_c$, the minimum of G is at $\vec{M} = 0$; for $T < T_c$, the unmagnetized state becomes unstable and new minima emerge at $|\vec{M}| = \sqrt{-\frac{r}{2u}} \sim (T_c - T)^{\frac{1}{2}}$. This is the *mean field theory* description of a second-order phase transition. It's not the right value of β (which is about 1/3) for the 3d Curie point, but it shows very simply how to get an answer that is not analytic at T_c.

LG Effective Action. Landau and Ginzburg can do even better. $G(M)$ with constant M is like the effective *potential*; if we let $M(\vec{x})$ vary in space, we can ask and answer what is the effective *action*, $G[M(\vec{x})]$. The Landau-Ginzburg effective action is

$$G[M] = \int d^d\vec{x}\left(r\vec{M}^2 + u\left(\vec{M}^2\right)^2 + Z\partial_i\vec{M}\cdot\partial_i\vec{M} + ...\right) \tag{3.3}$$

– now we are allowed to have gradients. Z is a new unknown function of T; let's set it to 1 by rescaling M. This just a scalar field theory (with several scalars) in euclidean space. Each field has a mass $\sqrt{r}$ (they are all the same as a consequence of the spin rotation symmetry). So $\frac{1}{\sqrt{r}}$ is a length scale, to which we turn next.

Definition of correlation length. Suppose we perturb the system by turning on an external (we pick it) magnetic field (source for $\vec{M}$) $\vec{H}$, which adds to the Hamiltonian

[8] In (3.2), I've averaged over all space; instead we could have averaged over just a big enough patch to make it look smooth. We'll ask 'how big is big enough?' next – the answer is 'the correlation length'.

by $-\vec{H}\cdot\vec{M}$. (So far we are doing Euclidean physics, which means equilibrium, no real time dependence.) Pick the field to be small, so its effect is small and we can study the linearized equations (let's do it for $T > T_c$, so we're expanding around $M = 0$):

$$\left(-\partial^2 + r\right)\vec{M} = \vec{H}\ .$$

Recall the Green's function G_2 of a massive scalar field: G_2 solves this equation in the case where H is a delta function. Since the equation is linear, that solution determines the solution for general H (this was why Green introduced Green's functions):

$$\begin{aligned} M(x) &= \int d^3y G_2(x,y)H(y) = \int d^3y \left(\int đ^3k \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2 + r}\right) H(y) \\ &= \int d^3y \frac{1}{4\pi|\vec{x}-\vec{y}|} e^{-\sqrt{r}|\vec{x}-\vec{y}|} H(y). \end{aligned} \tag{3.4}$$

The Green's function

$$G_2^{IJ}(x) = \left\langle \vec{M}^I(x)\vec{M}^J(0)\right\rangle = \delta^{IJ}\frac{1}{4\pi|\vec{x}|}e^{-\sqrt{r}|\vec{x}|}$$

is diagonal in the vector index I, J so I've suppressed it in (3.4). G_2 is the answer to the question: if I perturb the magnetization at the origin, how does it respond at x? The answer is that it dies off like

$$\left\langle \vec{M}(x)\vec{M}(0)\right\rangle \sim e^{-|x|/\xi}$$

– this relation defines the *correlation length* ξ, which will depend on the parameters. In the LG mean field theory, we find $\xi = \frac{1}{\sqrt{r}}$. The LG theory predicts the behavior of ξ as we approach the phase transition to be $\xi \sim \frac{1}{(T-T_c)^\nu}$ with $\nu = \frac{1}{2}$. Again the exponent is wrong in detail (we'll see why below), but it's a great start.

Now let's return to the microscopic model (3.1). Away from the special value of βJ, the correlation functions behave as

$$\langle s_i s_j\rangle_{\text{connected}} \sim e^{-\frac{r_{ij}}{\xi}}$$

where $r_{ij} \equiv$ distance between sites i and j. As $T \to T_c$, $\xi \to \infty$. Notice that the subscript *connected* means that we need not specify whether we are above or below T_c, since it subtracts out the disconnected bit $\langle s_i\rangle\langle s_j\rangle$ by which their form differs. From the more microscopic viewpoint, ξ is the length scale over which the values of the spins are highly correlated. This allows us to answer the question of how much coarse-graining we need to do to reach a continuum approximation: The continuum description in terms of

$$M(x) \equiv \frac{\sum_{i\in R_x}\langle s_i\rangle}{\text{Vol}(R_x)} \tag{3.5}$$

is valid if we average over regions R (centered around the point x) with linear size bigger than ξ.

Coarse-graining by block spins. We want to understand the connection between the microscopic spin model and the macroscopic description of the magnetization better, for example to systematically improve upon the quantitative failures of the LG mean field theory for the critical exponents. Kadanoff's idea is to consider a sequence of *blocking transformations*, whereby we group more and more spins together, to interpolate between the spin at a single site s_i, and the magnetization averaged over the whole system, passing through (3.5) on the way.

The blocking (or 'decimation') transformation can be implemented in more detail for Ising spins on the (2d) triangular lattice as follows (Fig. 3). Group the spins into blocks of three as shown; we will construct a new coarser Ising system, where the sites of the new lattice correspond to the blocks of the original one, and the spin at the new site is an average of the three. One way to do this is majority rule:

$$s_{\text{block},\,b} \equiv \text{sign}\left(\sum_{i\in\text{block},b} s_i\right) \ .$$

(The reason to study the triangular lattice is so that there can be no ties in the simplest block.)

We want to write our original partition function in terms of the averaged spins on a lattice with a larger lattice spacing (it's not exactly twice as large, but I'll call it that below). We'll use the identity

$$1 = \sum_{s_{\text{block}}} \delta\left(s_{\text{block}} - \text{sign}(\sum_{i\in\text{block}} s_i)\right) \ .$$

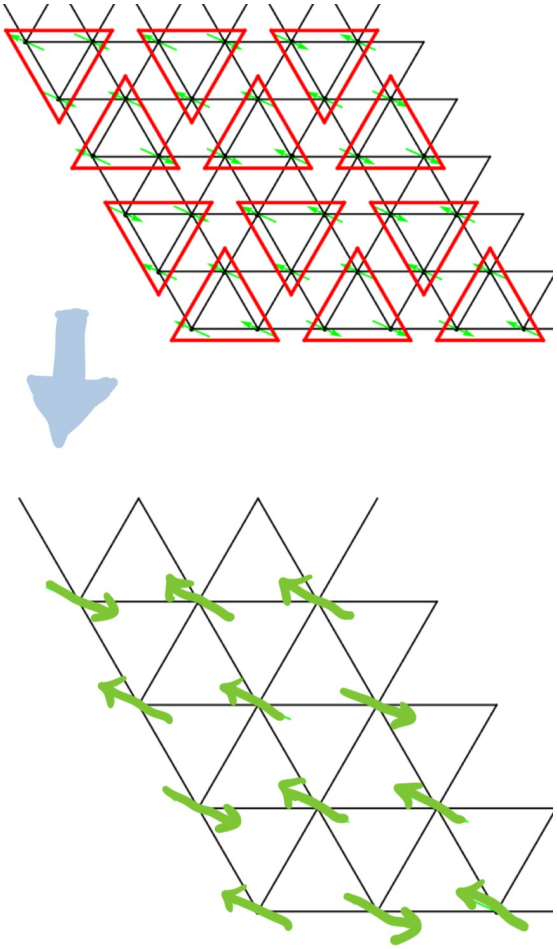


Figure 3: A blocking transformation.

This is true for each block; we can insert one of these for each block. Split the original sum into nested sums, the outer one over the blocks, and the inner one over the spins within the block:

$$Z = \sum_{\{s\}} e^{-\beta H[s_i]} = \sum_{\{s_{\text{block},\,b}\}} \sum_{s\in\text{block},b} \prod_{\text{blocks}} \delta\left(s_{\text{block},b} - \text{sign}\left(\sum_{i\in\text{block},b} s_i\right)\right) e^{-\beta H^{(a)}[s]} \ .$$

The superscript (a) on the Hamiltonian is intended to indicate that the lattice spacing is a. Now we interpret the inner sum as another example of integrating out stuff we don't care about to generate an effective interaction between the stuff we do care about:

$$\sum_{s\in\text{block},b\ \text{blocks}} \prod \delta\left(s^{(2a)} - \text{sign}\left(\sum_{i\in\text{block},b} s_i\right)\right) e^{-\beta H^a[s]} \equiv e^{-\beta H^{(2a)}[s^{(2a)}]}$$

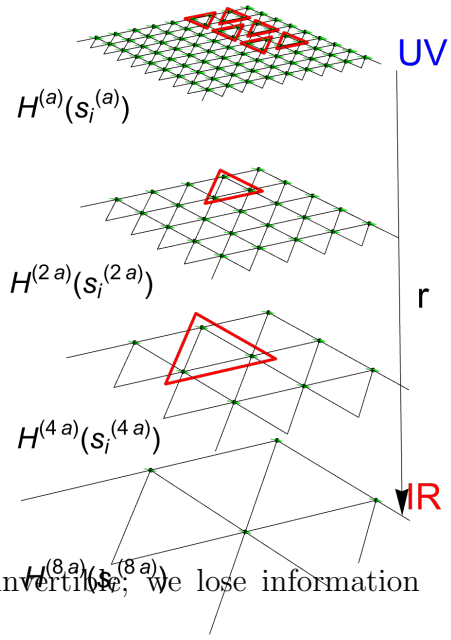
These sums are hard to actually do, except in 1d. But we don't need to do them to understand the form of the result.

As in our two-oscillator example from the first chapter, the new Hamiltonian will be less local than the original one – it won't just be nearest neighbors in general:

$$H^{(2a)}[s^{(2a)}] = J^{(2a)} \sum_{\langle i,j\rangle} s_i^{(2a)} s_j^{(2a)} + K^{(2a)} \sum_{\langle\langle i,j\rangle\rangle} s_i^{(2a)} s_j^{(2a)} + ...$$

where $\langle\langle i,j\rangle\rangle$ means next-neighbors. Notice that I've used the same labels i,j for the coarser lattice. We have rewritten the partition function as the same *kind* of model, on a coarser lattice, with different values of the couplings:

$$Z = \sum_{\{s^{(2a)}\}} e^{-\beta H^{(2a)}[s^{(2a)}]} \quad .$$



Now we can do it again. The decimation operation defines *a map on the space of (in this case Ising) Hamiltonians*:

$$H^{(a)} \mapsto H^{(2a)} \mapsto H^{(4a)} \mapsto H^{(8a)} \mapsto ...$$

The couplings $J, K...$ are coordinates on the space of Hamiltonians. Each time we do it, we double the lattice spacing; the correlation length in units of the lattice spacing gets halved, $\xi \mapsto \xi/2$. This operation is called a 'renormalization group transformation' but notice that it is very much not invertible; we lose information about the short-distance stuff by integrating it out.

RG fixed points. Where can it end? One thing that can happen is that the form of the Hamiltonian can stop changing:

$$H^{(a)} \mapsto H^{(2a)} \mapsto H^{(4a)} \mapsto H^{(8a)} \mapsto ... \mapsto H_\star \mapsto H_\star \mapsto H_\star \ ...$$

The fixed point hamiltionian $H_\star$, which is not changed by the rescaling operation, is scale invariant. What can its correlation length be if it is invariant under $\xi \to \xi/2$?

Either $\xi = 0$ (the mass of the fields go to infinity and there is nothing left to integrate) or $\xi = \infty$ (the mass goes to zero and we have more to discuss, we can call this a nontrivial fixed point).

Near a nontrivial fixed point, once $\xi \gg a$, the original lattice spacing, we are quite justified in using a continuum description, to which we return in subsection 3.2.

Perturbations of a fixed point. Before doing any more work, though, we can examine the possible behaviors of the RG flow near a fixed point. Consider a fixed point Hamiltonian $H_\star$, and move away from it slightly by changing one of the couplings a little bit:

$$H = H_\star + \delta g \mathcal{O}.$$

What does the RG do to this to leading order in δg? The possibilities are:

- If the flow takes it back to the original fixed point, $\mathcal{O}$ (and its associated coupling δg) is called *irrelevant*.
- If the flow takes it away from the original fixed point, $\mathcal{O}$ is called a *relevant* perturbation of $H_\star$.
- The new H might also be a fixed point, at least to this order in δg. Such a coupling (and the associated operator $\mathcal{O}$) is called *marginal*. If the new H really is a new fixed point, not just to leading order in δg, then $\mathcal{O}$ is called *exactly marginal*. Usually it goes one way or the other and is called *marginally relevant* or *marginally irrelevant*.

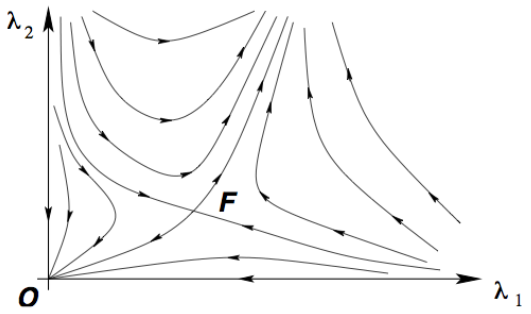


Figure 4: A possible set of RG flows for a system with two couplings $\lambda_{1,2}$. [from Álvarez-Gaumé and Vázquez-Mozo, hep-th/0510040]

Note the infrared-centric terminology.

Comment on *Universality*: The Ising model is a model of many microscopically-different-looking systems. It can be a model of spins like we imagined above. Or it could be a model of a lattice gas – we say spin up at site i indicates the presence of a gas molecule there, and spin down represents its absence. These different models will naturally have different microscopic interactions. But there will only be so many fixed points of the flow in the space of Hamiltonians on this system of 2-valued variables. This idea of the paucity of fixed points underlies Kadanoff and Wilson's explanation of

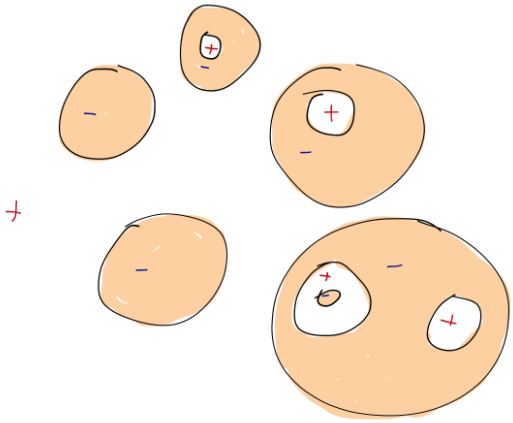
the experimental phenomenon of *universality*: the same critical exponents arise from very different-seeming systems (*e.g.* the Curie point of a magnet and the liquid-gas critical point).

The basic point is that there is a scale-invariant field theory (often a conformal field theory) that describes the intrinsic properties of the critical point; the critical exponents are dimensions of operators in this field theory.

3.2 Wilsonian perspective on renormalization

[Zee, §VI.8 (page 362 of 2d Ed.), Peskin §12.1]

Consider the ϕ^4 theory in Euclidean space, with negative m^2 (and no ϕ^k terms with odd k). This potential has two minima and a $\mathbb{Z}_2$ symmetry that interchanges them, $\phi \to -\phi$. If we squint at a configuration of ϕ, we can label regions of space by the sign of ϕ (as in the figure at right). The kinetic term for ϕ will make nearby regions want to agree, just like the $J\sum_{\langle ij\rangle}\sigma_i\sigma_j$ term in the ferromagnetic Ising model ($J>0$). The potential term discourages values of ϕ other than the two minima. So the critical point described by taking m^2 near zero is plausibly the same as the one obtained from the usual Ising model on a lattice.[9].



We will study the integral

$$Z_\Lambda \equiv \int_\Lambda [D\phi] e^{-\int d^D x \mathcal{L}(\phi)}. \tag{3.6}$$

Here the specification $\int_\Lambda$ says that we integrate over field configurations $\phi(x) = \int \text{đ}^D k e^{\mathbf{i}kx}\phi_k$ such that $\phi_k = 0$ for $|k| \equiv \sqrt{\sum_{i=1}^D k_i^2} > \Lambda$. Think of $2\pi/\Lambda$ as the lattice spacing[10] – there just aren't modes of shorter wavelength. We are using (again) a cutoff on the euclidean momenta $k_E^2 \le \Lambda^2$.

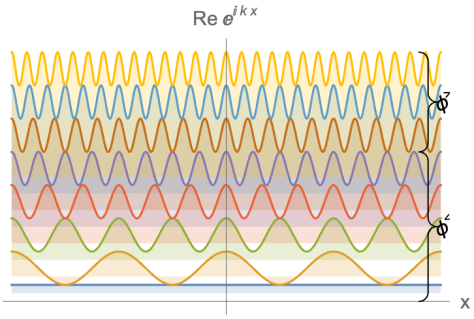
We want to understand (3.6) by a coarse-graining procedure, a continuum analog of blocking. It will be just like our discussion in §1, except instead of just two modes,

[9] For a more sophisticated argument for this equivalence, see pages 7-9 of Polyakov, *Gauge Fields and Strings*.

[10] This cutoff on momenta is not precisely the same as the effects of a lattice; with a lattice, the momentum space is periodic: $e^{\mathbf{i}kx_n} = e^{\mathbf{i}k(na)} = e^{\mathbf{i}\left(k+\frac{2\pi}{a}\right)(na)}$ for $n \in \mathbb{Z}$. Morally it is the same.

we'll do it for the whole field theory. But the idea is the same: do the integral over the high-energy modes first, for the reasons described in §1.
Break up the configurations into pieces:

$$\phi(x) = \int đk e^{ikx} \phi_k \equiv \phi^< + \phi^> \ .$$



Here $\phi^<$ has nonzero Fourier components only for $|k| \leq \Lambda - \delta\Lambda$ and $\phi^>$ has nonzero Fourier components only for $\Lambda - \delta\Lambda \leq |k| \leq \Lambda$. Zee calls the two parts 'smooth' and 'wiggly'. They could also be called 'slow' and 'fast' or 'light' and 'heavy'.
We want to do the integral over the heavy/wiggly/fast modes to develop an effective action for the light/smooth/slow modes:

$$Z_\Lambda = \int_{\Lambda-\delta\Lambda} [D\phi^<] e^{-\int d^D x \mathcal{L}(\phi^<)} \int [D\phi^>] e^{-\int d^D x \mathcal{L}_1(\phi^<, \phi^>)}$$

where $\mathcal{L}_1$ contains all the dependence on $\phi^>$ (and no other terms).

These integrals are hard to actually do, except in a gaussian theory. But we don't need to do them to understand the form of the result. First give it a name:

$$e^{-\int d^D x \delta L(\phi^<)} \equiv \int [D\phi^>] e^{-\int d^D x \mathcal{L}_1(\phi^<, \phi^>)} \tag{3.7}$$

so once we've done the integral we'll find

$$Z_\Lambda = \int_{\Lambda-\delta\Lambda} [D\phi^<] e^{-\int d^D x (\mathcal{L}(\phi^<) + \delta\mathcal{L}(\phi^<))} \ . \tag{3.8}$$

To get a feeling for the form of $\delta\mathcal{L}$ (and because there is little reason not to) consider the more general Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \sum_n g_n \phi^n + ... \tag{3.9}$$

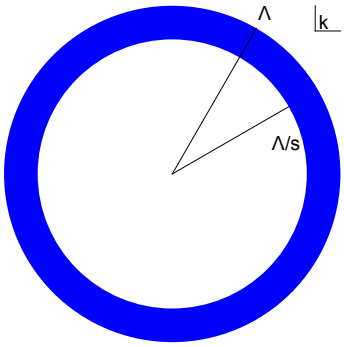
where we include *all* possible terms consistent with the symmetries (rotation invariance, maybe $\phi \to -\phi$...). Then we can find an explicit expression for $\mathcal{L}_1$:

$$\int d^D x \mathcal{L}_1(\phi^<, \phi^>) = \int d^D x \left(\frac{1}{2}(\partial\phi^>)^2 + \frac{1}{2} m^2 (\phi^>)^2 + ... \right)$$

(I write the integral so that I can ignore terms that integrate to zero such as $\partial\phi^< \partial\phi^>$.) This is the action for a scalar field $\phi^>$ interacting with itself and with a (slowly-varying) background field $\phi^<$. But what can the result $\delta\mathcal{L}$ be but something of the form (3.9)

again, with different coefficients? The result is to shift the couplings $g_n \to g_n + \delta g_n$. (This includes the coefficient of the kinetic term and also of the higher-derivative terms which are hidden in the ... in (3.9). You will see in a moment the logic behind which terms I hid.)

Finally, so that we can compare steps of the procedure to each other, we rescale our rulers. We'd like to change units so that the new $\int_{\Lambda-\delta\Lambda}$ is a $\int_\Lambda$ with different couplings.



We accomplish this by defining

$$\Lambda - \delta\Lambda \equiv \Lambda/s, \qquad s > 1.$$

In $\int_{\Lambda-\delta\Lambda}$, we integrate over fields with $|k| < \Lambda/s$. Change variables: $k = k'/s$ so now $|k'| < \Lambda$. So $x = x's, \partial' \equiv \partial/\partial x' = s\partial_x$ and wavefunctions are preserved $e^{\mathbf{i}kx} = e^{\mathbf{i}k'x'}$.

Plug this into the action

$$\int d^D x \mathcal{L}_{\text{eff}}(\phi^<) = \int d^D x' s^D \left(\frac{1}{2} s^{-2} \left(\partial'\phi^<\right)^2 + \sum_n \left(g_n + \delta g_n\right) \left(\phi^<\right)^n + ... \right)$$

We can make this look like $\mathcal{L}$ again by rescaling the field variable: $s^{D-2}\left(\partial'\phi^<\right)^2 \equiv \left(\partial'\phi'\right)^2$ (*i.e.* $\phi' \equiv s^{\frac{1}{2}(D-2)}\phi^<$):

$$\int d^D x \mathcal{L}_{\text{eff}}(\phi^<) = \int d^D x' \left(\frac{1}{2} \left(\partial'\phi'\right)^2 + \sum_n \left(g_n + \delta g_n\right) s^{D - \frac{n(D-2)}{2}} \left(\phi'\right)^n + ... \right)$$

So the end result is that integrating out a momentum shell of thickness $\delta\Lambda \equiv (1 - 1/s)\Lambda$ results in a change of the couplings to

$$g'_n = s^{-\frac{n(D-2)}{2} + D} \left(g_n + \delta g_n\right) \ .$$

This procedure produces a *flow* on the space of actions.

Ignore the interaction corrections, δg_n, for a moment. Then, since $s > 1$, the couplings with $\frac{n(D-2)}{2} - D > 0$ get smaller and smaller as we integrate out more shells. If we are interested in only the longest-wavelength modes, we can ignore these terms. They are *irrelevant*. Couplings ('operators') with $\frac{n(D-2)}{2} - D < 0$ get bigger and are *relevant*.

The mass term has $n = 2$ and $(m')^2 = s^2 m^2$ is always relevant for any $D < \infty$. So far, the counting is the same as our naive dimensional analysis. That's because we left out the δL term! This term can make an important difference, even in perturbation

theory, for the fate of marginal operators (such as ϕ^4 in $D = 4$), where the would-be-big tree-level term is agnostic about whether they grow or shrink in the IR.

Notice that starting from (3.6) we are assuming that the system has a rotation invariance in euclidean momentum. If one of those euclidean directions is time, this follows from Lorentz invariance. This simplifies the discussion. But for non-relativistic systems, it is often necessary to scale time differently from space. The relative scaling z in $\vec{x}' = \vec{x}/s, t' = s^{-z}t$ is called the *dynamical critical exponent*.

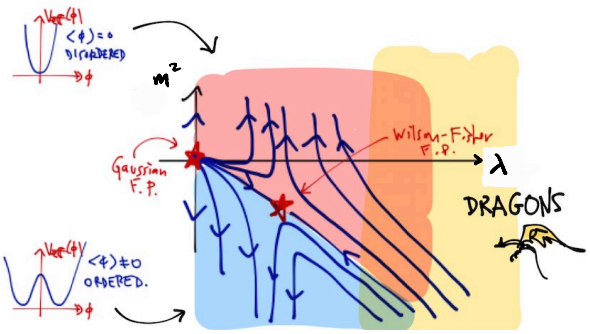
The definition of the beta function and of a fixed point theory are just as in the first lecture of last quarter.

If we do pick an example of an interaction with which to perturb the gaussian fixed point, we will indeed find other fixed points. An important family of such fixed points can be controlled by studying the theory in $D = 4 - \epsilon$ dimensions, just as in dim reg. For the case of a single scalar with an Ising ($\phi \to -\phi$) symmetry that we've been discussing, the beta function for the quartic term takes the form[11]

$$\beta_\lambda \equiv -s\partial_s \lambda(s) = \epsilon\lambda - a\lambda^2 + \mathcal{O}(\lambda^4) \tag{3.10}$$

with $a > 0$ a pure number, which has a zero at $\lambda = \epsilon/a$, which is small when ϵ is small, and hence the perturbative calculation that led to it is self-consistent. The calculation of β can be done by explicitly integrating out momentum shells using Wick's theorem, but in practice is most easily done by the methods we learned earlier – the answer is the same as we'll see below.

This fixed point that we find in perturbation theory is called the *Wilson-Fisher* fixed point, and is under perturbative control when ϵ is small. It has a single relevant perturbation preserving the $\phi \to -\phi$ symmetry, which is the mass term, and so we expect to reach it by tuning a single parameter. This fixed point gives a good description of the critical point of Ising magnets. The rates at which the couplings leave and enter the fixed point determine the critical exponents.



[End of Lecture 5]

[11] Here I am using the morally correct convention for the sign of the beta function (not the one in the high energy literature), where β points toward the IR.

The generalization to systems with an $\mathsf{O}(N)$ symmetry is obtained by adding an N-valued index to the scalar field. In the next subsection we will discuss the explicit calculation for general N.

3.3 The Wilson-Fisher fixed point

[Kardar, *Fields*, §5.5, 5.6; R. Shankar, *Rev. Mod. Phys.* **66** (1994) 129] We'll show that it's not actually necessary to ever do any momentum integrals to derive the RG equations.

Consider N-component scalar fields in D dimensions, with $\mathsf{O}(N)$ symmetry. We can define this model, for example, on a Euclidean lattice, by an action of the form

$$S[\phi^a] = \frac{1}{2}\sum_{a=1}^{N}\sum_{n,i}\left(\phi^a(n)-\phi^a(n+i)\right)^2 + \sum_n\left(r_0\sum_a\phi^a(n)^2 + u_0\left(\sum_a\phi^a(n)^2\right)^2\right). \tag{3.11}$$

Here n labels sites of some (*e.g.* hypercubic) lattice and i labels the (8 in the 4d hypercubic case) links connecting neighboring sites. We'll call the lattice spacing $2\pi/\Lambda_1$. In terms of Fourier modes, this is

$$S[\phi^a] = -\int_{|k|<\Lambda_0} \text{đ}^D k \phi^a(k) J(k) \phi^a(k) + S_{\text{int}} \ .$$

For the hyper-cubic lattice, we get (the second step is Taylor expansion)

$$J(k) = 2\left(\sum_{\mu=1}^{D}(\cos ak_\mu - 1)\right) \overset{ka\ll 1}{\simeq} \sum_\mu\left(a^2k_\mu^2 + \frac{a^4}{4\cdot 3}k_\mu^4 ...\right).$$

The energy function $J(k)$ only has the discrete rotation symmetries of the lattice (90° rotations for the hypercubic lattice). But the leading term at small wavenumber has full rotation invariance; in position space, this term is $a^2\partial_\mu\phi^a\partial^\mu\phi^a$. The next term $\int \text{đ}^D k\ a^4k^4|\phi_k|^2 = \int d^Dx\ a^4\phi^a\sum_\mu\partial_\mu^4\phi^a$, which breaks the rotation group to a discrete subgroup, is irrelevant by the counting we did above: $\int d^Dx\partial^4\phi^2 \sim s^{D-4-2\frac{D-2}{2}} = s^{-2}$. This means that rotation invariance *emerges* on its own. [12]

The path integral is defined by

$$Z \equiv \int \underbrace{[\text{d}\phi^a]_{|k|<\Lambda_0}}_{\equiv\prod_{|k|<\Lambda_0,a}\text{d}\phi^a(k)} e^{-S[\phi^a]} \ . \tag{3.12}$$

[12] Confession: the restriction on the momenta in the exact lattice model should be to a fundamental domain for the identification $k^\mu \equiv k^\mu + \Lambda_1^\mu$; I am going to replace this right away with a rotation-invariant cutoff on the magnitude $k^2 \equiv k^\mu k_\mu \leq \Lambda_0$ of the euclidean momentum. This is an unimportant lie for our purposes.

The $\mathsf{O}(N)$ symmetry acts by $\phi^a \to R^a_b \phi^b$, with $R^t R = \mathbb{1}_{N\times N}$. We will perturb about the Gaussian fixed point with (Euclidean) action

$$S_0[\phi] = \int_0^\Lambda \text{đ}^D k \underbrace{\phi^a(k)\phi^a(-k)}_{\equiv|\phi|^2(k)} \frac{1}{2}\left(r_0 + r_2 k^2\right). \tag{3.13}$$

The coefficient r_2 of the kinetic term is a book-keeping device that we may set to 1 if we choose.

With $u_0 = 0$, this is a bunch of gaussian integrals, and everything can be computed by Wick from the two-point function:

$$\left\langle \phi^a_>(q_1)\phi^b_>(q_2)\right\rangle_{0,>} = \frac{\delta^{ab}\not{\delta}(q_1+q_2)}{r_0 + q_1^2 r_2}. \tag{3.14}$$

I've defined $\not{\delta}(q) \equiv (2\pi)^D \delta^D(q)$. Notice that we are going to keep the mass perturbation r_0 in the discussion. (I've written $\phi_>$ here in anticipation of the fact that we are going to integrate only the fast modes at each RG step.)

Although this gaussian model is trivial, we can still do the RG to it. (We will turn on the interactions in a moment.) To review, an RG step has three ingredients:

1. Integrate out the fast modes, *i.e.* $\phi^>$, with $|k| \in (\Lambda - \delta\Lambda, \Lambda)$. I will call $\Lambda - \delta\Lambda \equiv \Lambda/s$, and $s > 1$, we will regard s as close to 1: $s - 1 \ll 1$.

$$\begin{aligned} Z &= \int \prod_{0\le|k|\le\Lambda/s} \text{d}\phi_<(k) \left(\int \prod_{\Lambda/s\le|k|\le\Lambda} \text{d}\phi_>(k) e^{-\left(\underbrace{S_0[\phi^<] + S_0[\phi^>]}_{\text{quadratic}} + \underbrace{S_{\text{int}}[\phi^<,\phi^>]}_{\text{mixes fast and slow}}\right)} \right) \\ &= \int [\text{d}\phi^<] e^{-S_0[\phi^<]} \underbrace{\left\langle e^{-S_{\text{int}}[\phi^<,\phi^>]}\right\rangle_{0,>}}_{\text{average over } \phi^>, \text{ with gaussian measure}} Z_{0,>} = \int [\text{d}\phi^<] e^{-\tilde{S}[\phi^<]}\,. \end{aligned} \tag{3.15}$$

   The factor of $Z_{0,>}$ is independent of $\phi^<$ and can be ignored.

2. Rescale momenta so that we may compare successive steps: $\tilde{k} \equiv sk$ lies in the same interval as we started with $|\tilde{k}| \in (0, \Lambda)$.

3. Are the actions $s(\phi) = r\phi^2 + u\phi^4$ and $\tilde{s}(\psi) = 4r\psi^2 + 16u\psi^4$ different? No: let $2\psi \equiv \phi$. We can rescale the field variable at each step:

$$\tilde{\phi}(\tilde{k}) \equiv \zeta^{-1}\phi_<(\tilde{k}/s).$$

   We will choose the 'wavefunction renormalization' factor ζ so that the kinetic terms are fixed.

RG for free field. If $S_{\rm int} = 0$, and $r_0 = 0$, then (3.15) gives

$$2\tilde{S}[\phi_<] = \int_{|k|<\Lambda/s} \text{đ}^D k \phi^a_<(k) k^2 \phi^a_<(k) \overset{\text{steps 2 and 3}}{=} s^{-D-2}\zeta^2 \int_{|\tilde{k}|<\Lambda} \tilde{\phi}^a(\tilde{k})\tilde{k}^2\tilde{\phi}^a(\tilde{k})\,\text{đ}^D\tilde{k}\ .$$

With $\zeta \equiv s^{\frac{D+2}{2}}$, the Gaussian action is a fixed point of the RG step:

$$\tilde{S}[\tilde{\phi}] = S[\phi] = S^\star.$$

Warning: the field $\phi(k)$ is the Fourier transform of the field $\phi(x)$ that we considered above. They are different by an integral over space or momenta: $\phi(x) = \int \text{đ}^D k \phi(k) e^{\mathbf{i}kx}$. So they scale differently. The result that $\zeta = s^{\frac{D+2}{2}}$ is perfectly consistent with our earlier result that $\phi(x)$ scales like $s^{\frac{2-D}{2}}$.

Now we consider perturbations of this fixed point. We'll only study those that preserve the $\mathsf{O}(N)$ symmetry. We can order them by their degree in ϕ. The first nontrivial case preserving the symmetry is

$$\delta S_2[\phi] = \int_{|k|<\Lambda} \text{đ}^D k \phi^a(k)\phi^a(k) r(k)/2\ .$$

Here $r(k)$ is a coupling *function*. If its position-space representation is local, it has a nice Taylor expansion about $k = 0$:

$$r(k) = \underbrace{r_0}_{\equiv m_0^2} + k^2 r_2 + \ldots$$

(I also assumed rotation invariance.) The same manipulation as above gives

$$\widetilde{2\delta S_2}[\tilde{\phi}(\tilde{k})] = s^{-D+\frac{D+2}{2}2=2} \int_{|\tilde{k}|<\Lambda} \tilde{\phi}^a(\tilde{k})\tilde{r}(\tilde{k})\tilde{\phi}^a(\tilde{k})\,\text{đ}^D\tilde{k}$$

with $\tilde{r}(\tilde{k}) = s^2 r(\tilde{k}/s)$, so that

$$\underbrace{\tilde{r}_0 = s^2 r_0}_{\text{relevant}}\ ,\qquad \underbrace{\tilde{r}_2 = s^0 r_2}_{\text{marginal by design}}\ ,\qquad \underbrace{\tilde{r}_4 = s^{-2} r_4}_{\text{irrelevant}} \ldots$$

This is taken care of by our Gaussian action.

Next we will consider the quartic perturbation, which will couple fast and slow modes. A tool at our disposal is the *cumulant expansion*, aka the exponentiation of the disconnected diagrams:

$$\left\langle e^{-\Omega} \right\rangle = e^{-\langle\Omega\rangle + \frac{1}{2}\left(\left\langle\Omega^2\right\rangle - \langle\Omega\rangle^2\right) + \ldots} \tag{3.16}$$

We break up our fields into slow and fast, and integrate out the fast modes:

$$Z_\Lambda = \int [D\phi_<] e^{-\int_0^{\Lambda/s} \text{đ}^D k |\phi_<(k)|^2 \left(\frac{r_0 + r_2 k^2}{2}\right)} Z_{0,>} \left\langle e^{-\mathcal{U}[\phi_<, \phi_>]} \right\rangle_{0,>} .$$

Again the $\langle ... \rangle_{0,>}$ means averaging over the fast modes with their Gaussian measure, and $Z_{0,>}$ is an irrelevant normalization factor, independent of the objects of our fascination, the slow modes $\phi_<$. I've written $|\phi(k)|^2 \equiv \sum_a^N \phi^a(k)\phi^a(-k)$. The cumulant expansion gives

$$\log \left\langle e^{-\mathcal{U}} \right\rangle_{0,>} = -\underbrace{\langle \mathcal{U} \rangle_{0,>}}_{1} + \underbrace{\frac{1}{2}\left(\langle \mathcal{U}^2 \rangle_{0,>} - \langle \mathcal{U} \rangle_{0,>}^2 \right)}_{2}$$

$$1 = \langle \mathcal{U}[\phi_<, \phi_>] \rangle_{0,>} = u_0 \int \prod_{i=1}^4 \text{đ}^D k_i \text{đ}\!\!\!/ (\sum_i k_i) \left\langle \prod_i^4 (\phi_< + \phi_>)_i \right\rangle_{0,>}$$

Diagramatically, these $2^4 = 16$ terms decompose as in Fig. 5.

The interesting terms are

$$1_3 = -u_0 \underbrace{2}_{\text{symmetry}} \underbrace{N}_{=\delta^{aa}} \int_0^{\Lambda/s} \text{đ}^D k |\phi_<(k)|^2 \int_{\Lambda/s}^{\Lambda} \text{đ}^D q \frac{1}{r_0 + r_2 q^2}$$

$$1_4 = \frac{4 \cdot 1}{2 \cdot N} 1_3,$$

the latter of which has a bigger symmetry factor but no closed flavor index loop. The result through $\mathcal{O}(u)$ is then

$$\boxed{r_0 \to r_0 + \delta r_0 = r_0 + 4u_0(N+2) \int_{\Lambda/s}^{\Lambda} \text{đ}^D q \frac{1}{r_0 + r_2 q^2} + \mathcal{O}(u_0^2)} .$$

r_2 and u are unchanged. RG step ingredients 2 (rescaling: $\tilde{q} \equiv sq$) and 3 (renormalizing: $\tilde{\phi} \equiv \zeta^{-1}\phi_<$) allow us to restore the original action; we can choose $\zeta = s^{1+D/2}$ to keep $\tilde{r}_2 = r_2$.

The second-order-in-u_0 terms are displayed in Fig. 6. The interesting part of the second order bit

$$2 = \frac{1}{2} \langle \mathcal{U}[\phi_<, \phi_>]^2 \rangle_{0,>,\text{connected}}$$

is the correction to $\mathcal{U}[\phi_<]$. There are less interesting bits which are zero or constant or two-loop corrections to the quadratic term. The correction to the quartic term at 2nd order is

$$\delta_2 S_4[\phi_<] = u_0^2(4N + 32) \int_0^{\Lambda/s} \prod_i^4 \left(\text{đ}^D k_i \phi_<(k_i) \right) \text{đ}\!\!\!/ (\sum k_i) f(k_1 + k_2)$$

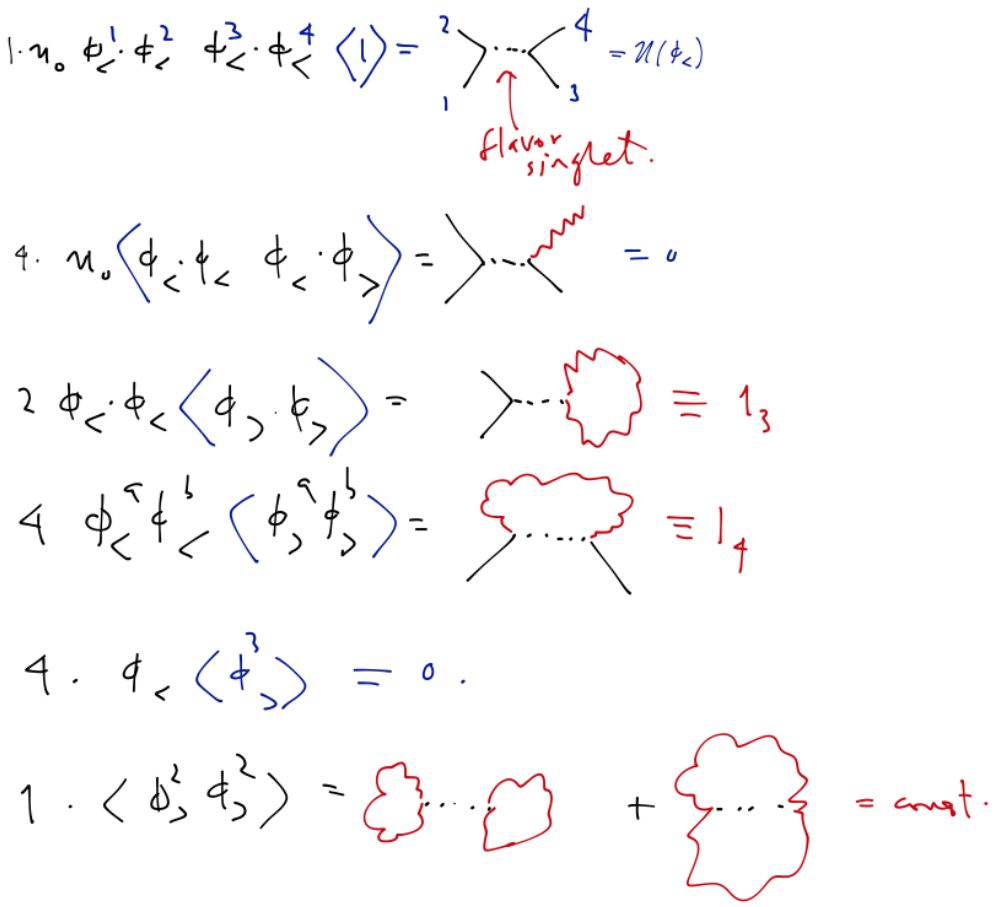


Figure 5: 1st order corrections from the quartic perturbation of the Gaussian fixed point of the O(N) model. Wiggly lines denote propagation of fast modes $\phi_>$, straight lines denote (external) slow modes $\phi_<$. A further refinement of the notation is that we split apart the 4-point vertex to indicate how the flavor indices are contracted; the dotted line denotes a direction in which no flavor flows, i.e. it represents a coupling between the two flavor singlets, $\phi^a\phi^a$ and $\phi^b\phi^b$. The numbers at left are multiplicities with which these diagrams appear. (The relative factor of 2 between 1_3 and 1_4 can be understood as arising from the fact that 1_3 has a symmetry that exchanges the fast lines but not the slow lines, while 1_4 does not.) Notice that closed loops of the wiggly lines represent factors of N, since we must sum over which flavor is propagating in the loop – the flavor of a field running in a closed loop is not determined by the external lines, just like the momentum.

with

$$f(k_1{+}k_2) = \int \mathrm{đ}^D q \frac{1}{(r_0 + r_2 q^2)(r_0 + r_2(k_1+k_2-q)^2)} \simeq \int \mathrm{đ}^D q \frac{1}{(r_0+r_2q^2)^2}\,(1+\mathcal{O}(k_1+k_2))$$

– the bits that depend on the external momenta give irrelevant derivative corrections, like $\phi_<^2 \partial^2 \phi_<^2$. We ignore them.

The full result through $\mathcal{O}(u_0^2)$ is then the original action, with the parameter re-

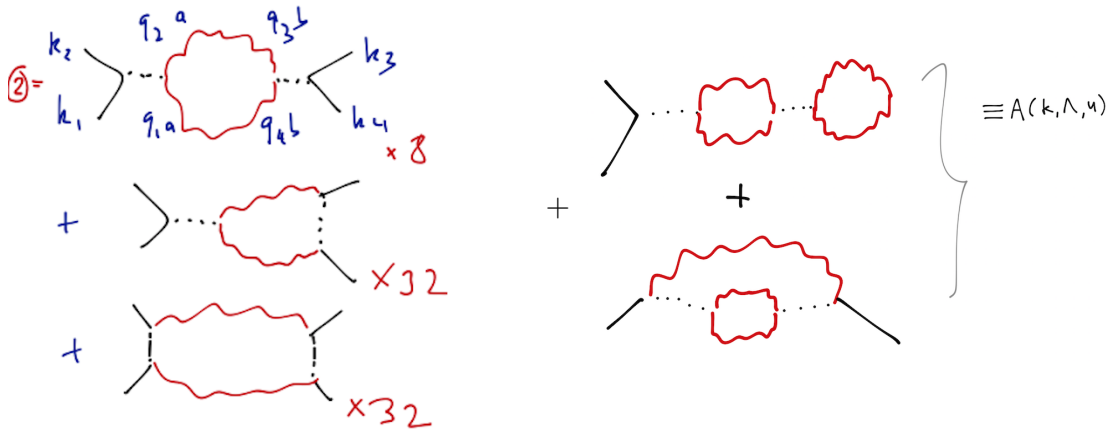


Figure 6: 2nd order corrections from the quartic perturbation of the Gaussian fixed point of the O(N) model. The left column of diagrams are corrections to the quartic interaction, and the right column correct quadratic terms. In fact the top right diagram is independent of the external momentum and hence only corrects r_0; the bottom right diagram (that looks like a sheep) also corrects the kinetic term (along with one more I didn't draw which differs in how the flavor indices are contracted). Notice that the diagram at right has two closed flavor loops, and hence goes like N^2, and it comes with two powers of u_0. You can convince yourself by drawing some diagrams that this pattern continues at higher orders. If you wanted to define a model with large N you should therefore consider taking a limit where $N \to \infty$, $u_0 \to 0$, holding $u_0 N$ fixed. The quantity $u_0 N$ is often called the 't Hooft coupling.

placement

$$\begin{pmatrix} r_2 \\ r_0 \\ u_0 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{r}_2 \\ \tilde{r}_0 \\ \tilde{u}_0 \end{pmatrix} = \begin{pmatrix} s^{-D-2}\zeta^2(r_2 + \delta r_2) \\ s^{-D}\zeta^2(r_0 + \delta r_0) \\ s^{-3D}\zeta^4\,(u_0 + \delta u_0) \end{pmatrix} + \mathcal{O}(u_0^3).$$

The shifts are:

$$\begin{cases} \delta r_2 = u_0^2 \frac{\partial_k^2 A(0)}{r_2} \\ \delta r_0 = 4u_0(N+2)\int_{\Lambda/s}^{\Lambda} \text{đ}^D q \frac{1}{r_0 + r_2 q^2} - A(0)u_0^2 \\ \delta u_0 = -\frac{1}{2}u_0^2(8N+64)\int_{\Lambda/s}^{\Lambda} \text{đ}^D q \frac{1}{(r_0 + r_2 q^2)^2} \end{cases} .$$

Here A is the two-loop ϕ^2 correction that we didn't compute (it contains the leading contribution to the wavefunction renormalization, $A(k) = A(0) + \frac{1}{2}k^2\partial_k^2 A(0) + ...$). We can choose to keep $\tilde{r}_2 = r_2$ by setting

$$\zeta^2 = \frac{s^{D+2}}{1 + u_0^2 \partial_k^2 A(0)/r_2} = s^{D+2}\left(1 + \mathcal{O}(u_0^2)\right).$$

Now let's make the RG step infinitesimal:

$$s = e^{\ell} \simeq 1 + \ell$$

$$\begin{cases} \frac{dr_0}{d\ell} = 2r_0 + \frac{4(N+2)K_D\Lambda^D}{r_0+r_2\Lambda^2}u_0 - Au_0^2 + \mathcal{O}(u_0^3) \\ \frac{du_0}{d\ell} = (4-D)u_0 - \frac{4(N+8)K_D\Lambda^D}{(r_0+r_2\Lambda^2)^2}u_0^2 + \mathcal{O}(u_0^3) \end{cases} \tag{3.17}$$

I defined $K_D \equiv \frac{\Omega_{D-1}}{(2\pi)^D}$.

To see how the previous thing arises, and how the integrals *all went away*, let's consider just the $\mathcal{O}(u_0)$ correction to the mass:

$$\begin{aligned} \tilde{r}_0 = r_0 + \ell\frac{dr_0}{d\ell} &= s^2\left(r_0 + 4u(N+2)\int_{\Lambda/s}^{\Lambda}\frac{\mathrm{d}^D q}{r_0+r_2q^2} + \mathcal{O}(u_0^2)\right) \\ &= (1+2\ell)\left(r_0 + 4u_0(N+2)\frac{\Omega_{D-1}}{(2\pi)^D}\Lambda^D\frac{1}{r_0+r_2\Lambda^2}\ell + \mathcal{O}(u_0^2)\right) \\ &= \left(2r_0 + \frac{4u_0(N+2)}{r_0+r_2\Lambda^2}K_D\Lambda^D\right)\ell + \mathcal{O}(u_0^2). \end{aligned} \tag{3.18}$$

where everything is up to $\mathcal{O}\left(\ell^2\right)$ terms.

Now we are home. The phase diagram for the $\mathsf{O}(N)$ model is in Fig. 7. (3.17) has two fixed points. One is the free fixed point at the origin where nothing happens. The other (Wilson-Fisher) fixed point is at

$$\begin{cases} r_0^\star = -\frac{2u_0^\star(N+2)K_D\Lambda^D}{r_0^\star+r_2\Lambda^2} \overset{D=4-\epsilon}{=} -\frac{1}{2}\frac{N+2}{N+8}r_2\Lambda^2\epsilon + \mathcal{O}(\epsilon^2) \\ u_0^\star = \frac{(r^\star+r_2\Lambda^2)^2}{4(N+8)K_D\Lambda^D}\epsilon \overset{D=4-\epsilon}{=} \frac{1}{4}\frac{r_2^2}{(N+8)K_4}\epsilon + \mathcal{O}(\epsilon^2) \end{cases}$$

which is at positive $u_0^\star$ if $\epsilon > 0$. In the second step we keep only leading order in $\epsilon = 4 - D$, in anticipation of the fact that $u_\star \sim \epsilon$, so that $\epsilon^2 \sim u_0^2$ is of the same order as terms we ignored.

[End of Lecture 6]

Important lessons.

- Elimination of modes does not introduce new singularities into the couplings. At each step of the RG, we integrate out a finite-width shell in momentum space – we are doing integrals that are convergent in the infrared and ultraviolet.
- The RG plays nicely with symmetries. In particular any symmetry *of the regulated model* is a symmetry of the long-wavelength effective action.[13]

[13]The extra qualifier about the regulated model is important because some symmetries of continuum classical field theories cannot be realized as symmetries of well-defined quantum field theories, *i.e.* anomalies exist. It is also possible that no degrees of freedom in the IR theory transform under (some part of) a symmetry.

- Some people conclude from the field theory calculation of the ϕ^4 beta function that ϕ^4 theory "does not exist" or "is trivial", in the sense that *if we demand that this description is valid up to arbitrarily short distances*, we would need to pick $\lambda(\Lambda = \infty) = \infty$ in order to get a finite interaction strength at long wavelengths. You can now see that this is a ridiculous conclusion. Obviously the theory exists in a useful sense. It can easily be defined at short distances (for example) in terms of the lattice model we mentioned at the beginning of this subsection. Similar statements apply to QED.

- The corrections to the mass of the scalar field are of order of the cutoff. This makes it hard to understand how you could arrive in the IR and find that an interacting scalar field has a mass that is much smaller than the cutoff. Yet, there seems to be a Higgs boson with $m \simeq 125$ GeV, and no cutoff on the Standard Model in sight. This is a mystery.

- As Tony Zee says, a more accurate (if less catchy) name than 'renormalization group' for what we've just described would be 'the trick of doing the path integral a little at a time'.

- The term 'renormalization group' is actually used for many rather different things in physics. The Wilsonian framework I've just described makes no reference to perturbation theory (so far) and is extremely general. In high energy physics, the term is often used much more narrowly as a procedure for summing logarithms in perturbation theory, like we did last quarter.

Critical exponents. Now we follow useful strategies for dynamical systems and *linearize* (3.17) near the W-F fixed point:

$$\frac{d}{d\ell}\begin{pmatrix}\delta r_0\\ \delta u_0\end{pmatrix} = M_\star \begin{pmatrix}\delta r_0\\ \delta u_0\end{pmatrix} \tag{3.19}$$

The matrix $M_\star$ is a 2x2 matrix whose eigensystem describes the flows near the fixed point. For the Wilson-Fisher fixed point, it looks like

$$M_\star = \begin{pmatrix} 2 - \frac{N+2}{N+8}\epsilon & \dots \\ \mathcal{O}(\epsilon^2) & -\epsilon \end{pmatrix} .$$

Its eigenvalues (which don't care about the off-diagonal terms because the lower left entry is $\mathcal{O}(\epsilon^2)$ are

$$y_r = 2 - \frac{N+2}{N+8}\epsilon + \mathcal{O}(\epsilon^2) > 0$$

which determines the instability of the fixed point and

$$y_u = -\epsilon + \mathcal{O}(\epsilon^2) < 0 \quad \text{for } D < 4$$

which is a stable direction. An implicit claim I am making here is that if we included any of the other possible operators (like ϕ^6 or $\vec{\nabla}^2\phi\vec{\nabla}^2\phi$) in our action, and therefore had a bigger $K \times K$ matrix $M_\star$ associated to K possible couplings, *all the other eigenvalues would be negative – i.e.* all the other operators are irrelevant at the fixed point.

We now turn to the correlation length exponent, ν. Recall that the correlation length is the length scale above which the relevant perturbation gets big and cuts off the critical fluctuations of the fixed point. As the actual fixed point is approached, this happens at longer and longer scales: ξ diverges at a rate determined by the exponent ν.
We can proceed as follows. First we relate the scaling of the correlation length to the scaling behavior of the relevant perturbation that takes us away from from the fixed point. The latter we will evaluate subsequently in our example. (The way we did this in §3.3 is easier, but I promise this will be instructive.)
Suppose we begin our RG procedure with a perturbation of a fixed-point Hamiltonian by a relevant operator $\mathcal{O}$:

$$H(\xi_1) = H_\star + \delta_1\mathcal{O} \quad .$$

Under a step of the RG, $\xi_1 \to s^{-1}\xi_1$, $\delta_1 \to s^\Delta\delta_1$, where I have defined Δ to be the scaling dimension of the operator $\mathcal{O}$. Then after N steps, $\delta = s^{N\Delta}\delta_1, \xi = s^{-N}\xi_1$. Eliminating s^N from these equations we get the relation

$$\xi = \xi_1 \left(\frac{\delta}{\delta_1}\right)^{-\frac{1}{\Delta}} \tag{3.20}$$

which is the definition of the correlation length exponent ν, and we conclude that $\nu = \frac{1}{\Delta}$.

So y_r determines the correlation length exponent, ν. Its eigenvector is $\begin{pmatrix}\delta r_0 \\ 0\end{pmatrix}$ to $\mathcal{O}(\epsilon^2)$. This makes sense: r_0 is the relevant coupling that must be tuned to stay at the critical point, and the correlation length exponent ν answers the question: how does the correlation length scale with our deviation from the critical point $\delta r_0(0)$? The correlation length can be found as follows (see also the discussion around Eq. (3.20)). ξ is the value of $s = s_1$ at which the relevant operator has turned on by an order-1 amount, *i.e.* by setting $\xi \sim s_1$ when $1 \sim \delta r_0(s_1)$. According to the linearized RG

equation, close to the fixed point, we have $\delta r_0(s) = s^{y_r}\delta r_0(0)$. Therefore

$$1 = \xi^{y_r}\delta r_0(0), \quad \xi = (\delta r_0(0))^{-\nu} \ .$$

This last equality is the definition of the correlation length exponent, ν. Therefore

$$\nu = \frac{1}{y_r} = \left(2\left(1 - \frac{1}{2}\frac{N+2}{N+8}\epsilon\right)\right)^{-1} + \mathcal{O}(\epsilon^2) \simeq \frac{1}{2}\left(1 + \frac{N+2}{2(N+8)}\epsilon\right) + \mathcal{O}(\epsilon^2).$$

The remarkable success of setting $\epsilon = 1$ in this expansion to get answers for $D = 3$ does not really have a good explanation. There is a sense in which the WF fixed point really is close to the Gaussian fixed point. See the references for more details on this; for refinements of this estimate, see Zinn-Justin's book.

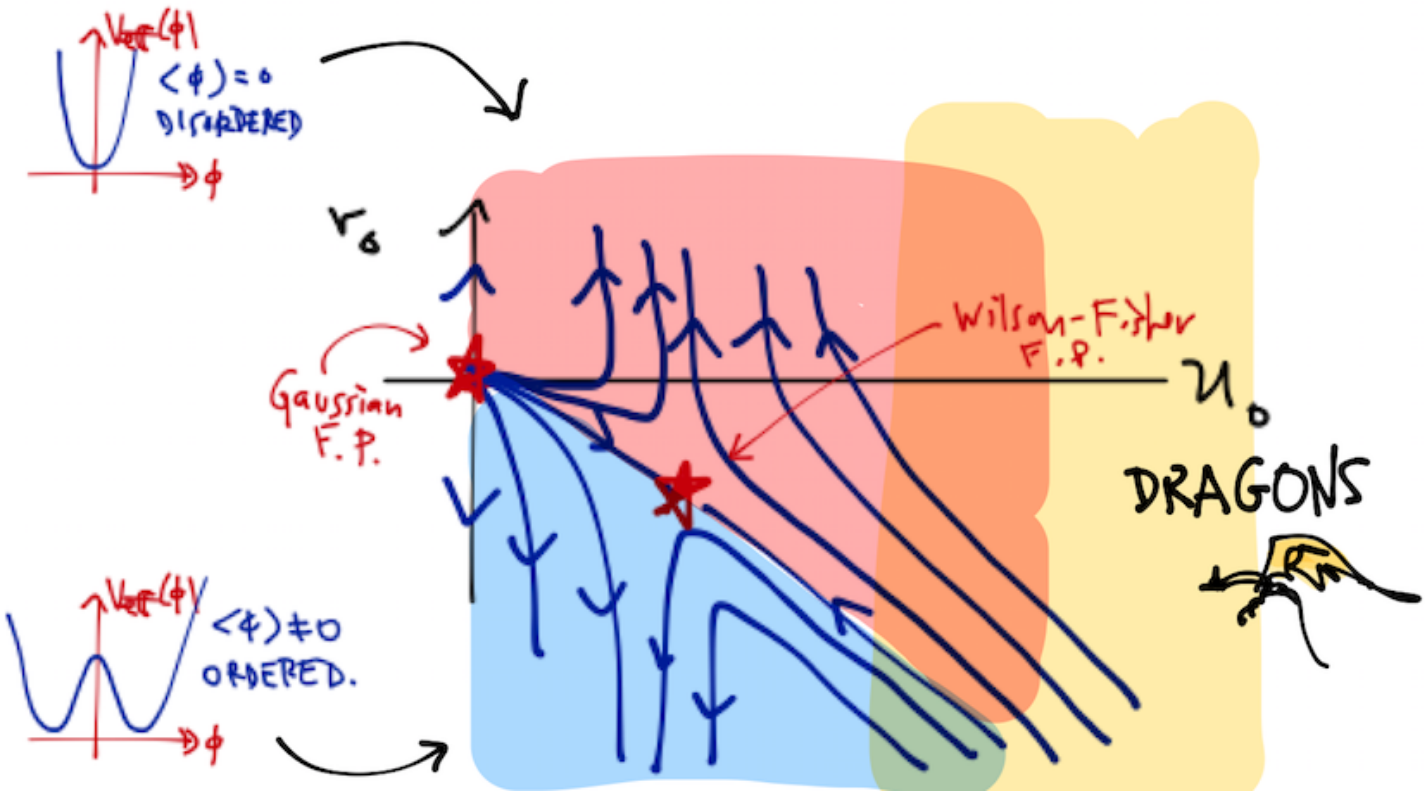


Figure 7: The ϕ^4 phase diagram. If $r_0(\ell = \infty) > 0$, the effective potential for the uniform 'magnetization' has a minimum at the origin; this is the disordered phase, where there is no magnetization. If $r_0(\ell = \infty) = V''_{\rm eff} < 0$, the effective potential has minima away from the origin, and the groundstate breaks the symmetry (here $\phi \to e^{i\theta}\phi$); this is the ordered phase.

The W-F fixed point describes a continuous phase transition between ordered and disordered phases. An external variable (roughly r_0) must be tuned to reach the phase

transition. A physical realization of this is the following: think of our euclidean path integral as a thermal partition function at temperature $1/\beta$:

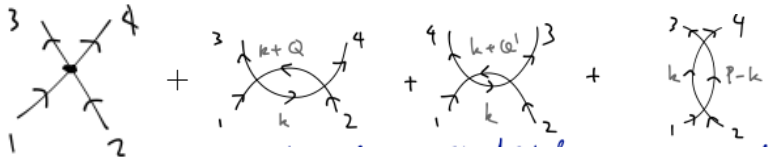
$$Z = \int [D\phi] e^{-\beta H[\phi]} \ ;$$

here we are integrating over thermal fluctuations of classical fields. (Various special cases have special names: $N = 1$ is the Ising model, the case with $\mathsf{O}(2)$ symmetry is called the XY model.) WLOG, we can choose normalize our fields so that the coefficient β determines r_0. The critical value of r_0 then realizes the critical temperature at which this system goes from a high-temperature disordered phase to a low-temperature ordered phase. For this kind of application, $D \leq 3$ is most interesting physically. We will see that the ϵ expansion about $D = 4$ is nevertheless quite useful.

You could ask me what it means for the number of dimensions D to be not an integer. One correct answer is that we have constructed various well-defined functions of continuous D simply by keeping D arbitrary; basically all we need to know is the volume of a D-sphere for continuous D (essentially, one of the dim reg axioms). An also-correct answer that some people (*e.g.* me) find more satisfying is is the following. Suppose we can define our QFT by a discrete model, defined on a discretized space (like in (3.11)). Then we can also put the model on a graph whose *fractal dimension* is not an integer. Evidence that this is a physical realization of QFT in non-integer dimensions is given in [Gefen-Meir-Mandelbrot-Aharony] and [Gefen-Mandelbrot-Aharony]. Some subtle and interesting issues about uniqueness and unitarity of the field theories so defined are raised here and here.

3.4 Comparison with renormalization by counterterms

Is this procedure the same as 'renormalization' in the high-energy physics sense of sweeping divergences under the rug of bare couplings? Let me answer this in the case $N = 2$ of the above calculation. Suppose we impose the renormalization condition that $\Gamma_4(k_4...k_1) \equiv \Gamma(4321)$, the 1PI 4-point vertex, is cutoff independent. Its leading contributions come from the diagrams:



(3.21)

(where now the diagrams denote amputated amplitudes, the arrows indicate flow of scalar charge (since we're studying the case with $\mathsf{O}(2)$ symmetry) and also momentum,

and the integrals run over all momenta up to the cutoff). Clearly there is already a big similarity. In more detail, this is

$$\Gamma(4321) = u_0 - u_0^2 \int_0^\Lambda \text{đ}^D k \left(\frac{1}{(k^2+r_0)(|k+k_3-k_1|^2+r_0)} + \frac{1}{(k^2+r_0)(|k+k_4-k_1|^2+r_0)} + \frac{1}{2}\frac{1}{(k^2+r_0)(|-k+k_1+k_2|^2+r_0)} \right)$$

And in particular, the bit that matters for the running of the coupling is

$$\Gamma(0000) = u_0 - u_0^2 \frac{5}{32\pi^2} \log \frac{\Lambda^2}{r_0} + \mathcal{O}(u_0^3).$$

Demanding that this be independent of the cutoff $\Lambda = e^{-\ell}\Lambda_0$,

$$0 = \partial_\ell \left(\Gamma(0000)\right) = -\Lambda \frac{d}{d\Lambda}\Gamma(0000)$$

gives

$$0 = \frac{du_0}{d\ell} + \frac{5}{16\pi^2}u_0^2 + \mathcal{O}(u_0^3)$$

$$\implies \boxed{\beta_{u_0} = -\frac{5}{16\pi^2}u_0^2} + \mathcal{O}(u_0^3)$$

as before. (The bit that would come from $\partial_\ell u_0^2$ in the second term is of order u_0^3 and so of the order of things we are already neglecting.)

I leave it to you to show that the flow for r_0 that results from demanding that $\langle \phi(k)\phi^\star(k)\rangle$ have a pole at $k^2 = -m^2$ (with m independent of the cutoff) gives the same flow we found above.

It is worth noting that although the continuum field theory perspective with counterterms is less philosophically satisfying, it is often easier for actual calculations than integrating momentum shells, mainly because we can use a convenient regulator like dim reg.

3.5 The operator product expansion and conformal perturbation theory

[Cardy, chapter 5] Some of the information in the beta functions depends on our choice of renormalization scheme and on our choice of regulator. Some of it does not: for example, the topology of the fixed points, and the critical exponents associated with them. Next we discuss a point of view which makes clear some of the data in the beta functions is universal. It also gives a more general perspective on the epsilon expansion and why it works. And it leads to the modern viewpoint on conformal field theory.

Operator product expansion (OPE). Suppose we want to understand a correlation function of local operators like

$$\langle \phi_i(x_1)\phi_j(x_2)\Phi\rangle$$

where $\{\Phi\}$ is a collection of other local operators at locations $\{x_l\}$; suppose that the two operators we've picked out are closer to each other than to any of the others:

$$|x_1 - x_2| \ll |x_{1,2} - x_l|,\ \forall l.$$

Then from the point of view of the collection Φ, $\phi_i\phi_j$ looks like a single local operator. But which one? Well, it looks like some sum over all of them:

$$\langle \phi_i(x_1)\phi_j(x_2)\Phi\rangle = \sum_k C_{ijk}(x_1 - x_2)\,\langle \phi_k(x_1)\Phi\rangle$$

where $\{\phi_k\}$ is some basis of local operators. By Taylor expanding we can move all the space-dependence of the operators to one point, *e.g.*:

$$\phi(x_2) = e^{(x_2-x_1)^\mu \frac{\partial}{\partial x_1^\mu}}\phi(x_1) = \phi(x_1) + (x_2 - x_1)^\mu \partial_\mu \phi(x_1) + \cdots .$$

A shorthand for this collection of statements (for any Φ) is the OPE

$$\phi_i(x_1)\phi_j(x_2) \sim \sum_k C_{ijk}(x_1 - x_2)\phi_k(x_1) \tag{3.22}$$

which is to be understood as an operator equation: true for all states, but only up to collisions with other operator insertions (hence the $\sim$ rather than $=$).

This is an attractive concept, but is useless unless we can find a good basis of local operators. At a fixed point of the RG, it becomes much more useful, because of scale invariance. This means that we can organize our operators according to their scaling dimension. Roughly it means two wonderful simplifications:

- We can find a special basis of operators $\{\mathcal{O}_i\}$ where

$$\langle \phi_i(x)\phi_j(0)\rangle_\star = \frac{\delta_{ij}}{r^{2\Delta_i}} \tag{3.23}$$

(here, for the simple case of scalar operators) where Δ_i is the *scaling dimension* of ϕ_i. The $\star$ indicates that this correlator is evaluated at the fixed point. (3.23) defines the multiplicative normalizations of the ϕ_k. This basis is the same as the operators multiplying eigenvectors of the scaling matrix $M_\star$ in (3.19), and the Δ_k are related to the eigenvalues (by $y_k = d - \Delta_k$).

Given (3.23), we can order the contributions to $\sum_k$ in the OPE (3.22) by increasing Δ_k, which means smaller contributions to $\langle\phi\phi\Phi\rangle$.

- Further, the form of C_{ijk} is fixed up to a number. Again for scalar operators,

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \sim \sum_k \frac{c_{ijk}}{|x_1-x_2|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(x_1) \tag{3.24}$$

where c_{ijk} is now a set of pure numbers, the *OPE coefficients* (or *structure constants*).

The structure constants c_{ijk} are universal data about the fixed point: they transcend perturbation theory. How do I know this? Because they can be computed from correlation functions of scaling operators *at the fixed point*: multiply the BHS of (3.24) by $\mathcal{O}_k(x_3)$ and take the expectation value at the fixed point:

$$\begin{aligned}\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle_\star &\overset{(3.24)}{=} \sum_{k'} \frac{c_{ijk'}}{|x_1-x_2|^{\Delta_i+\Delta_j-\Delta_k}} \langle \mathcal{O}_{k'}(x_1)\mathcal{O}_k(x_3)\rangle_\star \\ &\overset{(3.23)}{=} \frac{c_{ijk}}{|x_1-x_2|^{\Delta_i+\Delta_j-\Delta_k}} \frac{1}{|x_1-x_3|^{2\Delta_k}}\end{aligned} \tag{3.25}$$

(There is a better way to organize the RHS here, but let me not worry about that here.) The point here is that by evaluating the LHS at the fixed point, with some known positions $x_{1,2,3}$, we can extract c_{ijk}.

Confession: I (and Cardy) have used a tiny little extra assumption of *conformal invariance* to help constrain the situation here. It is difficult to have scale invariance without conformal invariance, so this is not a big loss of generality. We can say more about this later but for now it is a distraction.

[End of Lecture 7]

Conformal perturbation theory. Suppose we find a fixed point of the RG, $H_\star$. (For example, it could be the gaussian fixed point of N scalar fields.) Let us study

its neighborhood. (For example, we could seek out the nearby interacting Wilson-Fisher fixed point in $D < 4$ in this way.) Let's think about the equilibrium probability distribution

$$\rho = e^{-H}/Z$$

with $Z = \mathrm{tr} e^{-H}$. Here we set the temperature equal to 1 and include it in the couplings, so H is dimensionless. We can parametrize it as

$$H = H_\star + \sum_x \sum_i g_i a^{\Delta_i} \mathcal{O}_i(x) \tag{3.26}$$

where a is the short distance cutoff (*e.g.* the lattice spacing), and $\mathcal{O}_i$ has dimensions of length$^{-\Delta_i}$ as you can check from (3.23). So g_i are de-dimensionalized couplings which we will treat as small and expand in[14].

Then

$$\begin{aligned}
\rho &= Z_\star/Z \rho_\star e^{-\sum_x \sum_i g_i a^{\Delta_i} \mathcal{O}_i(x)} \\
&\overset{\sum_x \simeq \frac{1}{a^d}\int d^d r}{\simeq} Z_\star/Z \rho_\star \left(1 - \sum_i g_i \int \mathcal{O}_i(x) \frac{d^d x}{a^{d-\Delta_i}} \right. \\
&\quad + \frac{1}{2} \sum_{ij} g_i g_j \int \frac{d^d x_1 d^d x_2}{a^{2d-\Delta_i-\Delta_j}} \mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \\
&\quad \left. - \frac{1}{3!} \sum_{ijk} g_i g_j g_k \int\int\int \frac{\prod_{a=1}^3 d^d x_a}{a^{3d-\Delta_i-\Delta_j-\Delta_k}} \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3) + ... \right)
\end{aligned}$$

where $\rho_\star \equiv e^{-H_\star}/Z_\star$. Comments:

- We used the fact that near the fixed point, the correlation length is much larger than the lattice spacing to replace $\sum_x \simeq \frac{1}{a^d}\int d^d x$.
- There is still a UV cutoff on all the integrals – the operators can't get within a lattice spacing of each other: $|x_i - x_j| > a$.
- The integrals over space are also IR divergent; we cut this off by putting the whole story in a big box of size L. This is a physical size which should be RG-independent.
- The structure of this expansion does *not* require the initial fixed point to be a free fixed point; it merely requires us to be able to say something about the correlation functions. As we will see, the OPE structure constants c_{ijk} are quite enough to learn something.

[14] Don't be put off by the word 'conformal' in the name 'conformal perturbation theory' – it just means doing perturbation theory about a general fixed point, not necessarily the gaussian one.

Now let's do the RG dance. We'll take the high-energy point of view here: while preserving Z, we make an infinitesimal change of the cutoff,

$$a \to ba = (1+\ell)a, \;\; 0 < \delta l \ll 1 \; .$$

The price for preserving Z is letting the couplings run $g_i = g_i(b)$. Where does a appear?
(1) in the integration measure factors $a^{d-\Delta_i}$.
(2) in the cutoffs on $\int dx_1 dx_2$ which enforce $|x_1 - x_2| > a$.
(3) *not* in the IR cutoff – L is fixed during the RG transformation, independent of b .
The leading-in-ℓ effects of (1) and (2) are additive and so may be considered separately:

$$(1) \qquad \tilde{g}_i = (1+\ell)^{d-\Delta_i} g_i \simeq g_i + (d-\Delta_i) g_i \ell \equiv g_i + \delta_1 g_i$$

The effect of (2) first appears in the $\mathcal{O}(g^2)$ term, the *change* in which is

$$(2) \qquad \sum_{i,j} g_i g_j \int_{|x_1-x_2|\in(a,a(1+\ell))} \frac{d^d x_1 d^d x_2}{a^{2d-\Delta_i-\Delta_j}} \underbrace{\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle_\star}_{=\sum_k c_{ijk}|x_1-x_2|^{\Delta_k-\Delta_i-\Delta_j}\langle\mathcal{O}_k\rangle_\star}$$
$$= \ell \sum_{ijk} g_i g_j c_{ijk} \Omega_{d-1} a^{-2d+\Delta_k} \int \langle \mathcal{O}_k \rangle_\star$$

So this correction can be absorbed by a change in g_k according to

$$\delta_2 g_k = -\ell \frac{1}{2} \Omega_{d-1} \sum_{ij} c_{ijk} g_i g_j + \mathcal{O}(g^3)$$

where the $\mathcal{O}(g^3)$ term comes from triple collisions which we haven't considered here. Therefore we arrive at the following expression for evolution of couplings: $\frac{dg}{d\ell} = (\delta_1 g + \delta_2 g)/\ell$

$$\frac{dg_k}{d\ell} = (d-\Delta_k) g_k - \frac{1}{2}\Omega_{d-1} \sum_{ij} c_{ijk} g_i g_j + \mathcal{O}(g^3) \; . \tag{3.27}$$

[15] At $g = 0$, the linearized solution is $dg_k/g_k = (d-\Delta_k)d\ell \implies g_k \sim e^{(d-\Delta_k)\ell}$ which translates our understanding of relevant and irrelevant at the initial fixed point in terms of the scaling dimensions Δ_k: g_k is relevant if $\Delta_k < d$.

[15] To make the preceding discussion we considered the partition function Z. If you look carefully you will see that in fact it was not really necessary to take the expectation values $\langle\rangle_\star$ to obtain the result (3.27). Because the OPE is an operator equation, we can just consider the running of the operator e^{-H} and the calculation is identical. A reason you might consider doing this instead is that expectation values of scaling operators on the plane actually vanish $\langle \mathcal{O}_i(x)\rangle_\star = 0$. However, if we consider the partition function in finite volume (say on a torus of side length L), then the expectation values of scaling operators are not zero. You can check these statements explicitly for the normal-ordered operators at the gaussian fixed point introduced below. Thanks to Sridip Pal for bringing these issues to my attention.

(3.27) says that to find the interaction bit of the beta function for g_k, we look at all the OPEs between operators in the perturbed hamiltonian (3.26) which produce g_k on the RHS.

Let's reconsider the Ising model from this point of view:

$$\begin{aligned} H &= -\frac{1}{2}\sum_{x,x'} J(x-x')S(x)S(x') - h\sum_x S(x) \\ &\simeq -\frac{1}{2}\sum_{x,x'} J(x-x')S(x)S(x') - h\sum_x S(x) + \lambda\sum_x \left(S(x)^2-1\right)^2 \\ &\simeq \int d^dx \left(\frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + r_0 a^{-2}\phi^2 + u_0 a^{d-4}\phi^4 + h a^{-1-d/2}\phi\right) \end{aligned} \tag{3.28}$$

In the first step I wrote a lattice model of spins $S = \pm 1$; in the second step I used the freedom imparted by universality to relax the $S = \pm 1$ constraint, and replace it with a potential which merely discourages other values of S; in the final step we took a continuum limit.

In (3.28) I've temporarily included a Zeeman-field term hS which breaks the $\phi \to -\phi$ symmetry. Setting it to zero it stays zero (*i.e.* it will not be generated by the RG) because of the symmetry. This situation is called *technically natural*.

Now, consider for example as our starting fixed point the Gaussian fixed point, with

$$H_{\star,0} = \int d^dx \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 .$$

Since this is quadratic in ϕ, all the correlation functions (and hence the OPEs, which we'll write below) are determined by Wick contractions using

$$\langle\phi(x_1)\phi(x_2)\rangle_{\star,0} = \frac{\mathcal{N}}{|x_1-x_2|^{d-2}}.$$

It is convenient to rescale the couplings of the perturbing operators by $g_i \to \frac{2}{\Omega_{d-1}} g_i$ to remove the annoying $\Omega_{d-1}/2$ factor from the beta function equation. Then the RG equations (3.27) say

$$\begin{cases} \frac{dh}{d\ell} = & (1+d/2)h - \sum_{ij} c_{ijh} g_i g_j \\ \frac{dr_0}{d\ell} = & 2r_0 - \sum_{ij} c_{ijr_0} g_i g_j \\ \frac{du_0}{d\ell} = & \epsilon u_0 - \sum_{ij} c_{iju_0} g_i g_j \end{cases}$$

So we just need to know a few numbers, which we can compute by doing Wick contractions with free fields.

Algebra of scaling operators at the Gaussian fixed point. It is convenient to choose a basis of *normal-ordered* operators, which are defined by subtracting out

their self-contractions. The self-contractions are annoying both because they are more terms, and also because they are infinite. That is, let

$$\mathcal{O}_n \equiv :\phi^n: := \phi^n - (\text{self-contractions})$$

so that $\langle :\phi^n: \rangle = 0$, and specifically

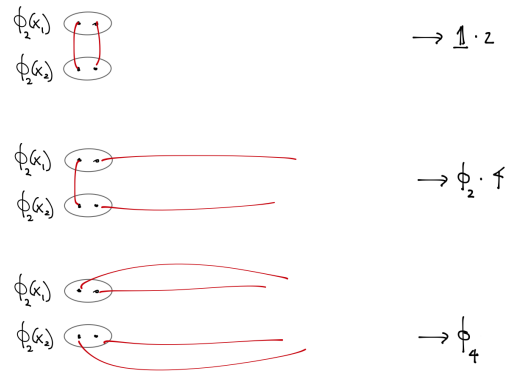
$$\mathcal{O}_2 = \phi^2 - \langle \phi^2 \rangle , \quad \mathcal{O}_4 = \phi^4 - 3 \langle \phi^2 \rangle \phi^2 . \tag{3.29}$$

Note that the contractions $\langle \phi^2 \rangle$ discussed here are defined on the plane. They are in fact quite UV sensitive and require some short-distance cutoff. When I write $\langle \phi^2 \rangle$, you can imagine that I am separating the locations of the two operators by some cutoff ϵ, so $\langle \phi^2 \rangle = \langle \phi(\epsilon)\phi(0) \rangle = \epsilon^{2-d}$; the goal is to subtract off all the bits which are singular as $\epsilon \to 0$, and then take the limit. This amounts to a shift in couplings $r_0 \to r_0 + 3u \langle \phi^2 \rangle_\star$.

To compute their OPEs, consider a correlator of the form above:

$$\langle \mathcal{O}_n(x_1)\mathcal{O}_m(x_2)\Phi \rangle$$

We do wick contractions with the free propagator, but the form of the propagator doesn't matter for the beta function, only the combinatorial factors. If we can contract all the operators making up $\mathcal{O}_n$ with those of $\mathcal{O}_m$, then what's left looks like the identity operator to Φ; that's the leading term, if it's there, since the identity has dimension 0, the lowest possible. More generally, some number of ϕs will be left over and will need to be contracted with bits of Φ to get a nonzero correlation function. For example, the contributions to $\mathcal{O}_2 \cdot \mathcal{O}_2$ are depicted at right. In determining the combinatoric factors, note that permuting the legs on the right does not change anything, they are identical.



The part of the result we'll need (if we set $h = 0$) can be written as (omitting the implied factors of $|x_1 - x_2|^{\Delta_i + \Delta_j - \Delta_k}$ necessary to restore dimensions):

$$\begin{cases} \mathcal{O}_2\mathcal{O}_2 & \sim 2\mathbb{1} + 4\mathcal{O}_2 + \mathcal{O}_4 + \cdots \\ \mathcal{O}_2\mathcal{O}_4 & \sim 12\mathcal{O}_2 + 8\mathcal{O}_4 + \cdots \\ \mathcal{O}_4\mathcal{O}_4 & \sim 24\mathbb{1} + 96\mathcal{O}_2 + 72\mathcal{O}_4 + \cdots \end{cases}$$

Notice that the *symmetric* operators (the ones we might add to the action preserving the symmetry) form a closed subalgebra of the operator algebra.

At $h = 0$, the result is (the $N = 1$ case of the result in §3.3)

$$\begin{cases} \frac{dr_0}{d\ell} = & 2r_0 - 4r_0^2 - 2 \cdot 12 r_0 u_0 - 96 u_0^2 \\ \frac{du_0}{d\ell} = & \epsilon u_0 - r_0^2 - 2 \cdot 8 r_0 u_0 - 72 u_0^2 \end{cases}$$

and so the ($N = 1$) WF fixed point occurs at $u_0 = u_0^\star = \epsilon/72, r_0 = \mathcal{O}(\epsilon^2)$.

The difference in numerical numbers in the values of the fixed point couplings relative to our previous calculation comes from our different parametrization (recall that we shifted the definition of r when we switched to a basis of normal-ordered operators in (3.29)) – that is not universal information. We can extract something universal and independent of our choices as follows. Linearizing the RG flow about the *new* fixed point,

$$\frac{dr_0}{d\ell} = 2r_0 - 24 u_0^\star r_0 + \cdots$$

gives

$$\frac{dr_0}{r_0} = (2 - \frac{24}{72}\epsilon) d\ell \quad \Longrightarrow \quad r_0 \sim e^{(2-\frac{24}{72}\epsilon)\ell} \equiv \left(e^\ell\right)^{\frac{1}{\nu}}$$

which gives $\nu = \frac{1}{2} + \frac{1}{12}\epsilon + \mathcal{O}(\epsilon^2)$.

3.6 Comments on critical exponents

[Zinn-Justin, chapter 25, Peskin, chapter 12.5, Stone, chapter 16, Cardy, and the classic Kogut-Wilson]

Recall that the Landau-Ginzburg mean field theory made a (wrong) prediction for the critical exponents at the Ising transition:

$$\langle M \rangle \sim (T_c - T)^\beta \qquad \text{for } T < T_c, \quad \xi \sim (T_c - T)^{-\nu}$$

with $\beta_{MFT} = \frac{1}{2}$, $\nu_{MFT} = \frac{1}{2}$. This answer was wrong (*e.g.* for the Ising transition in (euclidean) $D = 3$, which describes uniaxial magnets (spin is ± 1) or the liquid-gas critical point) because it simply ignored the effects of fluctuations of the modes of nonzero wavelength, *i.e.* the δL bit in (3.8). I emphasize that these numbers are worth getting right because they are universal – they are properties of a fixed point, which are completely independent of any microscopic details.

Now that we have learned to include the effects of fluctuations at all length scales on the long-wavelength physics, we can do better. We've done a calculation which includes

fluctuations at the transition for an N-component magnet with an $\mathsf{O}(N)$ symmetry that rotates them into each other. The answers had some interesting dependence on N. The mean field theory prediction for the exponents is the same as for the Ising case (recall that we also did the calculation for a magnetization field with an arbitrary number N of components, and in fact the mean field theory prediction is independent of $N \geq 1$).

In general there are many *scaling relations* between various critical exponents, which can be understood beginning from the effective action, and were understood before the correct calculation of the exponents. So not all of them are independent. For illustration, we will briefly discuss two independent exponents.

Order parameter exponent, η. The simplest critical exponent to understand from what we've done so far is η, the exponent associated with the anomalous dimension of the field ϕ itself. (It is not the easiest to actually calculate, however.) This can be defined in terms of the (momentum-space) amputated two-point function of ϕ (that is, $\Gamma_2(p) = 1/\tilde{G}(p)$) as

$$\Gamma_2(p) \overset{\xi^{-1} \ll p \ll \Lambda}{\simeq} \left(\frac{p}{\Lambda}\right)^{2-\eta}$$

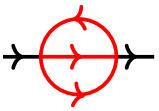
where ξ is the correlation length and Λ is the UV cutoff. This looks a bit crazy – at nonzero η, the full propagator has a weird power-law singularity instead of a $\frac{1}{p^2-m^2}$, and in position space it is a power law $G_2(x) \sim \frac{1}{|x|^{D-2+\eta}}$, instead of an exponential decay. An example where all the details can be understood is the operator $e^{\mathbf{i}\alpha X}$ made from the massless scalar field X in 1+1 dimensions (see the homework).

$\Gamma_2(p)$ is the 1PI momentum space 2-point vertex, *i.e.* the kinetic operator. We can interpret a nonzero η as saying that the dimension of ϕ, which in the free theory was $\Delta_0 = \frac{2-D}{2}$, has been modified by the interactions to $\Delta = \frac{2-D}{2} - \eta/2$. $\eta/2$ is the *anomalous dimension* of ϕ. Quantum mechanics violates (naive) dimensional analysis; it must, since it violates classical scale invariance. Of course (slightly more sophisticated) dimensional analysis is still true – the extra length scale is the UV cutoff, or some other scale involved in the renormalization procedure.

But how can this happen in perturbation theory? Consider physics near the gaussian fixed point, where η must be small, in which case we can expand:

$$\Gamma_2(p) \overset{\xi^{-1} \ll p \ll \Lambda, \eta \ll 1}{\simeq} \left(\frac{p}{\Lambda}\right)^2 \left(e^{-\eta \log(p/\Lambda)}\right) = \left(\frac{p}{\Lambda}\right)^2 (1 - \eta \log(p/\Lambda) + ...)$$

It comes from the wavefunction renormalization.
In the ϕ^4 theory, $\eta = 0$ at one loop. The leading correction to η comes from the 'sunrise' (or 'eyeball') diagram at right, at two loops. (I draw the $\phi^>$ lines in red and the $\phi^<$ lines in black.) So in this model, $\eta \sim g_\star^2 \sim \epsilon^2$.

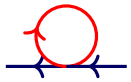


[Ma, *Modern Theory of Critical Phenomena*, p. 209] Tarun Grover gave me a hard time for not emphasizing enough the fact that at the Wilson-Fisher fixed point, the anomalous dimension of the order-parameter field is nonzero – it is not a free field. He called it "the central result." So here is another perspective on this calculation, which allows us to get the actual value. It is an example where it is easier to study field theory in real space, rather than momentum space.

Return to our expression for the correction to the effective action for the slow modes from integrating out the fast modes from the cumulant expansion, (3.16). But now write δS in position space:

$$\delta S[\phi^<] = \frac{u_0}{4}\left\langle \int d^D x |\phi^< + \phi^>|^4(x) \right\rangle_{>,0} - \frac{(u_0/4)^2}{2}\left\langle \int d^D x |\phi^< + \phi^>|^4(x) \int d^D y |\phi^< + \phi^>|^4(y) \right\rangle_{>,0,c} + ...$$

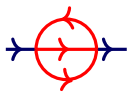
where the subscript c indicates connected. We will look for terms in this expansion that look like $\int d^D x \partial\phi_<^2$. The bit from the $\mathcal{O}(u_0)$ term is of the form



$$= u_0 \int dx \phi_<(x)^2 G_>(x-x)$$

and so doesn't give a correction to the kinetic term, only to the mass, as expected.

The terms with two slow modes involve six fast modes, and have exactly the form of the eyeball diagram above (but now interpreted as a position-space diagram)[16]:

$$\delta S[\phi^<] \ni$$  $$= -\frac{n+2}{2} u_0^2 \int d^D x \int d^D y \phi^<(x) G_>(x-y)^3 \phi^<(y). \qquad (3.30)$$

Since the fast modes involve only small wavelengths, their propagator $G_>$ must be short-ranged; therefore we can Taylor expand

$$\phi^<(y) = \phi^<(x) + \vec{r}\cdot\vec{\nabla}\phi^<(x) + \frac{1}{2}(\vec{r}\cdot\vec{\nabla})^2\phi^<(x) + \cdots, \qquad (3.31)$$

where $r \equiv y - x$. This gives

$$\delta S \ni -\frac{1}{2}u_0^2 \int d^D x \phi^<(x) \int d^D y G_>(r)^3 \left(\underbrace{\phi^<(x)}_{\text{mass correction}} + \underbrace{\vec{r}\cdot\vec{\nabla}\phi^<(x)}_{\text{vanishes by rotation symmetry}} + \frac{1}{2}(\vec{r}\cdot\vec{\nabla})^2\phi^<(x) + \cdots \right). \qquad (3.32)$$

[16] Although I drew the diagrams appropriate to the XY model, in this calculation, I have not been careful about the numerical prefactor, which depends on the number of components n of the order parameter field. This prefactor directly determines the numerical factor in η at the WF fixed point, which is a universal constant of nature, like π or e, and therefore worth determining. It's my factors of two that you should watch out for.

[End of Lecture 8]

So this last term is what we are looking for, and it takes the form (after an IBP)

$$+\int d^D x \; r_2 \left(\partial\phi^<(x)\right)^2 \delta Z$$

with

$$\delta Z = r_2^{-1} \frac{n+2}{4} u_0^2 \int d^D r \frac{r^2}{D} G_>(r)^3 \tag{3.33}$$

where we used rotation invariance: $\int d^D r r^i r^j f(|r|) = \int d^D r \frac{r^2 \delta^{ij}}{D} f(|r|)$.

Here we can evaluate $G_>$ directly in $D = 4$ (since the differences will be in the $\mathcal{O}(\epsilon^3)$ slush):

$$r_2 G_>(x-y) \equiv \int_{\Lambda/b}^{\Lambda} \text{đ}^D k \frac{e^{\mathbf{i}k(x-y)}}{k^2} \tag{3.34}$$

$$= \int_{\Lambda/b}^{\Lambda} k^{4-1-2} dk \frac{\Omega_2}{(2\pi)^4} \underbrace{\int_{-1}^{1} d\theta \sin^2\theta e^{\mathbf{i}kr\cos\theta}}_{=\frac{\pi J_1(kr)}{kr}} \tag{3.35}$$

$$= \frac{\Omega_2}{(2\pi)^4} \frac{\pi}{r} \int_{\Lambda/b}^{\Lambda} dk J_1(kr) \tag{3.36}$$

$$= \frac{4\pi}{(2\pi)^4} \frac{\pi}{r^2} \left(J_0(r\Lambda/b) - J_0(r\Lambda)\right) = \frac{1}{4\pi^2 r^2} \left(J_0(r\Lambda/b) - J_0(r\Lambda)\right) \tag{3.37}$$

where J_n are Bessel functions. (Note that the RG parameter b here is what I called s earlier, sorry .)

So we have

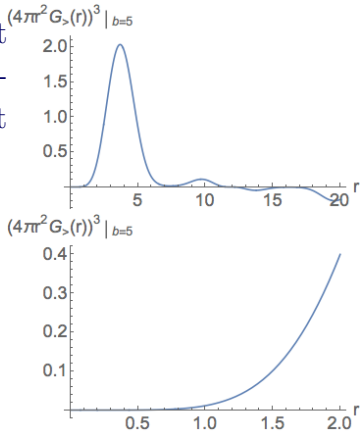
$$\delta Z = r_2^{-1} \frac{(n+2)u_0^2}{4D} \left(\frac{1}{4\pi}\right)^3 \Omega_3 \int_0^\infty dr r^{D-1} r^2 G_>(r)^3 \tag{3.38}$$

$$\overset{D\to 4}{=} r_2^{-4} \frac{(n+2)u_0^2}{16} \left(\frac{1}{4\pi^2}\right)^3 2\pi^2 \int_0^\infty \frac{dr}{r} \left(J_0(r\Lambda/b) - J_0(r\Lambda)\right)^3 \tag{3.39}$$

$$= r_2^{-4} \frac{(n+2)u_0^2}{2^{10}\pi^4} \int_0^\infty \frac{dr}{r} \left(J_0(r/b) - J_0(r)\right)^3 \tag{3.40}$$

where $r \equiv r\Lambda$ and the cutoff dependence drops out.

To find the dependence on b, again the crucial idea is that the $G_>(r)$ is short-ranged; the BesselJ oscillates, but the envelope decays when its argument is of order a few. To get the idea, treat $J_0(r) \sim \theta(1-r)$, $J_0(r/b) \sim \theta(b-r)$, so



$$(4\pi r^2 G_>(r))^3 = (J_0(r/b) - J_0(r))^3 \sim \theta(r-1)\theta(b-r)$$

and

$$\int_0^\infty \frac{dr}{r} (J_0(r/b) - J_0(r))^3 \sim \int_1^b \frac{dr}{r} = \ln b.$$

The oscillations give additive corrections to this answer, which are independent of b[17]. Therefore, we find

$$\delta Z = r_2^{-4} c u_0^2 \ln b,$$

where c is a numerical number made of 2s and πs. The anomalous dimension of ϕ is then $\eta = \partial_{\ln b} \delta Z = c r_2^{-4} u_0^2$.

Universal scaling functions and critical exponents. [Cardy §3.5] Let's find the singular part of the free energy density $f = -\frac{1}{N} \log Z$. Recall that we've set the temperature to 1 by absorbing it into the couplings, so that the partition function is

$$Z = \mathrm{tr}_s e^{-H_K(s)} = \mathrm{tr}_{s'} e^{-H_{K'}(s')} \tag{3.43}$$

where the primes denote the result of some RG step, whereby the N sites are reduced to $N' = b^{-D} N$ sites (blocks, $b > 1$). K and K' stand for the collection of couplings before and after this transformation. In the course of this transformation, various factors in the measure can be produced and the BHS of (3.43) are related by

$$e^{-Nf(K)} = C(K) e^{-N' f(K')} \tag{3.44}$$

where $C(K)$ is the garbage from the measure. The free energies before and after the RG step are related by

$$f(K) = g(K) + b^{-D} f(K') \tag{3.45}$$

[17] Actually, Mathematica can do the integrals

$$\int_0^\infty \frac{dr}{r} (J_0(r/b) - J_0(r)) = \ln b \tag{3.41}$$

$$\int_0^\infty \frac{dr}{r} (J_0(r/b) - J_0(r))^2 = \ln b \,, \tag{3.42}$$

and they both give *exactly* $\ln b$, but it doesn't like higher powers. The latter integral is the position space expression for the diagrams which correct u_0 at one loop, such as:  .

with $g(K) = \log(C(K))/N$. The interesting, singular part of the free energy density therefore satisfies the RG equation

$$f_s(K) = b^{-D} f_s(K'). \tag{3.46}$$

Let's look at this equation for couplings near (but not at) the Wilson-Fisher fixed point. Then it says

$$\begin{aligned} f_s(h, \delta r, \delta u ...) &= b^{-D} f_s(b^{y_h} h, b^{y_r} \delta r, b^{y_u} \delta u, ...) && (3.47) \\ &= b^{-nD} f_s(b^{n y_h} h, b^{n y_r} \delta r, b^{n y_u} \delta u, ...) && (3.48) \end{aligned}$$

as a function of the deviations from the WF fixed point in various directions ($\delta h = h$). In the second step we iterated the transformation n times. Some of the couplings (the relevant ones) are growing as n increases, and this will take us far from the fixed point. So we should stop before that happens. Pick some arbitrary fixed small-enough value $r_0 = |b^{y_r} \delta r|$ at which to stop. Then, ignoring the irrelevant couplings (which includes u for $D < 4$)

$$f_s(h, \delta r) = \left| \frac{\delta r}{r_0} \right|^{D/y_r} f_s(r_0, h(\delta r/r_0)^{-y_h/y_r}) \equiv \left| \frac{\delta r}{r_0} \right|^{D/y_r} \Phi\left(\frac{h/h_0}{(\delta r/r_0)^{y_h/y_t}} \right). \tag{3.49}$$

Looking at the LHS, you can see that this quantity is actually independent of r_0. $\Phi(x)$ is a universal scaling function, a whole function's worth of universal data. In particular it encodes much of the zoo of critical exponents, as follows.

The specific heat exponent α is determined by

$$c_V \sim \partial_{\delta r}^2 f|_{h=0} \sim |\delta r|^{D/y_r - 2} \equiv |\delta r|^{-\alpha}. \tag{3.50}$$

The spontaneous magnetization exponent β is determined by

$$M \sim \partial_h f|_{h=0} \sim (-\delta r)^{D - y_h/y_r} \equiv (-r)^{\beta}. \tag{3.51}$$

The susceptibility exponent γ is determined by

$$\chi = \partial_h^2 f|_{h=0} \equiv |\delta r|^{-\gamma}. \tag{3.52}$$

The relation between magnetization and applied field defines δ:

$$M \sim h^{1/\delta}, \delta = \frac{y_h}{D - y_h}. \tag{3.53}$$

Because they can be extracted from the scaling function, these exponents satisfy various relations such as $\alpha + 2\beta + \gamma = 2, \alpha + \beta(1 + \delta) = 2$.

Logs at the upper critical dimension. In the above, we could set the irrelevant perturbation $\delta u = 0$. At the upper critical dimension the story is different. [Cardy §5.6] When $D = 4$, the RG equations are (here $b = e^\ell$)

$$\frac{du}{d\ell} = -72u^2 + \cdots \tag{3.54}$$

$$\frac{dr}{d\ell} = 2r - 24ur + \cdots . \tag{3.55}$$

The first equation says that u is marginally irrelevant – although classically u is marginally, the quantum corrections (the u^2 term) pushes it to zero in the IR. But it does so quite slowly. Rather than an exponential behavior in ℓ, we get a power

$$u(\ell) = \frac{u(0)}{1 + 72u(0)\ell}. \tag{3.56}$$

Applying our RG equation for the singular part of the self-energy (3.47) to $D = 4$ we have

$$f_s(r, u) = e^{-D\ell} f_s(r(\ell), u(\ell)). \tag{3.57}$$

We might be tempted to set $u = 0$ from the beginning, since it is irrelevant. But then we would find that for $r < 0$, the world explodes, *i.e.* the field runs off to infinity. It is therefore called a *dangerously irrelevant* variable. So we have to keep a small nonzero value of u. As before, let's choose some r_0 not too big at which to stop the RG flow, *i.e.* we pick $\ell = \ell_0$ such that $r(\ell_0) = r_0$. Therefore

$$f_s(r, u(0)) = e^{-4\ell_0} f_s(r_0, u(\ell_0)). \tag{3.58}$$

This is the analog of the expression with the scaling function above.

We can find the relationship between r_0 and ℓ_0 as follows. Using $\frac{dr}{d\ell} \equiv \beta_r$, we have

$$\ln \frac{r_0}{r} = \int_r^{r_0} \frac{dr'}{r'} = \int_0^{\ell_0} \frac{d\ell}{r\beta_r} = \int_0^{\ell_0} d\ell \left(2 - \frac{24u(0)}{1 + 72u(0)\ell}\right) \tag{3.59}$$

$$= 2\ell_0 - \frac{1}{3}\ln\left(1 + 72u(0)\ell_0\right) \ . \tag{3.60}$$

Then we can solve this equation for ℓ_0 iteratively:

$$\ell_0 \simeq \frac{1}{2}\ln\frac{r_0}{r} + \frac{1}{6}\ln\left(1 + 36u(0)\ln\frac{r_0}{r}\right). \tag{3.61}$$

If we pick r_0 big enough, we can use mean field theory to find

$$f_s(r_0, u(\ell_0)) \sim \frac{r_0^2}{u(\ell_0)} \tag{3.62}$$

– it's just the saddle point value of $r_0\phi^2 + u(\ell_0)\phi^4$.

Putting this into (3.58), we find

$$f_s(r, u(0)) = \frac{r^2}{u_0}\left(1 + 36u(0)\ln\frac{r_0}{r}\right)^{1/3}. \tag{3.63}$$

It gives back the mean field answer when $u(0) \to 0$, but it includes some log corrections. It implies that the heat capacity has the singular behavior

$$c_V = \partial_r^2 f_s \sim |\ln\frac{r}{r_0}|^{1/3}. \tag{3.64}$$

Correlation length exponent, revisited. [Cardy pp. 49-51] Here is a better way to think about the correlation length. At the critical point, the two-point function of the order parameter $G(x) \equiv \langle\phi(x)\phi(0)\rangle$ is a power law in x, specified by η. Away from the critical point, there is another scale, namely the size of the perturbation – the deviation of the knob δ from its critical value, such as $T - T_c$. An RG equation analogous to the one we wrote above for f_s implies that $G(x)$ takes the form

$$G(x) = \frac{1}{|x|^{D-2}}\left(\frac{1}{|x|/a}\right)^{\eta}\Phi\left(|x|\delta^{1/\Delta}\right)$$

where the argument of the *scaling function* Φ is dimensionless. (I emphasized that some length scale a, such as the lattice spacing, must make up the extra engineering dimensions to allow for an anomalous dimension of the field at the critical point.) When $x \gg$ all other length scales, $G(x)$ should decay exponentially, and the decay length must then be $\xi \sim \delta^{-\frac{1}{\Delta}}$ which says $\nu = \frac{1}{\Delta}$.

In the case of ϕ^4 theory, r_0 is the parameter that an experimentalist must carefully tune to access the critical point (what I just called δ) – it is the coefficient of the relevant operator $\mathcal{O} = |\phi|^2$ which takes us away from the critical point; it plays the role of $T - T_c$.

At the free fixed point the dimension of $|\phi|^2$ is just twice that of ϕ, and we get $\nu^{-1} = \Delta^{(0)}_{|\phi|^2} = 2\frac{D-2}{2} = D-2$. At the nontrivial fixed point, however, notice that $|\phi|^2$ is a composite operator in an interacting field theory. In particular, its scaling dimension is not just twice that of ϕ! This requires a bit of a digression.

3.7 Renormalization of composite operators and the Callan-Symanzik equation

[Peskin §12.4] The Wilson-Fisher fixed point is an example of an interacting fixed point, which we happen to be able to describe (for small ϵ) using the same variables

as the Gaussian theory. Perturbing the Wilson-Fisher fixed point by the mass term, a seemingly-innocuous quadratic operator, is then no longer quite so innocent. In particular, we must define what we mean by the operator $|\phi|^2$! This is necessary to understand the correlation-length critical exponent, the power with which the correlation length diverges as we tune to the critical point.

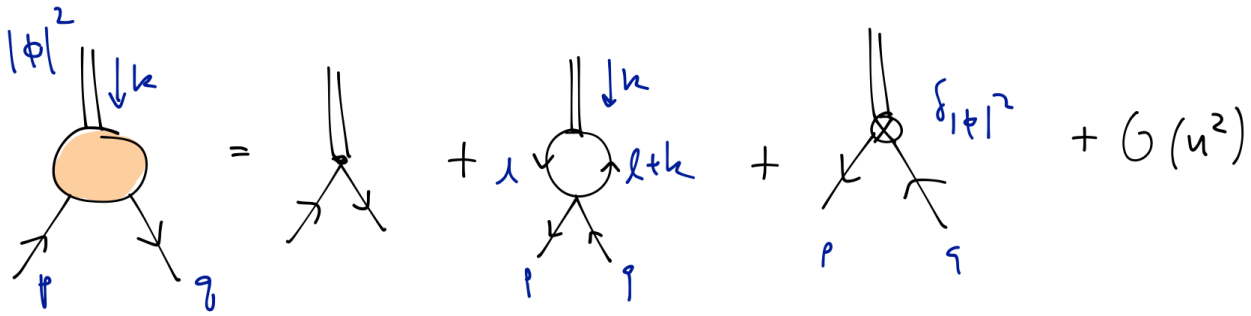
One way to define it (from the counterterms point of view, now, following Peskin and Zinn-Justin) is by adding an extra renormalization condition[18]. We can define the normalization of the composite operator $\mathcal{O}(k) \equiv |\phi|^2(k)$ by the condition that its (amputated) 3-point function gives

$$\langle \mathcal{O}_\Lambda(k)\phi(p)\phi^\star(q)\rangle_{\text{amputated}} = 1 \quad \text{at } p^2 = q^2 = k^2 = -\Lambda^2 \ .$$

The subscript on $\mathcal{O}_\Lambda(k)$ is to emphasize that its (multiplicative) normalization is defined by a renormalization condition at scale (spacelike momentum) Λ. Just like for the 'elementary fields', we can define a wavefunction renormalization factor:

$$\mathcal{O}_\Lambda \equiv Z_{\mathcal{O}}^{-1}(\Lambda)\mathcal{O}_\infty$$

where $\mathcal{O}_\infty \equiv \phi^\star\phi$ is the bare product of fields.



(3.65)

We can represent the implementation of this prescription diagramatically. In the diagram above, the double line is a new kind of thing – it represents the insertion of $\mathcal{O}_\Lambda$. The vertex where it meets the two ϕ lines is *not* the 4-point vertex associated with the interaction – two ϕs can turn into two ϕs even in the free theory. The one-loop, 1PI correction to this correlator is (the second diagram on the RHS of the figure)[19]

$$(-\lambda)\int_0^\infty \mathrm{đ}^D\ell \frac{1}{\ell^2}\frac{1}{(k+\ell)^2} = -\lambda\frac{c}{k^{4-D}}$$

[18] Note that various factors differ from Peskin's discussion in §12.4 because in this subsection, just for fun, I am discussing a complex field $\phi \neq \phi^\star$ (the case $N = 2$); this changes the symmetry factors – for $N = 1$ there is an extra factor of $\frac{1}{2}$.

[19] At higher order in u_0, the wavefunction renormalization of ϕ will also contribute to the renormalization of $|\phi|^2$.

where, using dim reg, $c = \frac{\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^2}$, and we know the k dependence of the integral by scaling.

Imposing the renormalization condition requires us to add a counterterm diagram (part of the definition of $|\phi|^2$, indicated by the $\otimes$ in the diagrams above) which adds

$$Z_{\mathcal{O}}^{-1}(\Lambda) - 1 \equiv \delta_{|\phi|^2} = \frac{\lambda c}{\Lambda^{4-D}} \ .$$

We can infer the dimension of (the well-defined) $|\phi|^2_\Lambda$ by writing a renormalization group equation for our 3-point function

$$G^{(2;1)} \equiv \left\langle |\phi|^2_\Lambda(k)\phi(p)\phi^\star(q)\right\rangle .$$

The resulting equation (3.67), named after Callan and Symanzik, is the demand that physics is independent of choices we've made in the renormalization procedure, in particular, of the arbitrary scale Λ at which we imposed the renormalization condition[20]. G is related to the correlation function of the bare fields by

$$G^{(2;1)} = Z^{-1}_{|\phi|^2}\sqrt{Z_\phi}\sqrt{Z_{\phi^\star}}\left\langle |\phi|^2_\infty(k)\phi_0(p)\phi_0^\star(q)\right\rangle . \tag{3.66}$$

The dependence on Λ is in the coupling λ, and in the renormalization factors Z. So:

$$0 \stackrel{!}{=} \Lambda\frac{d}{d\Lambda}G^{(n;1)} = \left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma_\phi - \gamma_{\mathcal{O}}\right)G^{(n;1)} \ . \tag{3.67}$$

$\gamma_{\mathcal{O}} \equiv \Lambda\frac{\partial}{\partial\Lambda}\log Z_{\mathcal{O}}(\Lambda)$ is the *anomalous dimension* of the operator $\mathcal{O}$, roughly the addition to its engineering dimension coming from the interactions (similarly $\gamma_\phi \equiv \frac{1}{2}\Lambda\frac{\partial}{\partial\Lambda}\log Z_\phi(\Lambda)$). In perturbation theory, our Green's function takes the schematic form

$$G = (\text{tree diagrams} + \text{1PI loop diagrams} + \text{counterterm} + \text{external leg corrections}) \tag{3.68}$$

(as you can see in (3.65)). The explicit dependence on Λ is all in the counterterms. For our example with $n = 2$ the anomalous dimension of $|\phi|^2$ is

$$\gamma_{|\phi|^2} = (4-D)\frac{\Gamma(2-D/2)}{16\pi^2}\lambda \stackrel{D\to 4}{=} \frac{2\lambda}{16\pi^2}.$$

Here's a good reason to care about the anomalous dimension, and which explains the name. What happens we add the operator $\mathcal{O}_\Lambda$ to the Lagrangian density:

$$\mathcal{L} = \mathcal{L}_0 + \Lambda^{d_g - D}g\mathcal{O}_\Lambda$$

[20]The same logic can be applied to correlation functions of only 'elementary operators'. For that discussion, see *e.g.* Peskin §12.2. The result is obtained just by leaving out the composite operators.

(where the factor of Λ is fixed by engineering dimensions so that g is dimensionless, so $d_g = D-2$ in the case of ϕ^2)? Let's compute $G = \langle \phi_1 \cdots \phi_n \rangle$ in this perturbed theory. We can count the number of insertions of $\mathcal{O}_\Lambda$ by counting powers of g and the Callan-Symanzik equation for the n point function of ϕ becomes

$$0 = \left(\Lambda\partial_\Lambda + \beta_\lambda(\lambda)\partial_\lambda + n\gamma_\phi(\lambda) + \underbrace{(-\gamma_{\mathcal{O}} + d_g - 4)g\partial_g}_{=\beta_{\mathcal{O}}\partial_g} \right) G \ . \tag{3.69}$$

So the anomalous dimension of $\mathcal{O}$ determines how it runs when we use it to perturb the action – it just gets added to its engineering dimension. What happened to dimensional analysis? Well, renormalization required us to introduce a new scale in the problem (in this case Λ), which doesn't go away.

One final comment about defining and renormalizing composite operators: if there are multiple operators with the same quantum numbers and the same scaling dimension, they will *mix* under renormalization. That is, in order to obtain cutoff-independent correlators of these operators, their definition must be of the form

$$\mathcal{O}^i_\Lambda = \left(Z^{-1}(\Lambda)\right)_{ij} \mathcal{O}^j_\infty$$

– there is a wavefunction renormalization *matrix*, and a matrix of anomalous dimensions

$$\gamma_{ij} = -\Lambda\partial_\Lambda \log \left(Z^{-1}(\Lambda)\right)_{ij} .$$

'Operator mixing' is really just the statement that correlation functions like $\langle \mathcal{O}^i \mathcal{O}^j \rangle$ are nonzero.

Solution of the Callan-Symanzik equation. In the free theory, the Green's function of n ϕs (say in position space) satisfies (by dimensional analysis) the scaling relation

$$G_n(\{sx_i\}, m^2) \equiv \langle \phi(sx_1) \cdots \phi(sx_n) \rangle_S = s^{n(2-D)/2} \langle \phi'(x_1) \cdots \phi'(x_n) \rangle_{S'} \tag{3.70}$$

where

$$x \equiv sx', \phi(x) \equiv s^{\frac{2-D}{2}} \phi'(x') \tag{3.71}$$

and S' is the action with the mass replaced by $m' = sm$. Notice that $s > 1$ takes us to the IR, where m' gets more important.

With interactions, we must also include a renormalization scale, and dimensional analysis in the renormalized theory implies

$$G_n(\{sx_i\}, \{g_I\}, \Lambda) = s^{n\left(\frac{2-D}{2}\right)} G_n(\{x_i\}, \{s^{4-d_I} g_I\}, s\Lambda) \tag{3.72}$$

where in the ϕ^4 theory $\{g_I\} = \{m^2, \lambda_4, \lambda_6 \cdots\}$, and $4 - d_I$ is the engineering dimensions of the coupling g, so $\{s^{4-d_I} g_I\} = \{s^2 m^2, \lambda_4, s^{-2}\lambda_6\}$.

So far this is just dimensional analysis. The Callan-Symanzik equation allows us to include the effects of fluctuations in this scaling relation. It is just an ODE in Λ:

$$(\Lambda \partial_\Lambda + \beta_I \partial_{g_I} + n\gamma_\phi)\, G_n = 0. \tag{3.73}$$

In terms of the running couplings

$$\Lambda \partial_\Lambda g_I(\Lambda) = \beta_I(g_I(\Lambda)) \tag{3.74}$$

the solution relates G at different renormalization points:

$$G_n(\{x\}, \{g_I(\Lambda_1)\}, \Lambda_1) = e^{-n\int_{\Lambda_1}^{\Lambda_2} \gamma_\phi(\Lambda) d\log\Lambda} G_n(\{x\}, \{g_I(\Lambda_2)\}, \Lambda_2). \tag{3.75}$$

Combining with the information from dimensional analysis:

$$G_n(\{sx\}, \{g_I(\Lambda)\}, \Lambda) \overset{(3.75)}{=} e^{-n\int_{\Lambda}^{\Lambda_2} \gamma_\phi(\Lambda') d\log\Lambda'} G_n(\{sx\}, \{g_I(\Lambda_2)\}, \Lambda_2) \tag{3.76}$$

$$\overset{(3.72)}{=} s^{n\left(\frac{2-D}{2}\right)} e^{-n\int_{\Lambda}^{\Lambda/s} \gamma_\phi(\Lambda') d\log\Lambda'} G_n(\{x\}, \{s^{4-d_I} g_I(\Lambda/s)\}, \Lambda) \tag{3.77}$$

where in the last step I set $\Lambda_2 \equiv \Lambda/s$. We learn that the effect of a rescaling $x \to sx$ has three parts: (1) the rescaling by the engineering dimensions, (2) the rescaling by the anomalous dimension, (3) the running of the coupling.

Consider for example the special case where a coupling sits at a fixed point $g = g^\star$ (and other couplings are zero). Then the anomalous dimension prefactor is

$$e^{-\int_{\Lambda_1}^{\Lambda_2} \gamma_\phi(\Lambda') d\log\Lambda'} = \left(\frac{\Lambda_2}{\Lambda_1}\right)^{-\gamma^\star} \tag{3.78}$$

with $\gamma^\star = \gamma_\phi(g^\star)$, so (3.77) becomes

$$G_n(\{sx_i\}, g^\star, \Lambda) = s^{n\left(\frac{2-D}{2} + \gamma^\star\right)} G_n(\{x_i\}, g^\star, \Lambda). \tag{3.79}$$

The system is scale invariant, but with scaling different from the result of a dimensional analysis that doesn't include the RG scale.

[End of Lecture 9]

4 Effective field theory

4.1 Introduction to effective field theory

[Some nice lecture notes on effective field theory can be found here: J. Polchinski, A. Manohar, I. Rothstein, D. B. Kaplan, H. Georgi, A. Manohar]

Having internalized Wilson's perspective on renormalization – namely that we should include all possible operators consistent with symmetries and let the dynamics decide which are important at low energies – we are led immediately to the idea of an *effective field theory* (EFT), or, *how to do physics without a theory of everything.* (You may notice that all the physics that has been done has been done without a theory of everything.) It is a weaponized version of selective inattention.

The basic idea is that the Hamiltonian (or the action) should contain all terms consistent with symmetries, organized according to an expansion in decreasing relevance to low energy physics. This is an implementation of the *totalitarian principle of physics*, that anything that can happen must happen.

Diatribe about 'renormalizability'. There is no reason to demand that a field theory that we have found to describe physics in some regime should be a valid description of the world to arbitrarily short (or long!) distances. This is a happy statement: there can always be new physics that has been so far hidden from us. Rather, an EFT comes with a regime of validity, and with necessary cutoffs. As we will discuss, in a useful implementation of an EFT, the cutoff implies a small parameter in which we can expand (and hence compute). (In the example of $S_{\rm eff}[q]$ of §1, the small parameter is ω/Ω.)

Caring about renormalizibility is pretending to know about physics at arbitrarily short distances. Which we definitely don't. Even when theories are renormalizable, this apparent victory is often false. For example, QED requires only two independent counterterms (for the mass and for the fine structure constant), and is therefore by the old-fashioned definition renormalizable, but it is superseded by the electroweak theory above 80GeV. Also: the coupling in QED actually increases logarithmically at shorter distances, and ultimately reaches a *Landau pole* at SOME RIDICULOUSLY HIGH ENERGY (of order $e^{+\frac{c}{\alpha}}$ where $\alpha \sim \frac{1}{137}$ is the fine structure constant (*e.g.* at the scale of atomic physics) and c is some numerical number. Plugging in numbers gives something like 10^{330} GeV, which is quite a bit larger than the Planck scale). This is of course completely irrelevant for physics and even in principle because of the previous remark about electroweak unification. And if not because of that, because of the Planck scale. A heartbreaking historical fact is that Landau and many other smart people gave up on QFT as a whole because of this silly fantasy about QED in

an unphysical regime.

We will see below that even in QFTs which are non-renormalizable in the strict sense, there is a more useful notion of renormalizability: effective field theories come with a small parameter (often some ratio of mass scales), in which we may expand the action. A useful EFT requires a finite number of counterterms *at each order in the expansion*.

Furthermore, I claim that this is *always* the definition of renormalizability that we are using, even if we are using a theory that is renormalizable in the traditional sense, which allows us to pretend that there is no cutoff. That is, there could always be corrections of order $\left(\frac{E}{E_{\text{new}}}\right)^n$ where E is some energy scale of physics that we are doing and E_{new} is some UV scale where new physics might come in; for large enough n, this is too small for us to have seen. The property of renormalizibility that actually matters is that we need a finite number of counterterms at each order in the expansion in $\frac{E}{E_{\text{new}}}$.

Renormalizable QFTs are in some sense *less* powerful than non-renormalizable ones – the latter have the decency to tell us when they are giving the wrong answer! That is, they tell us at what energy new physics *must* come in; with a renormalizable theory we may blithely pretend that it is valid in some ridiculously inappropriate regime like 10^{330} GeV.

Notions of EFT. There is a dichotomy in the way EFTs are used. Sometimes one knows a lot about the UV theory (*e.g.*

- electroweak gauge theory,
- QCD,
- electrons in a solid,
- water molecules

...) but it is complicated and unwieldy for the questions one wants to answer, so instead one develops an effective field theory involving just the appropriate and important dofs (*e.g.*, respectively,

- Fermi theory of weak interactions (or QED or ...),
- chiral lagrangian (or HQET or SCET or hydrodynamics of quark-gluon plasma or ...),
- Landau Fermi liquid theory (or the Hubbard model or a topological field theory or ...),

- hydrodynamics (or some theory of phonons in ice or ...)

...). As you can see from the preceding lists of examples, even a single UV theory can have many different IR EFTs depending on what phase it is in, and depending on what question one wants to ask. The relationship between the pairs of theories above is always coarse-graining from the UV to the IR, though exactly what plays the role of the RG parameter can vary wildly. For example, in the case of the Fermi liquid theory, the scaling is $\omega \to 0$, and momenta scale towards the Fermi surface, not $\vec{k} = 0$.

A second situation is when one knows a description of some low-energy physics up to some UV scale, and wants to try to infer what the UV theory might be. This is a common situation in physics! Prominent examples include: the Standard Model, and quantized Einstein gravity. Occasionally we (humans) actually learn some physics and an example of an EFT from the second category moves to the first category.

Instructions for EFT. Answer the following questions:

1. what are the dofs?
2. what are the symmetries?
3. where is the cutoff, Λ, on its validity?

Then write down all interactions between the dofs that preserve the symmetries, in an expansion in derivatives, with higher-dimension operators suppressed by more powers of the UV scale, Λ.

I must also emphasize two distinct usages of the term 'effective field theory' which are common, and which the discussion above is guilty of conflating (this (often slippery) distinction is emphasized in the review article by Georgi linked at the beginning of this subsection). The Wilsonian perspective advocated above produces a low-energy description of the physics which is really just a way of solving (if you can) the original model; very reductively, it's just a physically well-motivated order for doing the integrals. If you really integrate out the high energy modes exactly, you will get a non-local action for the low energy modes. This is to be contrasted with the local actions one uses in practice, by truncating the derivative expansion. It is the latter that is really the action of the effective field theory, as opposed to the *full theory*, with some of the integrals done already. The latter will give correct answers for physics below the cutoff scale, and it will give them much more easily.

Some more comments:

• Sometimes (in condensed matter circles) this approach of just writing all terms consistent with symmetries is called Landau theory or Landau-Ginzburg theory or maybe Landau-Ginzburg-Wilson.

• Do not underestimate the difficulty of step 1 of the EFT instructions. As we'll see in some examples below, the correct low-energy dofs can look nothing at all like the microscopic dofs.

• The Wilson RG justifies this procedure: coarse graining by integrating out short-wavelength modes produces all terms consistent with the symmetries.

• When we say "what are the symmetries?" we mean the symmetries G of the (regulated) microscopic theory. G must be a symmetry of the low-energy EFT[21]. Sometimes new symmetries can emerge at low energies. This procedure explains how this happens: if there are *no* relevant or marginal operators invariant under G that violate a symmetry K, then physics at lower and lower energies will be more and more K-symmetric.

Here are some interesting and/or important examples where EFT has been useful (some of which we will discuss in more detail below) and where you can learn about them:

- Hydrodynamics [Kovtun]
- Fermi liquid theory [J. Polchinski, R. Shankar, *Rev. Mod. Phys.* **66** (1994) 129]
- chiral perturbation theory [D. B. Kaplan, §4]
- heavy quark effective theory [D. B. Kaplan, §1.3, Manohar and Wise, *Heavy Quark Physics*]
- random surface growth (KPZ) [Zee, chapter VI]
- color superconductors [D. B. Kaplan, §5]
- gravitational radiation from binary mergers [Goldberger, Rothstein, Porto]
- soft collinear effective theory [Becher, Stewart]
- magnets [Zee, chapter VI.5, hep-ph/9311264v1]
- effective field theory of cosmological inflation [Senatore et al, Cheung et al, Porto]

[21] Actually, there is a dumb loophole here: it may be that G or some subgroup of G simply doesn't act on the low-energy degrees of freedom. For example, we could have a microscopic system with symmetry G and a completely trivial low-energy theory, with no degrees of freedom at all.

- effective field theory of dark matter direct detection [Fitzpatrick et al]
- here is some advocacy for the effective field theory viewpoint in biology: [Phillips]

There are many others, the length of this list was limited by how long I was willing to spend digging up references.

4.2 The color of the sky

[from hep-ph/9606222 and nucl-th/0510023] Why is the sky blue? Basically, it's because the blue light from the sun scatters in the atmosphere more than the red light, and you (I hope) only look at the scattered light.

Here is an understanding of this fact using the EFT logic. Consider the scattering of photons off atoms (in a gas) at low energies. Low energy means that the photon does not have enough energy to probe the substructure of the atom – it can't excite the electrons or the nuclei. This means that the atom is just a particle, with some mass M.

The dofs are just the photon field and the field that creates an atom.

The symmetries are Lorentz invariance and charge conjugation invariance and parity. We'll use the usual redundant description of the photon that has also gauge invariance.

The cutoff is the energy ΔE that it takes to excite atomic energy levels we've left out of the discussion. We allow no inelastic scattering. This means we require

$$E_\gamma \ll \Delta E \sim \frac{\alpha}{a_0} = \alpha^2 m_e \ll a_0^{-1} = \alpha m_e \ll m_e \ll M_{\rm atom} \tag{4.1}$$

where $a_0 = (\alpha m_e)^{-1}$ is the Bohr radius. Because of this separation of scales, we can also ignore the recoil of the atom, and treat it as infinitely heavy.

Let's call the field that destroys an atom with velocity v ϕ_v. $v^\mu v_\mu = 1$ and $v_\mu = (1,0,0,0)_\mu$ in the atom's rest frame. The (Lorentz-singlet) Lagrangian can depend on v^μ. We can write a Lagrangian for the free atoms as

$$L_{\rm atom} = \phi_v^\dagger \mathbf{i} v^\mu \partial_\mu \phi_v \quad .$$

This action is related by a boost to the statement that the atom at rest has zero energy – in the rest frame of the atom, the eom is just $\partial_t \phi_{v=(1,\vec{0})} = 0$. (If we didn't define the zero of energy to be at the rest mass, there would be an additional term $\gamma_v M_{\rm atom} \phi_v^\dagger \phi_v$, $\gamma_v \equiv \frac{1}{\sqrt{1-v^2}}$.) Notice that the kinetic term $\phi_v^\dagger \frac{\vec{\nabla}^2}{2M_{\rm atom}} \phi_v$ is a very small correction given our hierarchy of scales (4.1).

So the Lagrangian density is

$$L_{\rm Maxwell}[A] + L_{\rm atom}[\phi_v] + L_{\rm int}[A, \phi_v]$$

and we must determine $L_{\rm int}$. It is made from local, Hermitian, gauge-invariant, Lorentz invariant operators we can construct out of $\phi_v, F_{\mu\nu}, v_\mu, \partial_\mu$ (it can only depend on $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and not A_μ directly, by gauge invariance, because the atom, and hence

ϕ_v, is neutral.). It should actually only depend on the combination $\phi_v^\dagger \phi_v$ since we will not create and destroy atoms – there is a(n emergent) $\mathsf{U}(1)$ symmetry associated with atom number conservation. (Notice that we didn't have to specify the statistics of the atoms or ϕ_v.) Therefore

$$L_{\text{int}} = c_1 \phi_v^\dagger \phi_v F_{\mu\nu} F^{\mu\nu} + c_2 \phi_v^\dagger \phi_v v^\sigma F_{\sigma\mu} v_\lambda F^{\lambda\mu} + c_3 \phi_v^\dagger \phi_v \left(v^\lambda \partial_\lambda\right) F_{\mu\nu} F^{\mu\nu} + \dots$$

$\dots$ indicates terms with more derivatives and more powers of velocity (*i.e.* an expansion in $\partial \cdot v$). Which are the most important terms at low energies? Demanding that the Maxwell term dominate, we get the power counting rules (so time and space should scale the same way):

$$[\partial_\mu] = 1, \quad [F_{\mu\nu}] = 2$$

This then implies $[\phi_v] = 3/2, [v] = 0$ and therefore

$$[c_1] = [c_2] = -3, [c_3] = -4 \ .$$

These interactions are all irrelevant; terms with more partials are more irrelevant.

What makes up these dimensions in the couplings c_i? They must come from the length scales that we have integrated out to get this description – the size of the atom $a_0 \sim (\alpha m_e)^{-1}$ and the energy gap between the ground state and the electronic excited states $\Delta E \sim \alpha^2 m_e$. For $E_\gamma \ll \Delta E, a_0^{-1}$, we can just keep the two leading terms.

In the rest frame of the atom, these two leading terms $c_{1,2}$ represent just the scattering of $E^2 - B^2$ and E^2 respectively. To determine their coefficients one would have to do a matching calculation to a more complete theory (compute transition rates in a theory that does include extra energy levels of the atom). But a reasonable guess is just that the scale of new physics (in this case atomic physics) makes up the dimensions: $c_1 \simeq c_2 \simeq a_0^3$. (In fact the coefficient of B^2 comes with extra factor of v/c which suppresses it.) The scattering cross section then goes like $\sigma \sim c_i^2 \sim a_0^6$; dimensional analysis ($[\sigma] = -2$ is an area, $[a_0^6] = -6$) then tells us that we have to make up four powers with the only other scale around:

$$\sigma \propto E_\gamma^4 a_0^6.$$

(The factor of E_γ^2 in the amplitude arises from $\vec{E} \propto \partial_t \vec{A}$.) Blue light, which has about twice the energy of red light, is therefore scattered 16 times as much.

The leading term that we left out is the one with coefficient c_3. The size of this coefficient determines when our approximations break down. We might expect this to come from the next smallest of our neglected scales, namely ΔE. That is, we expect

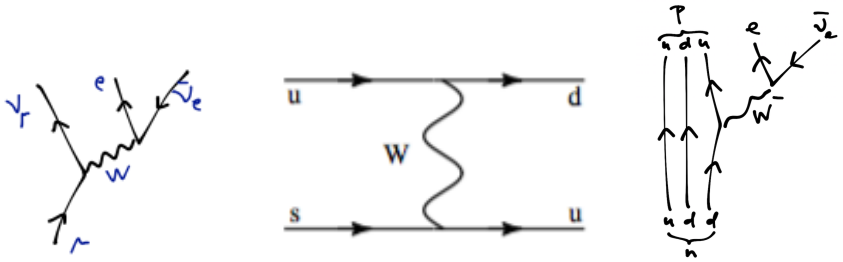
$$\sigma \propto E_\gamma^4 a_0^6 \left(1 + \mathcal{O}\left(\frac{E_\gamma}{\Delta E}\right)\right).$$

The ratio in the correction terms is appreciable for UV light.

4.3 Fermi theory of Weak Interactions

[from §5 of A. Manohar's EFT lectures] Let's think about part of the Standard Model as an example of EFT.

$$L_{EW} \ni -\frac{1}{2}\left(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu\right)\left(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}\right) + M_W^2 W^+_\mu W^{-\mu} \qquad (4.2)$$
$$-\frac{\mathbf{i}g}{\sqrt{2}}\bar{\psi}_i\gamma^\mu P_L\psi_j W^+_\mu V_{ij} \quad + \text{terms involving } Z \text{ bosons}$$



Some things intermediate, off-shell W bosons can do: μ decay, $\Delta S = 1$ processes, neutron decay

If we are asking questions with external momenta less than M_W, we can integrate out W and make our lives simpler:

$$\delta S_{eff} \sim \left(\frac{\mathbf{i}g}{\sqrt{2}}\right)^2 V_{ij}V^\star_{k\ell}\int \mathrm{d}^D p \frac{-\mathbf{i}g_{\mu\nu}}{p^2 - M_W^2}\left(\bar{\psi}_i\gamma^\mu P_L\psi_j\right)(p)\left(\bar{\psi}_k\gamma^\nu P_L\psi_\ell\right)(-p)$$

(I am lying a little bit about the W propagator in that I am not explicitly projecting out the fourth polarization with the negative residue. Also, the W carries electric charge, so the charges of $\bar{\psi}_i$ and ψ_j in (4.2) must differ by one.) This is non-local at scales $p \gtrsim M_W$ (recall the discussion of the subsection §1). But for $p^2 \ll M_W^2$,

$$\frac{1}{p^2 - M_W^2} \overset{p^2 \ll M_W^2}{\simeq} -\frac{1}{M_W^2}\left(1 + \underbrace{\frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \ldots}_{\text{derivative couplings}}\right) \qquad (4.3)$$

$$S_F = -\frac{4G_F}{\sqrt{2}}V_{ij}V^\star_{kl}\int d^4x \left(\bar{\psi}_i\gamma^\mu P_L\psi_j\right)(x)\left(\bar{\psi}_k\gamma_\mu P_L\psi_\ell\right)(x) + \mathcal{O}\left(\frac{1}{M_W^2}\right) + \text{kinetic terms for fermions} \qquad (4.4)$$

where $G_F/\sqrt{2} \equiv \frac{g^2}{8M_W^2}$ is the Fermi coupling. We can use this (Fermi's) theory to compute the amplitudes above, and it is much simpler than the full electroweak theory (for example I don't have to lie about the form of the propagator of the W-boson like I

did above). It was discovered first and used quite effectively long before the existence of Ws was suspected.

On the other hand, this theory is *not the same* as the electroweak theory; for example it is not renormalizable, while the EW theory is (at least if we included the Higgs sector, rather than just writing a mass term for the Ws). Its point in life is to help facilitate the expansion in $1/M_W$. There is something about the expression (4.4) that should make you nervous, namely the big red 1 in the $1/M_W^2$ corrections: what makes up the dimensions? The short answer is derivatives of the Fermi fields. This becomes an issue when we ask about loops in §4.4.

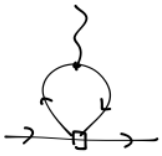
4.4 Loops in EFT

Suppose we try to define the Fermi theory S_F with a euclidean momentum cutoff $|k_E| < \Lambda$. We expect that we'll have to set $\Lambda \sim M_W$. A simple example that shows that this is problematic arises by asking about radiative corrections in the 4-Fermi theory to the coupling between the fermions and the photon (or the Z boson).

We are just trying to estimate the magnitude of this correction, so don't worry about the factors and the gamma matrices:

$$\sim I \equiv \underbrace{\frac{1}{M_W^2}}_{\propto G_F} \underbrace{\int^\Lambda \text{đ}^4 k \frac{1}{k}\frac{1}{k} \text{tr}\,(\gamma ...)}_{\sim \int^\Lambda k \text{d}k \sim \Lambda^2 \sim M_W^2} \sim \mathcal{O}(1).$$

Even worse, consider what happens if we use the vertex coming from the $\left(\frac{p^2}{M_W^2}\right)^\ell$ correction in (4.3)



$$\sim I_\ell \equiv \frac{1}{M_W^2} \int^\Lambda \text{đ}^4 k \frac{1}{k^2} \left(\frac{k^2}{M_W^2}\right)^\ell \sim \mathcal{O}(1)$$

– it's also unsuppressed by powers of ... well, anything. This is a problem.

Fix: A way to fix this is to use a "mass-independent subtraction scheme", such as dimensional regularization and minimal subtraction ($\overline{\text{MS}}$). The crucial feature is that the dimensionful cutoff parameter appears only inside logarithms ($\log \mu$), and not as free-standing powers (μ^2).

With such a scheme, we'd get instead

$$I \sim \frac{m^2}{M_W^2} \log \mu \qquad I_\ell \sim \left(\frac{m^2}{M_W^2}\right)^{\ell+1} \log \mu$$

where m is some mass scale *other* than the RG scale (like a fermion mass parameter, or an external momentum, or a dynamical scale like Λ_{QCD}).

We will give a more detailed example next. The point is that in a mass-independent scheme, the regulator doesn't produce new dimensionful things that can cancel out the factors of M_W in the denominator. It respects the 'power counting': if you see 2ℓ powers of $1/M_W$ in the coefficient of some term in the action, that's how many powers will suppress its contributions to amplitudes. This means that the EFT is like a renormalizable theory *at each order in the expansion* (here in $1/M_W$), in that there is only a finite number of allowed vertices that contribute at each order (counterterms for which need to be fixed by a renormalization condition). The insatiable appetite for counterterms is still insatiable, but it eats only a finite number at each order in the expansion. Eventually you'll get to an order in the expansion that's too small to care about, at which point the EFT will have eaten only a finite number of counterterms.

There is a price for these wonderful features of mass-independent schemes, which has two aspects:

- Heavy particles (of mass m) don't decouple when $\mu < m$. For example, in a mass-independent scheme for a gauge theory, heavy charged particles contribute to the beta function for the gauge coupling even at $\mu \ll m$.
- Perturbation theory will break down at *low* energies, when $\mu < m$; in the example just mentioned this happens because the coupling keeps running.

We will show both these properties very explicitly in the next subsection. The solution of both these problems is to integrate out the heavy particles by hand at $\mu = m$, and make a new EFT for $\mu < m$ which simply omits that field. Processes for which we should set $\mu < m$ don't have enough energy to make the heavy particles in external states anyway. (For some situations where you should still worry about them, see Aneesh Manohar's notes linked above.) [End of Lecture 10]

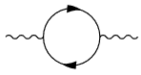
4.4.1 Comparison of schemes, case study

The case study we will make is the contribution of a charged fermion of mass m to the running of the QED gauge coupling.

First some recapitulation: Recall that the QED Lagrangian is

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}\left(\mathbf{i}\not{D} - m\right)\psi$$

with $D_\mu = \partial_\mu - \mathbf{i}eA_\mu$. By redefining the field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ by a constant factor we can move around where the e appears, *i.e.* by writing $\tilde{A} = eA$, we can make the

gauge kinetic term look like $\frac{1}{4e^2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$. This means that the charge renormalization really comes from the vacuum polarization, the correction to the photon propagator:  . Recall that, by the Ward identity for gauge invariance, the vacuum polarization takes the form

$$\Pi^{\mu\nu}(q^2) = \Pi(q^2)\left(q^2\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \equiv \Pi(q^2)P^{\mu\nu}(q).$$

In dim reg, the one-loop vacuum polarization correction satisfies the gauge invariance Ward identity $\Pi^{\mu\nu} = P^{\mu\nu}\delta\Pi_2$ (unlike the euclidean momentum cutoff which is not gauge invariant), with

$$\begin{aligned}\delta\Pi_2(p^2) &\overset{\text{Peskin p. 252}}{=} -\frac{8e^2}{(4\pi)^{D/2}}\int_0^1 dx x(1-x)\frac{\Gamma(2-D/2)}{\Delta^{2-D/2}}\bar{\mu}^\epsilon \\ &\overset{D\to 4}{=} -\frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\left(\frac{2}{\epsilon} - \log\left(\frac{\Delta}{\mu^2}\right)\right) . \end{aligned} \tag{4.5}$$

In the second line of (4.5), we expanded the Γ-function about $D = 4$; there are other singularities at other integer dimensions.

End of recapitulation. The choice of scheme shows up in our choice of renormalization condition to impose on $\Pi(p^2)$:

Mass-dependent scheme: subtract the value of the graph at $p^2 = -M^2$ (a very off-shell, euclidean, momentum). That is, we impose a renormalization condition that says

$$\Pi^{(M)}(p^2 = -M^2) \overset{!}{=} 0 . \tag{4.6}$$

In a mass-dependent scheme, we demand that the counterterm cancels $\delta\Pi_2$ when we set the external momentum to $p^2 = -M^2$, so that the whole contribution at order e^2 is :

$$0 \overset{(4.6)!}{=} \Pi_2^{(M)}(p^2 = -M^2) = \underbrace{\delta_{F^2}^{(M)}}_{\text{counterterm coefficient for } \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} + \delta\Pi_2(p^2 = -M^2)$$

$$\implies \Pi_2^{(M)}(p^2) = \frac{e^2}{2\pi^2}\int dx x(1-x)\log\left(\frac{m^2 - x(1-x)p^2}{m^2 + x(1-x)M^2}\right).$$

Notice that the μs go away in this scheme.

Mass-Independent scheme: This is to be contrasted with what we get in a mass-independent scheme, such as $\overline{\text{MS}}$, in which Π is defined by the rule that we *subtract*

the $1/\epsilon$ *pole.* This means that the counterterm is

$$\delta_{F^2}^{(\overline{\rm MS})} = -\frac{e^2}{2\pi^2}\frac{2}{\epsilon}\underbrace{\int_0^1 dx x(1-x)}_{=1/6}. \tag{4.7}$$

(Confession: I don't know how to state this in terms of a simple renormalization condition on Π_2. Also: the bar in $\overline{\rm MS}$ refers to the (not so important) distinction between $\bar{\mu}$ and μ.) The resulting vacuum polarization function is

$$\Pi_2^{(\overline{\rm MS})}(p^2) = \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\log\left(\frac{m^2 - x(1-x)p^2}{\mu^2}\right).$$

Next we will talk about beta functions, and verify the claim above about the failure of decoupling. First let me say some words about what is failing. What is failing – the price we are paying for our power counting – is the basic principle of the RG, namely that physics at low energies shouldn't care about physics at high energies, except for small corrections to couplings. An informal version of this statement is: you don't need to know about nuclear physics to make toast. A more formal version is the *Appelquist-Carazzone Decoupling Theorem*, which I will not state (Phys. Rev. D11, 28565 (1975)). So it's something we must and will fix.

Beta functions. $\boxed{M}$: First in the mass-dependent scheme. The fermion contribution to the beta function for the EM coupling is[22]

$$\beta_e^{(M)} = \frac{e}{2}M\partial_M\Pi_2^{(M)}(p^2) = -\frac{1}{2}\left(\frac{e^3}{2\pi}\right)\int_0^1 dx x(1-x)\left(\frac{-2M^2x(1-x)}{m^2 + M^2x(1-x)}\right) + \mathcal{O}(e^5)$$

[22] What I've written here is a fancy way of writing it, since the RHS naively depends on p^2, but does not. Here is a derivation of the beta function for QED in this scheme (following the same logic as we used in the discussion of the QCD beta function): The QED Lagrangian is $L = -\frac{1}{4e_R^2\mu^\epsilon}Z_3(F^0_{\mu\nu}) + \cdots$ where F^0 is the bare field. This means that the bare coupling is $e_0 = e_R\mu^{\epsilon/2}Z_3^{-1/2}$. Here $Z_3 = 1+\delta_{F^2}$. The bare coupling knows nothing about our choice of M, and so

$$0 = M\frac{d}{dM}e_0 = e_0\left(\frac{\epsilon}{2} + \frac{1}{e_R}\beta_e^{(M)} - \frac{1}{2}\frac{1}{Z_3}M\frac{d}{dM}Z_3\right). \tag{4.8}$$

Solving for β (and writing $e \equiv e_R$) gives

$$\beta_e^{(M)} = -\frac{e\epsilon}{2} + \frac{e}{2}M\frac{d}{dM}\delta_{F^2} + ... \tag{4.9}$$

In this scheme,

$$\delta_{F^2} = -\delta\Pi_2(p^2 = -M^2) = \frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\left(\frac{2}{\epsilon} - \log\left(\frac{m^2 + x(1-x)M^2}{\mu^2}\right)\right) \tag{4.10}$$

depends explicitly on M, and the bits where $M\frac{d}{dM}$ hits e are higher order.

$$\begin{cases} \overset{m\ll M}{\simeq} \frac{e^3}{2\pi^2}\int_0^1 dx x(1-x) = \frac{e^3}{12\pi^2} \\ \\ \overset{m\gg M}{\simeq} \frac{e^3}{2\pi^2}\int_0^1 dx x(1-x)\frac{M^2 x(1-x)}{m^2} = \frac{e^3}{60\pi^2}\frac{M^2}{m^2} \end{cases} . \tag{4.11}$$

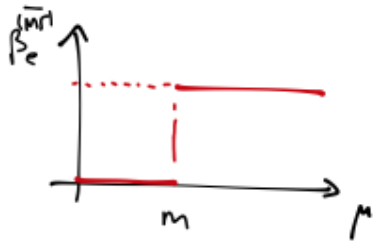
$$\boxed{\overline{\text{MS}}} : \beta_e^{(\overline{\text{MS}})} = \frac{e}{2}\mu\partial_\mu \Pi_2^{(\overline{\text{MS}})}(p^2) = -\frac{1}{2}\frac{e^3}{2\pi^2}\underbrace{\int_0^1 dx x(1-x)}_{=1/6}\underbrace{\mu\partial_\mu \log\frac{m^2 - p^2 x(1-x)}{\mu^2}}_{=-2}$$
$$= \frac{e^3}{12\pi^2}. \tag{4.12}$$

[23]

Also, the $\overline{\text{MS}}$ vacuum polarization behaves for small external momenta like

$$\Pi_2(p^2 \ll m^2) \simeq -\frac{e^3}{2\pi^2}\int_0^1 dx x(1-x) \underbrace{\log\frac{m^2}{\mu^2}}_{\gg 1, \text{for } \mu \ll m! \text{ bad!}}$$

As I mentioned, the resolution of both these problems is simply to define a new EFT for $\mu < m$ that omits the heavy field. Then the strong coupling problem goes away and the heavy fields do decouple. The price is that we have to do this by hand, and the beta function jumps at $\mu = m$; the coupling is continuous, though.



[23] Let me explain the expression for the beta function in the case of $\overline{\text{MS}}$ scheme. Following the same logic as the previous footnote, the bare coupling knows nothing about our choice of μ, and so

$$0 = \mu\frac{d}{d\mu}e_0 = e_0\left(\frac{\epsilon}{2} + \frac{1}{e_R}\beta_e^{(\overline{\text{MS}})} - \frac{1}{2}\frac{1}{Z_3}\mu\frac{d}{d\mu}Z_3\right). \tag{4.13}$$

Solving for β (and writing $e \equiv e_R$) gives

$$\beta_e^{(\overline{\text{MS}})} = -\frac{e\epsilon}{2} + \frac{e}{2}\mu\frac{d}{d\mu}\delta_{F^2} + ... \tag{4.14}$$
$$= -\frac{e\epsilon}{2} + \frac{e}{2}\beta_e^{\overline{\text{MS}}}\partial_e\delta_{F^2} + ... \tag{4.15}$$
$$= -\frac{e\epsilon}{2} + \frac{e}{2}\left(\frac{-e\epsilon}{2}\right)\partial_e\delta_{F^2} + ... \tag{4.16}$$

where in the last step we substituted the leading term for the beta function. In $\overline{\text{MS}}$ scheme, the counterterm, given in (4.7), goes like $\frac{1}{\epsilon}$ and we get the finite answer given above.

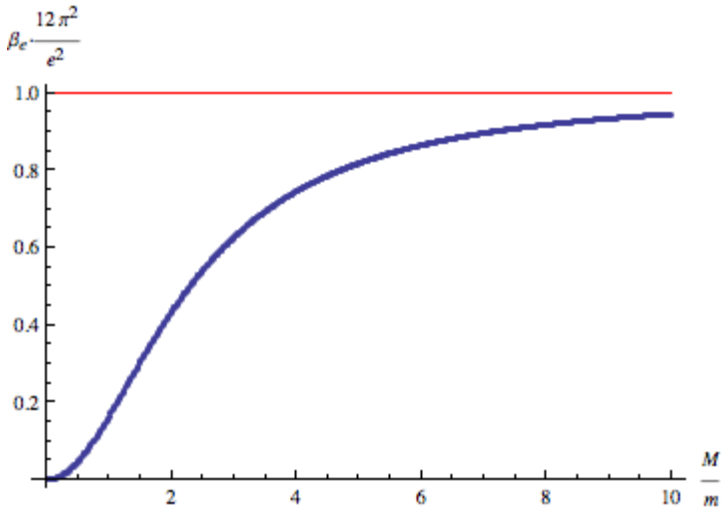
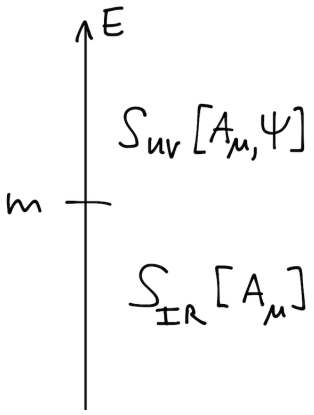


Figure 8: The blue curve is the mass-dependent-scheme beta function; at scales $M \ll m$, the mass of the heavy fermion, the fermion sensibly stops screening the charge. The red line is the $\overline{\text{MS}}$ beta function, which is just a constant, pinned at the UV value.

The couplings in the low energy EFT (here, a theory of just the photon) are determined by *matching*: this means compute a bunch of physical quantities in both descriptions, and solve for the couplings in the IR theory in terms of those of the UV theory.



Euler-Heisenberg Effective Action. What is this IR theory of just the photon, at energies below the mass of the electron? Let's play the EFT game. The dofs are just the photon. The symmetries are: Lorentz and charge conjugation symmetry ($A_\mu \to -A_\mu$) and parity and time-reversal symmetry. Also in the redundant description in terms of A_μ we must impose gauge invariance. These facts already mean that the Lagrangian is just a function of $F_{\mu\nu}$. The UV cutoff is the mass of the electron.

What's the action? Well, of course there is the Maxwell term.

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + c_1 F_{\mu\nu}\partial_\rho\partial^\rho F^{\mu\nu} + c_2 (F_{\mu\nu}F^{\mu\nu})^2 + c_3 (F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \cdots . \tag{4.17}$$

The cubic term $F^\nu_\mu F^\rho_\nu F^\mu_\rho$ is forbidden by C symmetry. $[c_1] = 2$ so $c_1 \propto \frac{1}{m_e^2}$. The F^4 operators have dimension eight, so $[c_2] = [c_3] = 4$, and we conclude that $c_{2,3} \propto \frac{1}{m_e^4}$.

In the UV theory, the $F\partial^2 F$ comes from the next term in the vacuum polarization in an expansion in q^2. The F^4 terms come from a loop of electrons with four external photon lines. This implies that $c_{2,3} \propto \frac{\alpha^2}{16\pi^2}$, where the $16\pi^2$ is always associated with a loop in four dimensions. Using the full QED theory we can of course compute the precise numerical factors. (The contributions from QCD are a topic of great current interest because this appears as a sub-diagram in $g-2$ of the muon, and dominates the current theory uncertainty in that quantity.)

The cross section for $\gamma\gamma \to \gamma\gamma$ is then

$$\sigma_{\gamma\gamma\to\gamma\gamma}(\omega) \sim \frac{\alpha^4\omega^6}{m_e^8}(1 + \mathcal{O}(\omega^2/m_e^2)). \tag{4.18}$$

The power of ω is determined by dimensional analysis so that $[\sigma] = -2$; the amplitude is $\mathcal{A} \sim \omega^4$ from the four factors of F and ω^{-2} comes from the phase space measure. This is a small cross section and this process has not yet been observed.

4.5 The Standard Model as an EFT.

The Standard Model. [Schwartz, §29]

	$L = \begin{pmatrix}\nu_L \\ e_L\end{pmatrix}$	e_R	ν_R	$Q = \begin{pmatrix}u_L \\ d_L\end{pmatrix}$	u_R	d_R	H
$\mathsf{SU}(3)$	-	-	-	□	□	□	-
$\mathsf{SU}(2)$	□	-	-	□	-	-	□
$\mathsf{U}(1)_Y$	$-\frac{1}{2}$	-1	0	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$

Table 1: The Standard Model fields and their quantum numbers under the gauge group. □ indicates fundamental representation, - indicates singlet. Except for the Higgs, each column is copied three times; each copy is called a *generation*. Except for the Higgs all the matter fields are Weyl fermions of the indicated handedness. Gauge fields as implied by the gauge groups. (Some people might leave out the right-handed neutrino, ν_R, which is totally neutral and therefore is hard to observe directly.)

The Lagrangian is just all the terms that are invariant under the gauge group $\mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)$ with dimension less than or equal to four – all renormalizable terms. This includes a potential for the Higgs, $V(|H|) = m_H^2|H|^2 + \lambda|H|^4$, where it turns out that $m_H^2 \leq 0$. The resulting vacuum expectation value higgses the Electroweak part of the gauge group down to electromagnetism (the subgroup preserving the doublet $\langle H \rangle$):

$$\mathsf{SU}(2) \times \mathsf{U}(1)_Y \overset{\langle H \rangle}{\rightsquigarrow} \mathsf{U}(1)_{EM}.$$

That is, the broken gauge bosons get masses from the Higgs kinetic term

$$|D_\mu H|^2|_{H=\begin{pmatrix}0 \\ v/\sqrt{2}\end{pmatrix}} \quad \text{with} \quad D_\mu H = \left(\partial_\mu - \mathbf{i}gW_\mu^a \tau^a - \frac{1}{2}\mathbf{i}g' Y_\mu\right) H$$

where Y_μ is the hypercharge gauge boson, and $W^a, a = 1, 2, 3$ are the $\mathsf{SU}(2)$ gauge bosons. There are two massive W-bosons with electric charge ± 1 (as described in §4.3), with $M_W = \frac{vg}{2}$. The photon and Z boson are the linear combinations of Y and W^3 that diagonalize the remaining mass terms:

$$\begin{pmatrix}A_\mu \\ Z_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w\end{pmatrix} \begin{pmatrix}Y_\mu \\ W_\mu^3\end{pmatrix}.$$

Here $\tan\theta_w \equiv \frac{g'}{g}$ defines the Weinberg angle. The masses are $M_\gamma = 0$ and $M_Z = \frac{M_W}{\cos\theta_w} < M_W$.

Fermion masses come from (dimension-four) Yukawa couplings

$$\mathcal{L}_{\text{Yukawa}} = -Y_{ij}^\ell \bar{L}_i H e_R^j - Y_{ij}^u \bar{Q}^i H d_R^j - Y_{ij}^d \bar{Q}^i \left(\mathbf{i}\tau^2 H^\star\right) u_R^j + h.c.$$

The contortion with the τ^2 is required to make a hypercharge-invariant. Plugging in the Higgs vev to *e.g.* the lepton terms gives $-m_e \bar{e}_L e_R + h.c.$ with $m_e = y_e v/\sqrt{2}$. There's lots of drama about the matrices Y which can mix the generations. The mass for the ν_R (which maybe could not exist – it doesn't have any charges at all) you can figure out on the homework later.

Whence the values of the charges under the $\mathsf{U}(1)$ ("hypercharge")? The condition $Y_L + 3Y_Q = 0$ (where Y is the hypercharge) is required by anomaly cancellation. This implies that electrons and protons $p = \epsilon_{ijk} u_i u_j d_k$ have exactly opposite charges of the same magnitude.

In fact, they are completely determined by demanding that the gauge group is not anomalous, *i.e.* that the $\mathsf{G}_1\mathsf{G}_2\mathsf{G}_3$ anomaly vanishes for all choices of $\mathsf{G}_i \in \{\mathsf{SU}(3), \mathsf{SU}(2), \mathsf{U}(1)_Y\}$ in the presence of gauge fields for all three gauge groups.

To check this, it is enough to ignore the Higgs field and the dynamics of the gauge fields. The coupling to the Higgs field produces masses for the fermions in a way that preserves all of the gauge invariance, despite the fact that $\mathsf{SU}(2) \times \mathsf{U}(1)_Y$ acts in a chiral manner. But the Higgs field is a scalar that transforms linearly, and so it doesn't contribute to the anomaly and we can just set it to zero and ignore it, and the calculation reduces to the one in the section on the chiral anomaly. A previous homework outlined all the choices of $\mathsf{G}_1\mathsf{G}_2\mathsf{G}_3$.

Here is a useful mnemonic for remembering the table of quantum numbers (possibly it is more than that): There are larger simple Lie groups that contain the SM gauge group as subgroups:

$$\begin{array}{ccccccc} \mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)_Y & \subset & \mathsf{SU}(5) & \subset & \mathsf{SO}(10) \\ \text{one generation} & = & 10 \oplus \bar{5} \oplus 1 & = & 16 \end{array}$$

The singlet of $\mathsf{SU}(5)$ is the right-handed neutrino, but if we include it, one generation is an irreducible (spinor) representation of $\mathsf{SO}(10)$. This idea is called grand unification. It is easy to imagine that the gauge group is actually the larger groups on the right, and another instance of the Higgs mechanism accomplishes the breaking down to the Standard Model. (The running of the respective gauge couplings go in the right direction with approximately the right rate to unify to a single value at $M_{GUT} \sim 10^{16} GeV$.)

Beyond the Standard Model with EFT. At what energy does the Standard Model stop working? Because of the annoying feature of renormalizibility, it doesn't tell us. However, we have experimental evidence against a cutoff on the Standard Model (SM) at energies less than something like 10 TeV. The evidence I have in mind is the absence of interactions of the form

$$\delta L = \frac{1}{M^2} \left(\bar{\psi} A \psi\right) \cdot \left(\bar{\psi} B \psi\right)$$

(where ψ represent various SM fermion fields and A, B can be various gamma and flavor matrices) with $M \lesssim 10$ TeV. Notice that I am talking now about interactions *other* than the electroweak interactions, which as we've just discussed, for energies above $M_W \sim 80$GeV cannot be treated as contact interactions – you can see the Ws propagate!

If such operators were present, we would have found different answers for experiments at LEP. But such operators would be present if we consider new physics in addition to the Standard Model (in most ways of doing it) at energies less than 10 TeV. For example, many interesting ways of coupling in new particles with masses that make them accessible at the LHC would have generated such operators.

A little more explicitly: the Standard Model Lagrangian L_0 contains all the renormalizable (*i.e.* engineering dimension ≤ 4) operators that you can make from its fields (though the coefficients of the dimension 4 operators do vary through quite a large range, and the coefficients of the two relevant operators – namely the identity operator which has dimension zero, and the Higgs mass, which has engineering dimension two, are strangely small, and so is the QCD θ angle).

To understand what lies beyond the Standard Model, we can use our knowledge that whatever it is, it is probably heavy (it could also just be very weakly coupled, which is a different story), with some intrinsic scale $\Lambda_{\rm new}$, so we can integrate it out and include its effects by corrections to the Standard Model:

$$L = L_0 + \frac{1}{\Lambda_{\rm new}}\mathcal{O}^{(5)} + \frac{1}{\Lambda_{\rm new}^2}\sum_i c_i \mathcal{O}_i^{(6)} + \cdots$$

where the $\mathcal{O}$s are made of SM fields, and have the indicated engineering dimensions, and preserve the necessary symmetries of the SM (Lorentz symmetry and gauge invariance).

In fact there is only one kind of operator of dimension 5 meeting these demands:

$$\mathcal{O}^{(5)} = c_5 \epsilon_{ij} \left(\bar{L}^c\right)^i H^j \epsilon_{kl} L^k H^l$$

where $H^i = (h^+, h^0)^i$ is the $\mathrm{SU}(2)_{EW}$ Higgs doublet and $L^i = (\nu_L, e_L)^i$ is an $\mathrm{SU}(2)_{EW}$ doublet of left-handed leptons, and $\bar{L}^c \equiv L^T C$ where C is the charge conjugation matrix. (I say 'kind of operator' because we can have various flavor matrices in here.) On the problem set you get to see from whence such an operator might arise, and what it does if you plug in the higgs vev $\langle H \rangle = (0, v)$. This term violates lepton number symmetry ($L \to e^{\mathbf{i}\alpha_L} L, Q \to Q, H \to H$).

[End of Lecture 11]

At dimension 6, there are operators that directly violate baryon number, such as

$$\epsilon_{\alpha\beta\gamma} (\bar{u}_R)^c_\alpha (u_R)_\beta \, (\bar{u}_R)^c_\gamma \, e_R.$$

You should read the above tangle of symbols as '$qqq\ell$' – it turns three quarks into a lepton. The epsilon tensor makes a color $\mathsf{SU}(3)$ singlet; this thing ϵqqq has the quantum numbers of a baryon, such as a proton or neutron. The long lifetime of the proton (you can feel it in your bones – see Zee p. 413) then directly constrains the scale of new physics appearing in front of this operator.

Three more comments about proton decay:

- The idea of Grand Unification means leptons and quarks are in the same representations of a larger gauge group – they can turn into each other by exchanging GUT gauge bosons. This predicts that the proton should not be perfectly stable, and integrating out the GUT gauge bosons should produce baryon-number violating operators like the ones above, suppressed by $M_{\text{GUT}} \simeq 10^{16}$ GeV.
- If we didn't know about the Standard Model, (but after we knew about QM and GR and EFT (the last of which people didn't know before the SM for some reason)) we should have made the estimate that dimension-5 Planck-scale-suppressed operators like $\frac{1}{M_{\text{Planck}}} p\mathcal{O}$ would cause proton decay (into whatever $\mathcal{O}$ makes). This predicts $\Gamma_p \sim \frac{m_p^3}{M_{\text{Planck}}^2} \sim 10^{-13} s^{-1}$ which is *not* consistent with our bodies not glowing. Actually it is a remarkable fact that there are no gauge-invariant operators made of SM fields of dimension less than 6 that violate baryon number symmetry ($L \to L, Q \to e^{\mathbf{i}\alpha_B} Q, H \to H$). This is an *emergent* symmetry, expected to be violated by the UV completion.
- Surely nothing can prevent $\Delta L \sim \left(\frac{1}{M_{\text{Planck}}}\right)^2 qqq\ell$. Happily, this is consistent with the observed proton lifetime.

There are also $\sim 10^2$ dimension 6 operators that preserve baryon number, and therefore are not as tightly constrained[24]. (Those that induce flavor-changing processes in the SM are more highly constrained and must have $\Lambda_{\text{new}} > 10^4$ TeV.) Two such operators are considered equivalent if they differ by something that vanishes by the tree-level SM equations of motion. This is the right thing to do, even for off-shell calculations (like green's functions and for fields running in loops). You know this from a previous problem set: the EOM are true as operator equations – Ward identities resulting from being free to change integration variables in the path integral[25].

[24] Recently, humans have gotten better at counting these operators. See this paper.

[25] There are a few meaningful subtleties here, as you might expect if you recall that the Ward identity is only true up to contact terms. The measure in the path integral can produce a Jacobian which renormalizes some of the couplings; the changes in source terms will drop out of S-matrix elements (recall our discussion of changing field variables in the Consequences of Unitarity section) but can change the form of Green's functions. For more information on the use of eom to eliminate redundant operators in EFT, see Arzt, hep-ph/9304230 and Georgi, "On-Shell EFT".

A special pair of dimension 6 operators lead to mixing between the various electroweak gauge bosons:

$$\Delta L \ni S\frac{\alpha}{\sin\theta_W\cos\theta_W v^2}H^\dagger W_a^{\mu\nu}\tau^a H B_{\mu\nu} - T\frac{2\alpha}{v^2}|H^\dagger D_\mu H|^2 \tag{4.19}$$

where $B_{\mu\nu}$ is the hypercharge gauge field strength. These 'oblique parameters' S and T were introduced by Peskin and Takeuchi and are very strongly constrained by the large amount of data from e^+e^- collisions at the Z resonance. They are expected to be large in technicolor models. For a systematic discussion, see for example here.

4.6 General relativity as an EFT

Let's continue playing the effective field theory game. Recall that the game is played by filling out the following survey:

1. What are the dofs?
2. What are the symmetries and what are the redundancies of the description?
3. What is the cutoff?

Then the output is an action, which is a sum of all terms made from the dofs, respecting the symmetries and redundancies, organized as a derivative expansion with higher order terms suppressed by more powers of the cutoff.

Einstein sort of played this game in 1915 in building a theory of gravity. His answer for the dofs was: a metric on spacetime $g_{\mu\nu}(x)$. This is a coordinate-dependent description of a line element $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ that gives the distances between spacetime points. This description is redundant in that the same line element can be written in different coordinate systems (such as $dx^2+dy^2 = dr^2+r^2d\theta^2$). The cutoff is the mass scale appearing in Newtonian gravity: $G_N = \frac{\#}{M_P^2}$, the Planck mass. (Here I'm using units with $\hbar = c = 1$.)

The demand that physics is independent of the choice of coordinate system is highly constraining, and the only terms one can write down are

$$S[g_{\mu\nu}] = M_P^2\int d^4x\left(\#M_P^2 + \#R + \frac{\#}{M_P^2}R^2 + \frac{\#}{M_P^4}R^3 + \frac{\#}{M_P^4}DRDR\cdots\right) \tag{4.20}$$

where R is the Ricci scalar, and R^n represents various possible contractions of n powers of the Riemann tensor.

This is the order in which Einstein should have written the terms, if he were (teleologically) following Wilson's rules. The first term is the cosmological constant, the

constant operator of dimension zero. Here it matters quite a bit, because it changes the equations of motion of the metric. It is observed to be very small in units of M_P^2. We don't know why this is the case. It is a gross violation of the rules of EFT.

The next term is the Einstein-Hilbert term, which is the only one Einstein included. The higher-order terms are too small to have any effect on any observation so far. The observations so far have all been done in a regime where the curvature R is small compared to M_P^2. I've normalized the terms the way I have in (4.20) because the EH term gives the kinetic term as well as a specific set of (irrelevant) interactions – rescaling the fields to have canonical kinetic terms, these interactions come with coefficients of negative mass dimension. All the rest of the terms are also irrelevant perturbations of the free-graviton fixed point.

The coupling to matter is also largely determined by demanding coordinate invariance (replace $\eta^{\mu\nu}$ by $g^{\mu\nu}$, replace ∂_μ by covariant derivative D_μ, and for spinors do the thing we described above). Included in the derivative expansion should also be terms involving matter fields and curvatures, like RH^2, where H is the higgs field.

Notice, very importantly, that we only have integer powers of R, not anything like $\sqrt{R}$ or $1/R$. Why is that? Well, this action is arising by integrating out small-wavelength modes. Integrating out such modes can't produce anything non-analytic, either in the momenta or in the fields themselves.

What are the loopholes in this argument? Well, the statement that the dimensionless numerical coefficients (all written as '#' above) are order one may not be correct – the cosmological constant is already a violation of this rule, so maybe some of the higher derivative terms could be important. Another loophole is in the choice of dofs. There could be other light dofs, like a scalar field, that we should include in our game (though then we have to explain why the mass for the scalar is small compared to M_P.)

I emphasize that this is a perfectly good quantum field theory. It is nice enough to be non-renormalizable and to tell us its (maximal) regime of validity. (Of course it could break down at a scale lower than M_P if we are missing some important other dofs.) It can be studied in perturbation theory about some vacuum geometry (such as flat space for $\Lambda = 0$, or anti-de Sitter space (AdS) for $\Lambda < 0$ or de Sitter space for $\Lambda > 0$). The tree-level approximation, *i.e.* classical physics, has been good enough for all observations so far.

The problem of Quantum Gravity arises in asking what is a more microscopic theory for which this is a low-energy EFT. The only candidate answer to that question that we have is string theory. The physics questions for we need to answer such a question involve large curvature or otherwise-strong fields (such as inside black holes, or in the very early universe) or if we care about which values of the coefficients # (or what

choices of matter coupled to gravity) are possible. Notice that, based on our experience with other examples of EFT (especially the next one), there is no good reason to think that the dofs of that UV theory should have anything to do with $g_{\mu\nu}$, nor to expect that there is something special about the Einstein-Hilbert term from the UV point of view.

4.7 Superconductors and superfluids

Who is Φ? Last quarter, we developed an effective (Landau-Ginzburg) description of superconductors which reproduces the Meissner effect (that magnetic flux is expelled or collimated into flux tubes); it is called the Abelian Higgs model: for time-independent configurations,

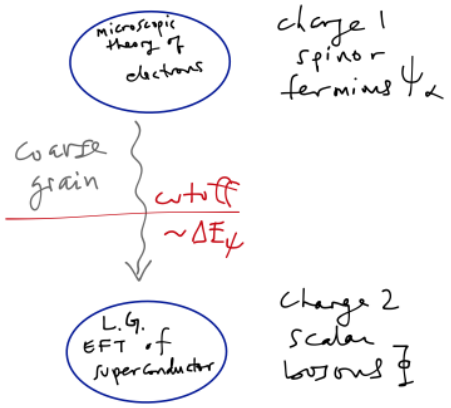
$$\mathcal{L} = \frac{1}{4}F_{ij}F_{ij} + |D_i\Phi|^2 + a|\Phi|^2 + \frac{1}{2}b|\Phi|^4 + ... \tag{4.21}$$

with $D_i\Phi \equiv (\partial_i - 2e\mathbf{i}A_i)\,\Phi$. Here A is the photon field. This is a slight modification of the previous expression to indicate that the Higgs field Φ has electric charge two. (We'll discuss the time-derivative terms later.) We could have guessed this description by playing the EFT game, knowing that the dofs involved are the photon and a charge-two scalar field. But who is this charge-two scalar field? (Relatedly: what is the cutoff on the validity of this description?)

New IR dofs. A feature of this example that I want you to notice: the microscopic description of a real superconductor involves electrons – charge $1e$ spinor fermions, created by some fermionic operator ψ_α, $\alpha = \uparrow, \downarrow$.

We are describing the low-energy physics of a system of electrons in terms of a bosonic field, which (in simple 's-wave' superconductors) is roughly related to the electron field by

$$\Phi \sim \psi_\alpha \psi_\beta \epsilon^{\alpha\beta} \; ; \tag{4.22}$$



Φ is called a Cooper pair field. At least, the charges and the spins and the statistics work out. The details of this relationship are not the important point I wanted to emphasize. Rather I wanted to emphasize the dramatic difference in the correct choice of variables between the UV description (spinor fermions) and the IR description (scalar bosons). One reason that this is possible is that it costs a large energy to make a fermionic excitation of the superconductor.

This can be understood roughly as follows: The microscopic theory of the electrons looks something like (ignoring the coupling to electromagnetism for now, except for a screened (and therefore short-ranged) repulsion which ultimately is the Coulomb interaction)

$$S[\psi] = S_2[\psi] + \int \mathrm{d}t\mathrm{d}^d x \; u\psi^\dagger\psi\psi^\dagger\psi + h.c. \tag{4.23}$$

where

$$S_2 = \int \mathrm{d}t \int \mathrm{đ}^d k \psi_k^\dagger \left(\mathbf{i}\partial_t - \epsilon(k)\right) \psi_k.$$

Spin is important here so that $\psi_\uparrow^\dagger\psi_\uparrow\psi_\downarrow^\dagger\psi_\downarrow$ is nonzero. A mean field theory description of the condensation of Cooper pairs (4.22) is obtained by replacing the quartic term in (4.23) by expectation values:

$$\begin{aligned} S_{MFT}[\psi] &= S_2[\psi] - \int \mathrm{d}t\mathrm{d}^d x \; u \langle\psi\psi\rangle \psi^\dagger\psi^\dagger + h.c. \\ &= S_2[\psi] - \int \mathrm{d}t\mathrm{d}^d x \; u\Phi\psi^\dagger\psi^\dagger + h.c. \end{aligned} \tag{4.24}$$

So an expectation value for Φ is a mass for the fermions. It is a funny kind of symmetry-breaking mass, but if you diagonalize the quadratic operator in (4.24) (actually it is done below) you will find that it costs an energy of order $\Delta E_\psi = u\langle\Phi\rangle$ to excite a fermion. That's the cutoff on the LG EFT.

A general lesson from this example is: the useful degrees of freedom at low energies can be very different from the microscopic dofs.

4.7.1 Lightning discussion of BCS.

I am sure that some of you are nervous about the step from $S[\psi]$ to $S_{MFT}[\psi]$ above. To make ourselves feel better about it, I will say a few more words about the steps from the microscopic model of electrons (4.23) to the LG theory of Cooper pairs (these steps were taken by Bardeen, Cooper and Schreiffer (BCS)).

First recall the *Hubbard-Stratonovich transformation* aka *completing the square*. In 0+0 dimensional field theory:

$$e^{-\mathbf{i}ux^4} = \frac{1}{\sqrt{\mathbf{i}\pi u}} \int_{-\infty}^{\infty} \mathrm{d}\sigma \; e^{-\frac{1}{\mathbf{i}u}\sigma^2 - 2\mathbf{i}x^2\sigma} \; . \tag{4.25}$$

At the cost of introducing an extra field σ, we turn a quartic term in x into a quadratic term in x. The RHS of (4.25) is gaussian in x and we know how to integrate it over x. (The version with $\mathbf{i}$ is relevant for the real-time integral.) Notice the weird extra factor of $\mathbf{i}$ lurking in (4.25). This can be understood as arising because we are trying

to use a scalar field, σ, to mediate a repulsive interaction (which it is, for positive u) (see Zee p. 193, 2nd Ed).

Actually, we'll need a complex H-S field:

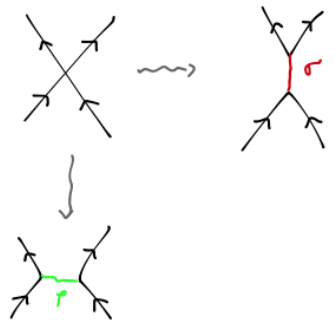
$$e^{-\mathbf{i}ux^2\bar{x}^2} = \frac{1}{\mathbf{i}\pi u}\int_{\mathbb{C}} \mathrm{d}^2\sigma \; e^{-\frac{1}{\mathbf{i}u}|\sigma|^2 - \mathbf{i}x^2\bar{\sigma} + \mathbf{i}\bar{x}^2\sigma} \;, \tag{4.26}$$

where $\int_{\mathbb{C}} \mathrm{d}^2\sigma ... \equiv \int_{-\infty}^{\infty} d\mathrm{Re}\,\sigma \int_{-\infty}^{\infty} d\mathrm{Im}\,\sigma ...$ (The field-independent prefactor is, as usual, not important for path integrals.)

We can use a field theory generalization of (4.26) to 'decouple' the 4-fermion interaction in (4.23):

$$Z = \int [D\psi D\psi^\dagger] e^{\mathbf{i}S[\psi]} = \int [D\psi D\psi^\dagger D\sigma D\sigma^\dagger] e^{\mathbf{i}S_2[\psi] + \mathbf{i}\int d^Dx \left(\bar{\sigma}\psi_\uparrow\psi_\downarrow + h.c.\right) - \int \mathrm{d}^Dx \frac{|\sigma|^2(x)}{\mathbf{i}u}} \;. \tag{4.27}$$

The point of this is that now the fermion integral is gaussian. At the saddle point of the σ integral (which is exact because it is gaussian), σ is the Cooper pair field, $\sigma_{\text{saddle}} = u\psi_\uparrow\psi_\downarrow$.



Notice that we made a choice here about in which 'channel' to make the decoupling – we could have instead introduces a different auxiliary field ρ and written $S[\rho,\psi] = \int \rho\psi^\dagger\psi + \int \frac{\rho^2}{2u}$, which would break up the 4-fermion interaction in the t-channel (as an interaction of the fermion density $\psi^\dagger\psi$) instead of the s (BCS) channel (as an interaction of Cooper pairs ψ^2). At this stage both are correct, but they lead to different mean-field approximations below. That the BCS mean field theory wins is a consequence of the RG, as I'll describe below.

How can you resist doing the fermion integral in (4.27)? Let's study the case where the single-fermion dispersion is $\epsilon(k) = \frac{\vec{k}^2}{2m} - \mu$.

$$I_\psi[\sigma] \equiv \int [D\psi D\psi^\dagger] e^{\mathbf{i}\int \mathrm{d}t\mathrm{d}^dx \left(\psi^\dagger\left(\mathbf{i}\partial_t - \frac{\nabla^2}{2m} - \mu\right)\psi + \bar{\sigma}\psi\psi + \bar{\psi}\bar{\psi}\sigma\right)}$$

The action here can be written as the integral of

$$L = \begin{pmatrix}\bar{\psi} & \psi\end{pmatrix} \begin{pmatrix} \mathbf{i}\partial_t - \epsilon(-\mathbf{i}\nabla) & \sigma \\ \bar{\sigma} & -\left(\mathbf{i}\partial_t - \epsilon(-\mathbf{i}\nabla)\right)\end{pmatrix} \begin{pmatrix}\psi \\ \bar{\psi}\end{pmatrix} \equiv \begin{pmatrix}\bar{\psi} & \psi\end{pmatrix} M \begin{pmatrix}\psi \\ \bar{\psi}\end{pmatrix}$$

so the functional integral is

$$I_\psi[\sigma] = \det M = e^{\mathrm{tr}\log M(\sigma)}.$$

If σ is constant (which will lower the energy), the matrix M is diagonal in momentum space, and the integral remaining to be done is

$$\int [D\sigma D\sigma^\dagger] e^{-\int \mathrm{d}^D x \frac{|\sigma(x)|^2}{2\mathbf{i}u} + \int đ^D k \log\left(\omega^2 - \epsilon_k^2 - |\sigma|^2\right)}.$$

It is often possible to do this integral by saddle point. This can be justified, for example, by the largeness of the volume of the Fermi surface, $\{k|\epsilon(k) = \mu\}$, or by a large number N of species of fermions. The result is an equation that determines σ, which as we saw earlier determines the fermion gap.

$$0 = \frac{\delta \text{exponent}}{\delta \bar{\sigma}} = \mathbf{i}\frac{\sigma}{2u} + \int đ\omega đ^d k \frac{2\sigma}{\omega^2 - \epsilon_k^2 - |\sigma|^2 + \mathbf{i}\epsilon} \ .$$

[End of Lecture 12]

We can do the frequency integral by residues:

$$\int đ\omega \frac{1}{\omega^2 - \epsilon_k^2 - |\sigma|^2 + \mathbf{i}\epsilon} = \frac{1}{2\pi} 2\pi\mathbf{i} \frac{1}{2\sqrt{\epsilon_k^2 + |\sigma|^2}}.$$

The resulting equation is naturally called the *gap equation*:

$$1 = -2u \int đ^d p' \frac{1}{\sqrt{\epsilon(p')^2 + |\sigma|^2}} \tag{4.28}$$

which you can imagine solving self-consistently for σ[26]. Plugging back into the action (4.27) says that σ determines the energy cost to have electrons around; more precisely, σ is the energy required to break a Cooper pair.

Comments:

- Notice that a solution of (4.28) requires $u < 0$, an *attractive* interaction. Superconductivity happens because the u that appears here is not the bare interaction between electrons, which is certainly repulsive (and long-ranged). This is where the phonons come in in the BCS discussion.

[26] I should have said: and in fact one can solve it. As we will learn in the next section, the integral is dominated by the behavior near the Fermi surface, near which $\epsilon(p') \simeq v_F \ell \equiv \varepsilon$; this approximation is valid for $|\epsilon| < E_D$, some UV cutoff on this description. The result is

$$1 = -2u \int \frac{đ^d p'}{\sqrt{\epsilon(p')^2 + |\sigma|^2}} \simeq -2u \int_{FS} \frac{đ^{d-1} k}{v_F} \int_{-E_D}^{E_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + |\sigma|^2}} = Nu \log\left(\frac{E_D + \sqrt{E_D^2 + |\sigma|^2}}{|\sigma|}\right)$$

where $N \equiv \int_{FS} \frac{đ^{d-1}k}{2\pi v_F}$ is the density of states at the Fermi surface. The largeness of this N justifies the saddle-point approximation. The solution for σ is

$$|\sigma| = \frac{2E_D e^{\frac{1}{2Nu}}}{e^{\frac{1}{Nu}} - 1} \overset{Nu \ll 1}{\simeq} 2E_D e^{-\frac{1}{Nu}}.$$

Notice that this is non-perturbative in the coupling strength u.

- If we hadn't restricted to a delta-function 4-fermion interaction $u(p,p') = u_0$ at the outset, we would have found a more general equation like

$$\sigma(\vec{p}) = -\frac{1}{2}\int \mathrm{đ}^d p' \frac{u(p,p')\sigma(\vec{p}')}{\sqrt{\epsilon(p')^2 + |\sigma(p')|^2}} \ .$$

- A conservative perspective on the preceding calculation is that we have made a variational ansatz for the groundstate wavefunction, and the equation we solve for σ is minimizing the variational energy – finding the best wavefunction within the ansatz.

- I haven't included here effects of the fluctuations of the sigma field about its saddle point. In fact, they make the four-fermion interaction that leads to Cooper pairing marginally *relevant*. This breaks the degeneracy in deciding how to split up the $\psi\psi\psi^\dagger\psi^\dagger$ into *e.g.* $\psi\psi\sigma$ or $\psi^\dagger\psi\rho$. BCS wins. This is explained beautifully in Polchinski, lecture 2, and R. Shankar. I will summarize the EFT framework for understanding this in §4.8.

- I've tried to give the most efficient introduction I could here. I left out any possibility of k-dependence or spin dependence of the interactions or the pair field, and I've conflated the pair field with the gap. In particular, I've been sloppy about the dependence on k of σ above.

- You can study a very closely related manipulation on the problem set, in examples (the $\mathsf{O}(N)$ model and the Gross-Neveu model) where the saddle point is justified by large N.

4.7.2 Non-relativistic scalar fields

[Zee §III.5, V.1, Kaplan nucl-th/0510023 §1.2.1] In the previous discussion of the EFT for a superconductor, I just wrote the free energy, and so we didn't have to think about whether the complex scalar in question was relativistic or not.

It is not. In real superconductors, at least. How should we think about a non-relativistic field? A simple answer comes from realizing that a relativistic field that can make a boson of mass m can certainly make a boson of mass m that is moving slowly, with $v \ll c$. By taking a limit of the relativistic model, then, we can make a description that is useful for describing the interactions of an indefinite number of bosons moving slowly in some Lorentz frame. A situation that calls for such a description, for example, is a large collection of ^{4}He atoms.

Reminder: Non-relativistic limit of a relativistic scalar field. A non-relativistic particle in a relativistic theory (like the ϕ^4 theory that we've been spending time with) has energy

$$E = \sqrt{p^2 + m^2} \overset{\text{if } v \ll c}{=} m + \frac{p^2}{2m} + ...$$

This means that the field that creates and annihilates it looks like

$$\phi(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}}} \left(a_{\vec{k}} e^{\mathbf{i}E_{\vec{k}}t - \mathbf{i}\vec{k}\cdot\vec{x}} + h.c. \right)$$

In particular, we have

$$\dot{\phi}^2 \simeq m^2 \phi^2$$

and the BHS of this equation is large. To remove this large number let's change variables:

$$\phi(x,t) \equiv \frac{1}{\sqrt{2m}} \left(e^{-\mathbf{i}mt} \underbrace{\Phi(x,t)}_{\text{complex}, \dot{\Phi} \ll m\Phi} + h.c. \right) .$$

Notice that Φ is complex, even if ϕ is real.

Let's think about the action governing this NR sector of the theory. We can drop terms with unequal numbers of Φ and $\Phi^\star$ since such terms would come with a factor of $e^{\mathbf{i}mt}$ which gives zero when integrated over time. Starting from $(\partial\phi)^2 - m^2\phi^2 - \lambda\phi^4$ we get:

$$L_{\text{real time}} = \Phi^\star \left(\mathbf{i}\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \Phi - g^2 \left(\Phi^\star \Phi\right)^2 + ... \tag{4.29}$$

with $g^2 = \frac{\lambda}{4m^2}$.

Notice that Φ is a complex field and its action has a U(1) symmetry, $\Phi \to e^{\mathbf{i}\alpha}\Phi$, even though the full theory did not. The associated conserved charge is the number of particles:

$$j_0 = \Phi^\star \Phi, j_i = \frac{\mathbf{i}}{2m} \left(\Phi^\star \partial_i \Phi - \partial_i \Phi^\star \Phi \right), \quad \partial_t j_0 - \nabla \cdot \vec{j} = 0 . \tag{4.30}$$

Notice that the 'mass term' $\Phi^\star\Phi$ is then actually the chemical potential term, which encourages a nonzero density of particles to be present.

This is another example of an *emergent* symmetry (like baryon number in the SM): a symmetry of an EFT that is not a symmetry of the microscopic theory. The ... in (4.29) include terms that break this symmetry, but they are irrelevant.

To see more precisely what we mean by irrelevant, let's think about scaling. To keep this kinetic term fixed we must scale time and space differently:

$$x \to \tilde{x} = sx, \; t \to \tilde{t} = s^2 t, \; \Phi \to \tilde{\Phi}(\tilde{x}, \tilde{t}) = \zeta \Phi(sx, s^2 t) .$$

A fixed point with this scaling rule has dynamical exponent $z = 2$. The scaling of the bare action (with no mode elimination step) is

$$S_E^{(0)} = \int \underbrace{dtd^d\vec{x}}_{=s^{d+z}d\tilde{t}d^d\tilde{x}} \left(\Phi^\star\left(sx, s^2t\right) \underbrace{\left(\partial_t - \frac{\vec{\nabla}^2}{2m}\right)}_{=s^{-2}\left(\tilde{\partial}_t - \frac{\tilde{\vec{\nabla}}^2}{2m}\right)} \Phi(sx, s^2t) - g^2\left(\Phi^\star\Phi(sx, s^2t)\right)^2 + ... \right)$$

$$= \underbrace{s^{d+z-2}\zeta^{-2}}_{\stackrel{!}{=}1 \implies \zeta = s^{-d/2}} \int \mathrm{d}\tilde{t}\mathrm{d}^d\tilde{x} \left(\tilde{\Phi}^\star\left(\tilde{\partial}_t - \frac{\tilde{\vec{\nabla}}^2}{2m}\right)\tilde{\Phi} - \zeta^{-2}g^2\left(\tilde{\Phi}^\star\tilde{\Phi}(\tilde{x}, \tilde{t})\right)^2 + ... \right) \quad (4.31)$$

From this we learn that $\tilde{g} = s^{2-d}g \to 0$ in the IR – the quartic term is irrelevant in $D = d+1 = 3+1$ with nonrelativistic scaling! Where does it become marginal? (Hint: look back at the first lecture of last quarter.)

Number and phase angle. In the NR theory, the canonical momentum for Φ is just $\frac{\partial L}{\partial \dot{\Phi}} \sim \Phi^\star$, with no derivatives. This statement becomes more shocking if we change variables to $\Phi = \sqrt{\rho}e^{\mathbf{i}\varphi}$ (which would be useful *e.g.* if we knew ρ didn't want to be zero); the action density is

$$L = \frac{\mathbf{i}}{2}\partial_t\rho - \rho\partial_t\varphi - \frac{1}{2m}\left(\rho\left(\nabla\varphi\right)^2 + \frac{1}{4\rho}\left(\nabla\rho\right)^2\right) - g^2\rho^2. \quad (4.32)$$

The first term is a total derivative. The second term says that the canonical momentum for the phase variable φ is $\rho = \Phi^\star\Phi = j_0$, the particle number density. Quantumly, then:

$$[\hat{\rho}(\vec{x}, t), \hat{\varphi}(\vec{x}', t)] = \mathbf{i}\delta^d(\vec{x} - \vec{x}').$$

Number and phase are canonically conjugate variables. If we fix the phase, the amplitude is maximally uncertain.

If we integrate over space, $N \equiv \int d^dx\rho(\vec{x}, t)$ gives the total number of particles, which is time independent, and satisfies $[N, \varphi] = \mathbf{i}$.

This relation explains why there's no Higgs boson in most non-relativistic superconductors and superfluids (in the absence of some extra assumption of particle-hole symmetry). In the NR theory with first order time derivative, the would-be amplitude mode that oscillates about the minimum of $V(\rho)$ is actually just the conjugate momentum for the goldstone boson!

4.7.3 Superfluids.

[Zee §V.1, Wen §3.3.3] Let me amplify the previous remark. A superconductor is just a superfluid coupled to an external U(1) gauge field, so we've already understood something about superfluids.

The effective field theory has the basic lagrangian (4.32), with $\langle\rho\rangle = \bar{\rho} \neq 0$. This nonzero density can be accomplished by adding an appropriate chemical potential to (4.32); up to an uninteresting constant, this is

$$L = \frac{\mathbf{i}}{2}\partial_t\rho - \rho\partial_t\varphi - \frac{1}{2m}\left(\rho\left(\nabla\varphi\right)^2 + \frac{1}{4\rho}\left(\nabla\rho\right)^2\right) - g^2\left(\rho-\bar{\rho}\right)^2.$$

Expand around such a condensed state in small fluctuations $\sqrt{\rho} = \sqrt{\bar{\rho}}+h$, $h \ll \sqrt{\bar{\rho}}$:

$$L = -2\sqrt{\bar{\rho}}h\partial_t\varphi - \frac{\bar{\rho}}{2m}\left(\vec{\nabla}\varphi\right)^2 - \frac{1}{2m}\left(\vec{\nabla}h\right)^2 - 4g^2\bar{\rho}h^2 + ...$$

Notice that h, the fluctuation of the amplitude mode, is playing the role of the canonical momentum of the goldstone mode φ. The effects of the fluctuations can be incorporated by doing the gaussian integral over h (What suppresses self-interactions of h?), and the result is

$$\begin{aligned} L &= \bar{\rho}\partial_t\varphi\frac{1}{4g^2\bar{\rho} - \frac{\nabla^2}{2m}}\bar{\rho}\partial_t\varphi - \frac{\bar{\rho}}{2m}\left(\vec{\nabla}\varphi\right)^2 \\ &= \frac{1}{4g^2}\left(\partial_t\varphi\right)^2 - \frac{\bar{\rho}}{2m}\left(\nabla\varphi\right)^2 + ... \end{aligned} \tag{4.33}$$

where in the second line we are expanding in the small wavenumber k of the modes, that is, we are constructing an action for Goldstone modes whose wavenumber is $k \ll \sqrt{8g^2\bar{\rho}m}$ so we can ignore higher gradient terms.

The linearly dispersing mode in this superfluid that we have found, sometimes called the phonon, has dispersion relation

$$\omega^2 = \frac{2g^2\bar{\rho}}{m}\vec{k}^2.$$

This mode has an emergent Lorentz symmetry with a lightcone with velocity $v_c = g\sqrt{2\bar{\rho}/m}$. The fact that the sound velocity involves g – which determined the steepness of the walls of the wine-bottle potential – is a consequence of the non-relativistic dispersion of the bosons. In the relativistic theory, in contrast, we have $L = \partial_\mu\Phi^\star\partial^\mu\Phi - \kappa\left(\Phi^\star\Phi - v^2\right)^2$ and we can take $\kappa \to \infty$ fixing v and still get a linearly dispersing mode by plugging in $\Phi = e^{\mathbf{i}\varphi}v$.

I've put the following paragraphs in an ugly color because they don't explain what I thought they explained.

What does this have to do with the phenomenology of superfluids, like dissipationless flow? The importance of the linearly dispersing phonon mode of the superfluid is that there is *no other* low energy excitation of the fluid. With a classical pile of (*e.g.* non interacting) bosons, a chunk of moving fluid can donate some small momentum $\vec{k}$ to a single boson at energy cost $\frac{(\hbar\vec{k})^2}{2m}$. A quadratic dispersion means more modes at small k than a linear one (the density of states is $N(E) \propto k^{D-1}\frac{dk}{dE}$). With only a linearly dispersing mode at low energies, there is a critical velocity below which a non-relativistic chunk of fluid cannot give up any momentum [Landau]: conserving momentum $M\vec{v} = M\vec{v}' + \hbar\vec{k}$ says the change in energy (which must be negative for this to happen on its own) is (eliminate $v' = v - \hbar k/M$):

$$\frac{1}{2}M(v')^2 + \hbar\omega(k) - \frac{1}{2}Mv^2 = -\hbar kv + \frac{(\hbar k)^2}{2M} + \hbar\omega(k) = (-v + v_c)k + \frac{(\hbar k)^2}{2M} \ .$$

For small k, this is only negative when $v > v_c$.

You can ask: an ordinary liquid also has a linearly dispersing sound mode; why doesn't Landau's argument mean that it has superfluid flow? The answer is that it has *other* modes with softer dispersion (so more contribution at low energies), in particular diffusion modes, with $\omega \propto k^2$ (there is an important factor of $\mathbf{i}$ in there).

The Goldstone boson has a compact target space, $\varphi(x) \equiv \varphi(x) + 2\pi$, since, after all, it is the phase of the boson field. This is significant because it means that as the phase wanders around in space, it can come back to its initial value after going around the circle – such a loop encloses a *vortex*. Somewhere inside, we must have $\Phi = 0$. And actually, our discussion of the vortices of the Abelian Higgs model did not depend on the form of the time-derivative terms. There is much more to say about this.

[Wen §3.7.3] The above argument about the Landau critical velocity does not really explain the phenomenon of superflow, where if we set up a current it keeps going for a very long time. One way to see this is that there are superfluids and superconductors where there are other light degrees of freedom besides the linearly-dispersing phonon. For example, sometimes the condensate fails to gap out the fermion excitations.

Here's the real reason for superflow. It happens entirely because the spatial components of the particle-number current have the form (plugging the form of the field into (4.30)):

$$\vec{j} = \frac{\rho}{m}\vec{\nabla}\varphi \tag{4.34}$$

where φ is a compact field $\varphi \simeq \varphi + 2\pi$. Consider the situation where the x direction is a circle $x \simeq x + L$ (for example if the superfluid lives in an annular region). Think about what is required to set up a flow of such a system in the x direction: we must

have

$$\varphi(x) = \alpha x \tag{4.35}$$

for some constant α, with $j_x = \frac{\rho\alpha}{m}$. But compactness of the boson and space requires that $\alpha L \in 2\pi\mathbb{Z}$ is quantized. This integer is the *vorticity* of the configuration. The reason is that the only way it can change is if a *vortex* (a point where $\Phi = 0$, so that φ is ill-defined) appears in the sample. (See the figure below.)

But as we've seen, vortices are costly. In a superfluid (where there is no dynamical gauge field), they are also confined, in the sense that a single vortex has infinite energy, and only a vortex-antivortex pair has finite energy. The difficulty of producing vortices is what makes the superflow configuration a long-lived metastable state.

Notice that in a superconductor, only the combination $\vec{A} + \vec{\nabla}\varphi$ is gauge invariant, so (4.34) is the same as the London equation

$$\vec{j} = \frac{\rho}{m}(\vec{A} + \vec{\nabla}\varphi) \tag{4.36}$$

(φ can be set to zero by choosing unitary gauge). This equation implies the Meissner effect, as you can see by sticking it into the Maxwell equation.

In many accounts of the subject, the above explanation involves some discussion of Galilean invariance. This is not necessary, but it is useful to understand how the configuration (4.35) arises from a slightly more microscopic point of view. The key point is that in order to preserve the action $S = \int d^d x dt \mathcal{L}$,

$$\mathcal{L} = \Phi^\star \left(\mathbf{i}\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \Phi - V(\Phi^\star \Phi) \ , \tag{4.37}$$

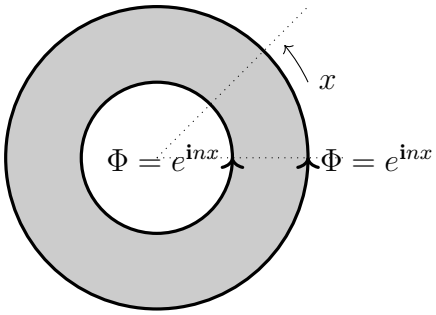
the Galilean boost

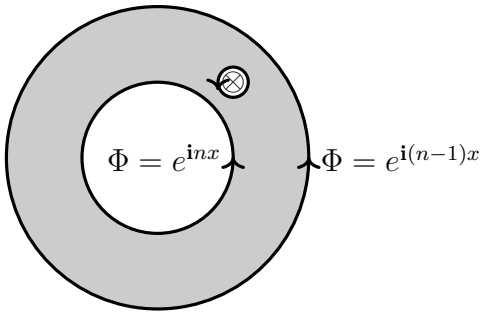
$$x_i' = x_i - v_i t, \quad t' = t \tag{4.38}$$

acts on the non-relativistic field Φ as

$$\Phi(x,t) \to \Phi'(x',t'), \quad \Phi(x,t) = e^{-\frac{1}{2}\mathbf{i}mv^2 t + \mathbf{i}mv_i x^i} \Phi'(x',t'). \tag{4.39}$$

At fixed time, a boost therefore winds up the phase of Φ to $e^{\mathbf{i}mvx}\Phi$. If $|\Phi| \neq 0$, this winding cannot be removed continuously.





[End of Lecture 13]

4.8 Effective field theory of metal

In previous subsections, we gave various descriptions of superconductors, appropriate at increasing energies. At the lowest energies, there was just a massive photon. At higher energies, there was a Cooper-pair field, (4.21). At even higher energies, where we can break apart Cooper pairs, there are electrons (4.23). In this subsection, we peel away one more layer of the onion: at even higher energies, those electrons are no longer paired up and constitute a metal.

[Polchinski, lecture 2 (I recommend these notes very strongly), and R. Shankar] Let us appreciate the remarkable phenomenon that is *metal*. An arbitrarily small electric field $\vec{E}$ leads to a nonzero current $\vec{j} = \sigma\vec{E}$. This means that there must be gapless modes with energies much less than the natural cutoff scale in the problem.

Scales involved: The Planck scale of solid state physics (made by the logic by which Planck made his quantum gravity energy scale, namely by making a quantity with dimensions of energy out of the available constants) is

$$E_0 = \frac{1}{2}\frac{e^4 m}{\hbar^2} = \frac{1}{2}\frac{e^2}{a_0} \sim 13\text{eV}$$

(where $m \equiv m_e$ is the electron mass and the factor of 2 is an abuse of outside information) which is the energy scale of *chemistry*. Chemistry is to solids as the melting of spacetime is to high-energy physics. As with high-energy physics, however, there are other scales involved. In particular a solid involves a lattice of nuclei, each with $M \gg m$ (of order the proton mass). So m/M is a useful small parameter which controls the coupling between the electrons and the lattice vibrations. Also, the actual speed of light $c \gg v_F$ can generally be treated as ∞ to first approximation. v_F/c

suppresses spin-orbit couplings that break $\mathsf{SU}(2)_{\text{spin}} \times \mathsf{SO}(3)_{\text{spatial rotations}}$ down to the diagonal (though large atomic numbers enhance them: $\lambda_{\text{SO}} \propto Z v_F/c$).

Let us attempt to construct a Wilsonian-natural effective field theory of this phenomenon. The answer is called Landau Fermi Liquid Theory. What are the right low-energy degrees of freedom? Let's make a guess that they are like electrons – fermions with spin and electric charge. They will not have exactly the properties of free electrons, since they must incorporate the effects of interactions with all their colleagues. The 'dressed' electrons are called quasielectrons, or more generally quasiparticles.

Given the strong interactions between so many particles, why should the dofs have anything at all to do with electrons? Landau's motivation for this description (which is not always correct) is that we can imagine starting from the free theory and adiabatically turning up the interactions. If we don't encounter any phase transition along the way, we can follow each state of the free theory, and use the same labels in the interacting theory.

We will show that there is a nearly-RG-stable fixed point describing gapless quasielectrons. Notice that we are not trying to match this description directly to some microscopic lattice model of a solid; rather we will do bottom-up effective field theory.

Having guessed the necessary dofs, let's try to write an action for them consistent with the symmetries. A good starting point is the free theory:

$$S_{\text{free}}[\psi] = \int dt\, \mathrm{d}^d p \left(\mathbf{i} \psi^\dagger_\sigma(p) \partial_t \psi_\sigma(p) - (\epsilon(p) - \epsilon_F)\, \psi^\dagger_\sigma(p) \psi_\sigma(p) \right)$$

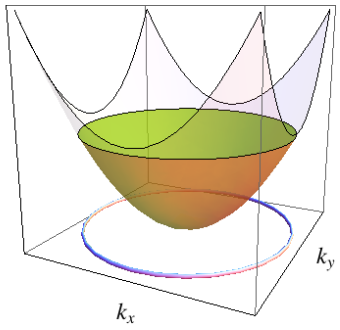
where σ is a spin index, ϵ_F is the Fermi energy (zero-temperature chemical potential), and $\epsilon(p)$ is the single-particle dispersion relation. For non-interacting non-relativistic electrons in free space, we have $\epsilon(p) = \frac{p^2}{2m}$. It will be useful to leave this as a general function of p. [27] [28]

[27] Notice that we are assuming translation invariance. I am not saying anything at the moment about whether translation invariance is discrete (the ions make a periodic potential) or continuous.

[28] We have chosen the normalization of ψ to fix the coefficient of the ∂_t term (this rescaling may depend on p).

The groundstate of the free theory is the filled Fermi sea:

$$|\text{gs}\rangle = \prod_{p|\epsilon(p)<\epsilon_F} \psi_p^\dagger |0\rangle\,, \quad \psi_p |0\rangle = 0, \quad \forall p.$$



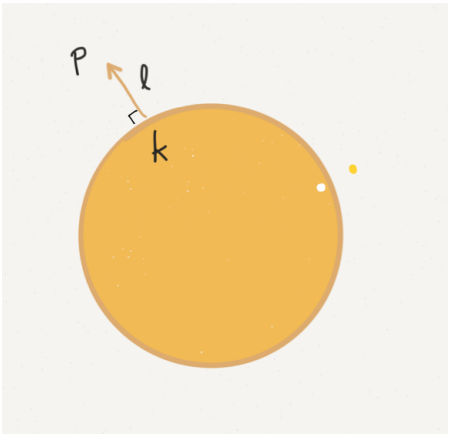
(If you don't like continuous products, put the system in a box so that p is a discrete label.) The Fermi surface is the set of points in momentum space at the boundary of the filled states:

$$\text{FS} \equiv \{p|\epsilon(p) = \epsilon_F\}.$$

The low-lying excitations are made by adding an electron just above the FS or removing an electron (creating a hole) just below.

In order to define the power-counting rules for our EFT, we would like to define a scaling transformation that focuses on the low-energy excitations. We scale energies by a factor $E \to bE, b < 1$. In relativistic QFT, $\vec{p}$ scales like E, toward zero, $\vec{p} \to b\vec{p}$, since all the low-energy stuff is near the single special point $\vec{p} = 0$. Here the situation is much more interesting because there is a whole surface of low-energy stuff on the FS. This will lead to what's called *hyperscaling violation* – we can't just count powers of momentum.

One way to implement this is to introduce a hierarchical labeling of points in momentum space, by breaking the momentum space into *patches* around the FS. (An analogous strategy of labeling is also used in heavy quark EFT and in SCET.)



We'll use a slightly different strategy, following Polchinski. To specify a point $\vec{p}$, we pick the nearest point $\vec{k}$ on the FS, $\epsilon(\vec{k}) = \epsilon_F$ (draw a line perpendicular to the FS from $\vec{p}$), and let

$$\vec{p} = \vec{k} + \vec{\ell}.$$

So $d-1$ of the components are determined by $\vec{k}$ and one is determined by ℓ. (There are some exceptional cases if the FS gets too wiggly. Ignore these for now.)

$$\epsilon(p) - \epsilon_F = \ell v_F(\vec{k}) + \mathcal{O}(\ell^2), \qquad v_F \equiv \partial_p \epsilon|_{p=k}.$$

So a scaling transformation of the field that accomplishes our goal of focusing on the FS is

$$\psi(t, \vec{p} = \vec{k} + \vec{\ell}) \mapsto \zeta \psi(t/s, \vec{k} + s\vec{\ell}) \tag{4.40}$$

(with ζ to be determined). That is, we scale the momenta toward the Fermi surface. Under this transformation,

$$S_{\text{free}} \mapsto \zeta^2 \int \underbrace{dt\; d^{d-1}\vec{k}\; d\vec{\ell}}_{=\tilde{t}/sd^{d-1}\vec{k}d\tilde{\ell}s} \left(\mathbf{i}\psi^\dagger(t/s, k+s\ell) \underbrace{\partial_t}_{=s^{-1}\partial_{\tilde{t}}} \psi(t/s, k+s\ell) - \underbrace{\ell v_F(k)}_{\tilde{\ell}/s} \psi^\dagger(t/s, k+s\ell)\psi(t/sk+s\ell) \right)$$
$$= \zeta^2 s^{-1} S_{\text{free}}. \tag{4.41}$$

Here I changed the dummy integration variables to $\tilde{t} \equiv t/s, \tilde{\ell} \equiv s\ell$. In order to make this go like s^0 we require $\zeta = s^{+\frac{1}{2}}$. Notice that the $\ell^{n\geq 2}$ corrections to the dispersion then produce terms that go like s^{n-1}, and are irrelevant.

Next we will play the EFT game. To do so we must enumerate the symmetries we demand of our EFT:

1. Particle number, $\psi \to e^{\mathbf{i}\theta}\psi$
2. Spatial symmetries: time-translation invariance, and either (a) continuous translation invariance and rotation invariance (as for *e.g.* liquid ^{3}He) or (b) lattice symmetries. This means that momentum space is periodically identified, roughly $p \simeq p + 2\pi/a$ where a is the lattice spacing (the set of independent momenta is called the Brillouin zone (BZ)) and p is only conserved modulo an inverse lattice vector $2\pi/a$. There can also be some remnant of rotation invariance preserved by the lattice. Case (b) reduces to case (a) if the Fermi surface does not go near the edges of the BZ.
3. Spin rotation symmetry, $\mathsf{SU}(n)$ if $\sigma = 1..n$. In the limit with $c \to \infty$, this is an internal symmetry, independent of rotations.
4. Let's assume that $\epsilon(p) = \epsilon(-p)$, which is a consequence of *e.g.* parity invariance (or, on the lattice, an inversion symmetry).

Now we enumerate all terms analytic in ψ and its momenta (since we are assuming that there are no other low-energy dofs integrating out which is the only way to get non-analytic terms in ψ) and consistent with the symmetries; we can order them by the number of fermion operators involved. Particle number symmetry means every ψ comes with a $\psi^\dagger$. The possible **quadratic terms** are[29]:

$$\int \underbrace{dt\; d^{d-1}\vec{k}\; d\vec{\ell}}_{\sim s^0}\; \mu(k)\; \underbrace{\psi^\dagger_\sigma(p)\psi_\sigma(p)}_{\sim s^{+1}} \sim s^{+1}$$

[29]Here and below I will just write the factors of s that arise from doing the scale transformation rather than writing out the whole transformation law of the term as in (4.41).

is *relevant*. This is like a mass term. It looks like we have lost the EFT game in the sense that our candidate fixed point has relevant operators and therefore seems fine-tuned, whereas we are trying to describe a whole phase of matter. But don't panic: it just shifts the FS around. The *existence* of a Fermi surface is Wilson-natural (*i.e.* a stable assumption given generic coefficients of all possible terms in the action); any precise location or shape (modulo something enforced by symmetries, like roundness) is not.

Adding one extra ∂_t or factor of ℓ costs a s^{-1} and makes the operator marginal; those terms are already present in S_{free}. Adding more than one makes it irrelevant.

Quartic terms:

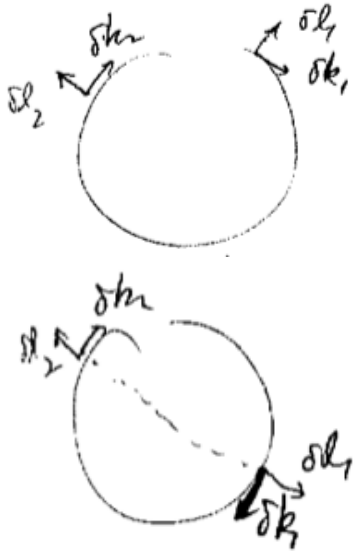
$$S_4 = \int \underbrace{dt \prod_{i=1}^{4} d^{d-1}\vec{k}_i d\vec{\ell}_i}_{\sim s^{1-4}} u(4\cdots 1) \underbrace{\psi^\dagger_\sigma(p_1)\psi_\sigma(p_3)\psi^\dagger_{\sigma'}(p_2)\psi_{\sigma'}(p_4)}_{\sim s^{+\frac{1}{2}\cdot 4}} \underbrace{\delta^d(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)}_{\sim s^{\Delta}}$$

The minus signs on $p_{3,4}$ is because $\psi(p)$ *removes* a particle with momentum p. We assume u depends only on k, σ, so does not scale – this will give the most relevant piece. How does the delta function scale?

$$\delta^d\left(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4\right) = \delta^d\left(k_1+k_2-k_3-k_4+\ell_1+\ell_2-\ell_3-\ell_4\right) \overset{?}{\simeq} \delta^d\left(k_1+k_2-k_3-k_4\right)$$

In the last (questioned) step, we used the fact that $\ell \ll k$ to ignore the contributions of the ℓs. If this is correct then the delta function does not scale (since ks do not), and $S_4 \sim s^{-1+\Delta} = s^{-1}$ is irrelevant (and quartic interactions with derivatives are moreso). If this were correct, the free-fixed point would be exactly stable.

There are two important subtleties: (1) the questioned equality above is questionable because of kinematics of the Fermi surface, and (2) there exist phonons. We will address these two issues in order.



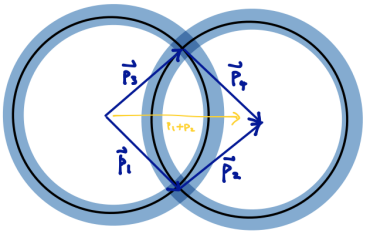
The **kinematic subtlety** in the treatment of the scaling of $\delta(p_1+p_2-p_3-p_4)$ arises because of the geometry of the Fermi surface. Consider scattering between two points on the FS, where (in the labeling convention above)

$$p_3 = p_1 + \delta k_1 + \delta\ell_1, \quad p_4 = p_2 + \delta k_2 + \delta\ell_2,$$

in which case the momentum delta function is

$$\delta^d(p_1+p_2-p_3-p_4) = \delta^d(\delta k_1 + \delta\ell_1 + \delta k_2 + \delta\ell_2).$$

For generic choices of the two points $p_{1,2}$ (top figure at left), δk_1 and δk_2 are linearly independent and the $\delta\ell$s can indeed be ignored as we did above. However, for two points with $p_1 = -p_2$ (they are called *nested*, as depicted in the bottom figure at left), then one component of $\delta k_1 + \delta k_2$ is automatically zero, revealing the tiny $\delta\ell$s to the force of (one component of) the delta function. In this case, $\delta(\ell) = \delta(\tilde{\ell}/s) = s\delta(\tilde{\ell})$ scales like s^{+1}, and for this particular kinematic configuration the four-fermion interaction is (classically) marginal. Classically marginal means quantum mechanics has a chance to make a big difference.



A useful visualization is at right ($d = 2$ with a round FS is shown; this is what's depicted on the cover of the famous book by Abrikosov-Gorkov-Dzyaloshinski): the blue circles have radius k_F; the yellow vector is the sum of the two initial momenta $p_1 + p_2$, both of which are near the FS; the condition that $p_3 + p_4$, each also on the FS, add up to the same vector means that p_3 must lie on the intersection of the two circles (spheres in $d > 2$). But when $p_1 + p_2 = 0$, the two circles are on top of each other so they intersect everywhere! Comments:

1. We assumed that both p_1 and $-p_2$ could actually both be near the FS. This is automatic if $\epsilon(p) = \epsilon(-p)$, *i.e.* if ϵ is only a function of p^2.

2. This discussion works for any $d > 1$.

3. **Forward scattering.** There is a similar phenomenon for the case where $p_1 = p_3$ (and hence $p_2 = p_4$). This is called *forward scattering* because the final momenta are the same as the initial momenta. (We could just as well take $p_1 = p_4$ (and hence $p_2 = p_3$).) In this case too the delta function will constrain the ℓs and will therefore scale.

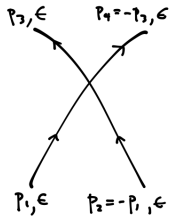
The tree-level-marginal 4-Fermi interactions at special kinematics leads to a *family* of fixed points labelled by 'Landau parameters'. In fact there is whole *function*'s worth of fixed points. In 2d, the points on the FS are parametrized by an angle θ, and the fixed point manifold is parametrized by the forward-scattering function

$$F(\theta_1, \theta_2) \equiv u(\theta_4 = \theta_2, \theta_3 = \theta_1, \theta_2, \theta_1)$$

(Fermi statistics implies that $u(\theta_4 = \theta_1, \theta_3 = \theta_2, \theta_2, \theta_1) = -F(\theta_1, \theta_2)$) and the BCS-channel (nesting) interaction:

$$V(\theta_1, \theta_3) = u(\theta_4 = -\theta_3, \theta_3, \theta_2 = -\theta_1, \theta_1).$$

Now let's think about what decision the fluctuations make about the fate of the nested interactions. The most interesting bit is the renormalization of the BCS interaction:



$= -\mathbf{i}V$

The electron propagator, obtained by inverting the kinetic operator in S_{free}, is

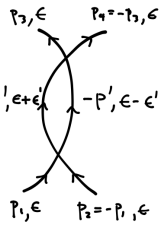
$$G(\epsilon, p = k + l) = \frac{\mathbf{i}}{\epsilon(1 + \mathbf{i}\eta) - v_F(k)\ell + \mathcal{O}(\ell)^2}$$

where I used $\eta \equiv 0^+$ for the infinitesimal specifying the contour prescription. [30]

Let's assume rotation invariance. Then $V(\theta_3, \theta_1) = V(\theta_3 - \theta_1)$, $V_l = \int \text{đ}\theta e^{\mathbf{i}l\theta} V(\theta)$. Different angular momentum sectors decouple from each other at one loop.

We will focus for simplicity on the s-wave bit of the interaction, so V is independent of momentum. We will integrate out just a shell in energy (depicted by the blue shaded shell in the Fermi surface figures). The interesting contribution comes from the following diagram:

$-\mathbf{i}\delta^{(1)}V =$



(4.43)

$$\begin{aligned}
&= -(-\mathbf{i}V)^2 \int_{\epsilon_0/s}^{\epsilon_0} \text{đ}\epsilon' \text{đ}^{d-1}k' \text{đ}\ell' \frac{\mathbf{i}^2}{\left((\epsilon+\epsilon')(1+\mathbf{i}\eta) - v_F(k')\ell'\right)\left((\epsilon-\epsilon')(1+\mathbf{i}\eta) - v_F(k')\ell'\right)} \\
&\text{do } \textstyle\int d\ell' \text{ by residues} \qquad = -V^2 \int \frac{\text{đ}\epsilon' \text{đ}^{d-1}k'}{(2\pi)} \frac{2\pi\mathbf{i}}{v_F(k')} \left(\underbrace{\epsilon - \epsilon' - (\epsilon + \epsilon')}_{=-2\epsilon'} \right)^{-1} \\
&= +\mathbf{i}V^2 \underbrace{\int_{\epsilon_0/s}^{\epsilon_0} \frac{d\epsilon'}{\epsilon'}}_{=\log(s)} \underbrace{\frac{1}{2} \int \frac{\text{đ}d-1k'}{v_F(k')}}_{\text{dos at FS}} .
\end{aligned} \tag{4.44}$$

Don't forget the fermion loop minus sign (in red, because I forgot it at first). Between the first and second lines, we did the ℓ' integral by residues. The crucial point is that

[30] It's in a unfamiliar place. But this is the $\mathbf{i}\epsilon$ (rather, $\mathbf{i}\eta$) prescription that we get by analytic continuation from Euclidean time. Think about the integral

$$\int \text{đ}\omega \frac{e^{i\omega t}}{\omega(1 + \mathbf{i}\eta) - \omega_0} = -\mathbf{i}\theta(t\text{sgn }(\omega_0))e^{i\omega_0 t}. \tag{4.42}$$

So it's the retarded green's function for particles and the advanced green's function for holes.

we are interested in external energies $\epsilon \sim 0$, but we are integrating out a shell near the cutoff, so $\epsilon' > \epsilon$ and the sign of $\epsilon + \epsilon'$ is opposite that of $\epsilon - \epsilon'$; therefore there is a pole on either side of the real ℓ axis and we get the same answer by closing the contour either way. On one side the pole is at $\ell' = \frac{1}{v_F(k')}(\epsilon + \epsilon')$. (In the t-channel diagram (what Shankar calls ZS), the poles are on the *same* side and it therefore does not renormalize the four-fermion interaction.)

The result to one-loop is then

$$V(s) = V - V^2 N \log(s) + \mathcal{O}(V^3)$$

with $N \equiv \frac{1}{4\pi}\int \frac{\mathrm{d}^{d-1}k'}{v_F(k')}$ is the density of states at the Fermi surface. From this we derive the beta function (recall that $s \to \infty$ in the IR as before)

$$s\frac{d}{ds}V(s) = \beta_V = -NV^2(s) + \mathcal{O}(V^3)$$

and the solution of the flow equation at $E = bE_1$ is

$$V(E) = \frac{V_1}{1 + NV_1 \log(E_1/E)} \begin{cases} \to 0 & \text{in IR for } V_1 > 0 \text{ (repulsive)} \\ \to -\infty & \text{in IR for } V_1 < 0 \text{ (attractive)} \end{cases} \tag{4.45}$$

There is therefore a very significant dichotomy depending on the sign of the coupling at the microscopic scale E_1, as in this phase diagram:  V

The conclusion is that if the interaction starts attractive at some scale it flows to large attractive values. The thing that is decided by our perturbative analysis is that (if $V(E_1) < 0$) the decoupling we did with σ ('the BCS channel') wins over the decoupling with ρ ('the particle-hole channel').

What happens at $V \to -\infty$? Here we need non-perturbative physics. The non-perturbative physics is in general hard, but we've already done what we can in §4.7.

[End of Lecture 14]

The remaining question is: Who is V_1 and why would it be attractive (given that Coulomb interactions between electrons, while screened and therefore short-ranged, are repulsive)? The answer is:

Phonons. The lattice of positions taken by the ions making up a crystalline solid spontaneously break many spacetime symmetries of their governing Hamiltonian. This implies a collection of gapless Goldstone modes in any low-energy effective theory of such a solid[31]. The Goldstone theorem is satisfied by including a field

$\vec{D}(\vec{r}) \propto$ (local) displacement $\delta\vec{r}$ of ions near $\vec{r}$ from their equilibrium positions

[31] Note that there is a subtlety in counting Goldstone modes from spontaneously broken spacetime symmetries: there are more symmetry generators than Goldstones. Basically it's because the associ-

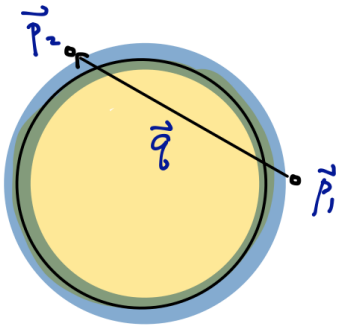
Most microscopically we have a bunch of coupled springs:

$$L_{\rm ions} \sim \frac{1}{2} M \sum_I \left(\delta \dot{\vec{r}}_I\right)^2 - k_{ij}^{IJ} \delta r_I^i \delta r_J^j + ...$$

where $\delta \vec{r}_J$ is the displacement from equilibrium of ion J. We don't want to ask about the spring constants k, except to say that they are independent of the nuclear mass M. It is useful to introduce a canonically normalized field in terms of which the action is

$$S[\vec{D}(r_I) = (M)^{1/2}\, \delta \vec{r}_I] = \frac{1}{2} \int dt d^d q \left(\partial_t D_i(q) \partial_t D_i(-q) - \omega_{ij}^2(q) D_i(q) D_j(-q)\right).$$

Here $\omega^2 \propto M^{-1}$. Their status as Goldstones means that the eigenvalues of $\omega_{ij}^2(q) \sim |q|^2$ at small q: moving everyone by the same amount does not change the energy. This also constrains the coupling of these modes to the electrons: they can only couple through derivative interactions.



For purposes of their interactions with the electrons, a nonzero q that keeps the e^- on the FS must scale like $q \sim s^0$. Therefore

$$dt d^d q \left(\partial_t D\right)^2 \sim s^{-1-2[D]} \quad \Longrightarrow \quad D \sim s^{+\frac{1}{2}}$$

(*i.e.* $D(t,q) \to s^{\frac{1}{2}} D(t/s, q)$ under a scale transformation), and the restoring force term $dt d^d q D^2 \omega^2(q) \sim s^{+2}$ is relevant, and dominates over the ∂_t^2 term for

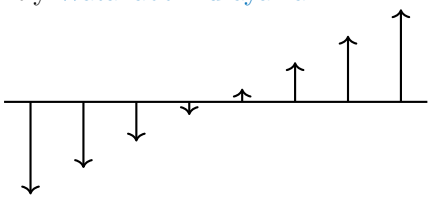
$$E < E_D = \sqrt{\frac{m}{M}} E_0 \quad \text{the Debye energy.}$$

(For the more traditional derivation of the relation between E_D and E_0, see *e.g.* DeGennes' *Superconductivity of Metals and Alloys*, pages 99-102.) This means that phonons mediate *static* interactions below E_D – we can ignore retardation effects, and their effects on the electrons can be fully incorporated by the four-fermion interaction we used above (with some $\vec{k}$ dependence). How do they couple to the electrons?

$$S_{\rm int}[D, \psi] = \int dt d^d q d^{d-1} k_1 d\ell_1 d^{d-1} k_2 d\ell_2 \; M^{-\frac{1}{2}} g_i(q, k_1, k_2) D_i(q) \psi_\sigma^\dagger(p_1) \psi_\sigma(p_2) \delta^d(p_1 - p_2 - q)$$

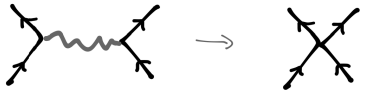
ated currents differ only by functions of spacetime; but a localized Goldstone particle is anyway made by a current times a function of spacetime, so you can't sharply distinguish the resulting particles. Some useful references on this subject are Low-Manohar and more recently Watanabe-Murayama.

The picture at right explains the idea: a rotation is just a position-dependent translation. $T_{\hat{y}}(ax) = R(\arctan(a))$.



$$\sim s^{+1-1-1+3/2} = s^{+1/2} \tag{4.46}$$

– here we took the delta function to scale like s^0 as above. This is relevant when we use the $\dot{D}^2$ scaling for the phonons; when the restoring force dominates we should scale D differently and this is irrelevant for generic kinematics. This is consistent with our previous analysis of the four-fermion interaction.

The summary of this discussion is: phonons do not destroy the Fermi surface, but they *do* produce an attractive contribution to the 4-fermion interaction, which is relevant in some range of scales (above the Debye energy). Below the Debye energy, it amounts to an addition to V that goes like $-g^2$: 

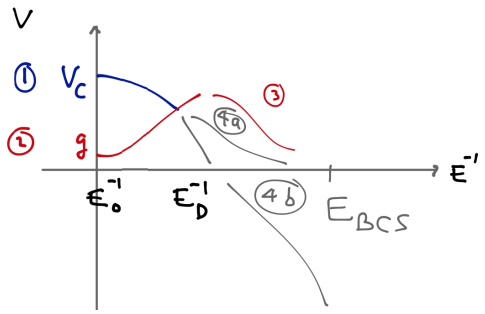
– an *attractive* contribution to the 4-fermion interaction.

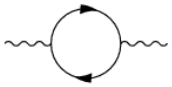
Notice that the scale at which the coupling V becomes strong ($V(E_{\rm BCS}) \equiv 1$ in (4.45)) is

$$E_{\rm BCS} \sim E_D e^{-\frac{1}{NV_D}}.$$

Two comments about this: First, it is non-perturbative in the interaction V_D, and in fact the same function we found earlier. Second, it provides a nice way to verify the role of phonons, since $E_D \sim M^{-1/2}$ can be varied by studying the same material with different isotopes and studying how the critical superconducting temperature ($\sim E_{\rm BCS}$) scales with the nuclear mass.

Actually, we can make some headway towards understanding the result of this interaction going strong. Because the diagrams with the special kinematics are marginal and hence unsuppressed, while all other interactions flow to zero at low energy, certain diagrams dominate. In particular, bubble-chains dominate.



Here's the narrative, proceeding as a function of decreasing energy scale, beginning at E_0, the Planck scale of solids: (1) Electrons repel each other by the Coulomb interaction. However, in a metal, this interaction is *screened* by processes like this: 

(the intermediate state is an electron-hole pair) and is short-ranged. It is still repulsive, however. As we coarse-grain more and more, we see more and more electron-hole pairs and the force weakens. (2) While this is happening, the electron-phonon interaction is relevant and growing. This adds an attractive bit to V. This lasts until E_D. (3) At E_D the restoring force term in the phonon lagrangian dominates (for the purposes of their

interactions with the electrons) and we can integrate them out. (4) What happens next depends on the sign of $V(E_D)$. If it's positive, V flows harmlessly to zero. If it's negative, it becomes moreso until we exit the perturbative analysis around $E_{\rm BCS}$, and vindicate our choice of Hubbard-Stratonovich channel above.

Further brief comments, for which I refer you to Shankar:

1. Putting back the possible angular dependence of the BCS interaction, the result at one loop is

$$\frac{dV(\theta_1-\theta_3)}{d\ell} = -\frac{1}{8\pi^2}\int_0^{2\pi} \text{đ}\theta V(\theta_1-\theta)V(\theta-\theta_3)$$

   or in terms of angular momentum components,

$$\frac{dV_l}{d\ell} = -\frac{V_l^2}{4\pi}.$$

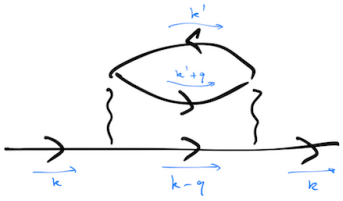
2. This example is interesting and novel in that it is a (family of) fixed point(s) characterized by a dimensionful quantity, namely k_F. This leads to a phenomenon called *hyperscaling violation* where thermodynamic quantities need not have their naive scaling with temperature.

3. The one loop analysis gives the right answer to all loops in the limit that $N \sim (k_F/\Lambda)^{d-1} \gg 1$, where Λ is the UV cutoff on the momentum.

4. The forward scattering interaction (for any choice of function $F(\theta_{13})$) is not renormalized at one loop. This means it is exactly marginal at leading order in N.

5. Like in ϕ^4 theory, the sunrise diagram at two loops is the first appearance of wavefunction renormalization. In the context of the Fermi liquid theory, this leads to the renormalization of the effective mass which is called $m^\star$. It also leads to a finite lifetime of the Landau quasiparticle, as follows.

Another consequence of the FS kinematics which I should emphasize more: it allows the quasiparticle to be stable. The leading contribution to the decay rate of a one-quasiparticle state with momentum k can be obtained applying the optical theorem to the following process.



In the figure, the object  is the four-fermion vertex (the wiggly line is just for clarity). The intermediate state is two electrons with momenta $k'+q$ and

$k-q$, and one *hole* with momentum k'. (To understand the contour prescription for the propagator, it is useful to begin with

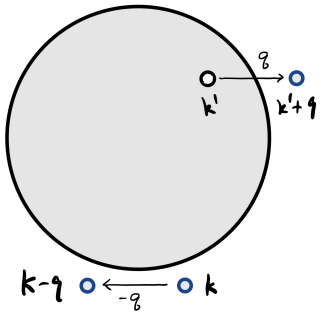
$$G(t,p) = \langle \mathrm{gs}| \mathcal{T} c_p^\dagger(t) c_p(0) |\mathrm{gs}\rangle \,, \quad c_p^\dagger(t) \equiv e^{-\mathbf{i}\mathbf{H}t} c_p^\dagger e^{\mathbf{i}\mathbf{H}t}$$

and use the free-fermion fact $[\mathbf{H}, c_p^\dagger] = \epsilon_p c_p^\dagger$. For more details, see the steps leading up to equation (7.7) of AGD (Abrikosov, Gorkov, Dzyaloshinski, *Methods of QFT in Statistical Physics.*)) Notice that this is the eyeball diagram which gives the lowest-order contribution to the wavefunction renormalization of a field with quartic interactions.

After doing the frequency integrals by residues, we get something of the form

$$\Sigma(k,\epsilon) = \int đq\, đk' \frac{|u_q|^2}{D} \theta(\epsilon_{k'}\epsilon_{k'+q})\theta(\epsilon_{k'}\epsilon_{k-q})$$

$$D \equiv \epsilon_k(1+\mathbf{i}\eta) + \epsilon_{k'}(1-\mathbf{i}\eta) - \epsilon_{k'+q}(1+\mathbf{i}\eta) - \epsilon_{k-q}(1+\mathbf{i}\eta)$$



By the optical theorem, its imaginary part is the (leading contribution to the) inverse-lifetime of the quasiparticle state with fixed k:

$$\tau^{-1}(k) = \mathrm{Im}\,\Sigma(k,\epsilon) = \pi \int đq\, đk' \delta(D) |u_q|^2 f(-\epsilon_{k'}) f(\epsilon_{k'+q}) f(\epsilon_{k-q})$$

where

$$f(\epsilon) = \lim_{T\to 0} \frac{1}{e^{\frac{\epsilon-\epsilon_F}{T}}+1} = \theta(\epsilon < \epsilon_F)$$

is the Fermi function. This is just the demand that a particle can only scatter into an empty state and a hole can only scatter into a filled state. These constraints imply that all the energies are near the Fermi energy: both $\epsilon_{k'+q}$ and $\epsilon_{k'}$ lie in a shell of radius ϵ about the FS; the answer is proportional to the density of possible final states, which is thus

$$\tau^{-1} \propto \left(\frac{\epsilon}{\epsilon_F}\right)^2 .$$

So the width of the quasiparticle resonance is

$$\tau^{-1} \propto \epsilon^2 \ll \epsilon$$

much smaller than its frequency – it is a sharp resonance, a well-defined particle.

The fact that the single-particle lifetime goes like ω^2 implies (as long as the decay of quasiparticles is the main source of current dissipation) that the electrical resistivity goes like $\rho(T) \sim T^2$. Rather, this is the contribution from electron-electron scattering.

Disorder, in the form of static impurities, contributes an additive constant. In $d = 3$, phonons contribute $\rho \sim T^5$ (T^3 from the density of states of bosons with linear dispersion and an extra factor of T^2 for the derivative coupling), for $T < T_D$. For $T > T_D$, the phonons can be treated as classical springs and contribute $\rho \sim T$.

In high-temperature superconductors (at optimal doping), in sharp contrast, the resistivity goes like $\rho \sim T$ in a large range of temperatures, including temperatures well below the Debye temperature. The above analysis shows that some other EFT must describe them. A metal that is not described by the Landau Fermi liquid theory is called a non-Fermi liquid.

One significant loophole is that there could be other light degrees of freedom besides the electronic quasiparticles and the phonons. One way in which extra bosonic degrees of freedom can arise is at a quantum critical point.

4.9 Pions

[Schwartz §28.1] Below the scale of electroweak symmetry breaking, we can forget the W and Z bosons. Besides the 4-Fermi interactions, the remaining drama is QCD and electromagnetism:

$$\mathcal{L}_{QCD_2} = -\frac{1}{4}F_{\mu\nu}^2 + \mathbf{i}\sum_{\alpha=L,R}\sum_f \bar{q}_{\alpha f}\not{D}q_{\alpha f} - \bar{q}Mq.$$

Here f is a sum over quark flavors, which includes the electroweak doublets, u and d (the subscript '2' is for 2 flavors). Let's focus on just these two lightest flavors, u and d. We can diagonalize the mass matrix by a field redefinition (this is what makes the CKM matrix meaningful): $M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$. If it were the case that $m_u = m_d$, we would have *isospin* symmetry

$$\begin{pmatrix} u \\ d \end{pmatrix} \to U \begin{pmatrix} u \\ d \end{pmatrix}, \quad U \in \mathsf{SU}(N_f = 2).$$

If, further, there were no masses $m = 0$, then L and R decouple and we would also have chiral symmetry, $q \to e^{\mathbf{i}\gamma_5\alpha}q$, *i.e.*

$$q_L \to Wq_L, q_R \to W^{-1}q_R, \; W \in \mathsf{SU}(N_f = 2).$$

Why do I restrict W to $\mathsf{SU}(2)$ and not $\mathsf{U}(2)$? The central bit of the axial symmetry $\mathsf{U}(1)_A$ is anomalous – its divergence is proportional to the *gluon* theta term operator $F \wedge F$, which has all kinds of nonzero matrix elements. It's not a symmetry (see Peskin page 673 for more detail). The missing non-Goldstone boson is called the η'. The central bit of the vectorlike transformation $q \to e^{\mathbf{i}\alpha}q$ is *baryon number*, B. (Actually this is anomalous in the presence of electroweak gauge fields, but $B - L$ is not).

The groundstate of QCD is mysterious, because of infrared slavery. Here's one piece of input from experiment and numerical simulation. Apparently it is the case that in the groundstate

$$\langle \bar{q}_f q_f \rangle = V^3 \tag{4.47}$$

independent of flavor f. This condensate spontaneously breaks

$$\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R \to \mathsf{SU}(2)_{\text{isospin}}, \tag{4.48}$$

the diagonal combination. $\begin{pmatrix} u \\ d \end{pmatrix}$ is a doublet. Since $p = u_\alpha u_\beta d_\gamma \epsilon_{\alpha\beta\gamma}, n = u_\alpha d_\beta d_\gamma \epsilon_{\alpha\beta\gamma}$, this means that $\begin{pmatrix} p \\ n \end{pmatrix}$ is also a doublet. This symmetry is (explicitly) weakly broken by

the difference of the masses $m_d = 4.7\text{MeV} \neq m_u = 2.15\text{MeV}$ and by the electromagnetic interactions, since $q_d = -1/3 \neq q_u = 2/3$.

This symmetry-breaking structure enormously constrains the dynamics of the color singlets which are the low-energy excitations above the QCD vacuum (hadrons). Let us use the EFT strategy. We know that the degrees of freedom must include (pseudo-)Goldstone bosons for the symmetry breaking (4.48) ('pseudo' because of the weak explicit breaking). [End of Lecture 15]

Effective field theory. Since QCD is strongly coupled in this regime, let's use the knowing-the-answer trick: the low energy theory must include some fields that represent the breaking of the symmetry (4.48). One way to do this is to introduce a matrix field Σ that transforms like

$$\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R : \Sigma \to g_L \Sigma g_R^\dagger, \quad \Sigma^\dagger \to g_R \Sigma^\dagger g_L^\dagger$$

(this will be called a *linear* sigma model, because Σ transforms linearly) – we have in mind $\bar{q}_\alpha q_\beta \sim \Sigma_{\alpha\beta}$, a bit like the Hubbard-Stratonovich variable ρ in the density-density channel. We can make singlets (hence an action) out of $\Sigma_{\alpha\beta}\Sigma^\dagger_{\beta\alpha} = \text{tr}\Sigma\Sigma^\dagger \equiv |\Sigma|^2$:

$$\mathcal{L} = |\partial_\mu \Sigma|^2 - \mathcal{V}(\Sigma) + \cdots, \mathcal{V}(\Sigma) = -m^2 \text{tr}\Sigma\Sigma^\dagger + \frac{\lambda}{4}\left(\text{tr}\Sigma\Sigma^\dagger\right)^2 + g\text{tr}\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger, \qquad (4.49)$$

which is designed to have a minimum at $\langle\Sigma\rangle = \frac{V}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, with (when $g \to 0$) $V = 2m/\sqrt{\lambda}$ (here V is from (4.47)), which preserves $\mathsf{SU}(2)_{\text{isospin}}$ (under which $\Sigma \to g\Sigma g^\dagger$). We can parametrize the fluctuations about this configuration as

$$\Sigma(x) = \frac{V + \sigma(x)}{\sqrt{2}} e^{\frac{2\mathbf{i}\pi^a(x)\tau^a}{F_\pi}}$$

where $F_\pi = V = \frac{2m}{\sqrt{\lambda}}$ is chosen to give $\pi^a(x)$ canonical kinetic terms. The π^a parametrize the directions of field space in which the potential is flat (like the field θ that goes around the minimum of a wine-bottle potential). Under $g_{L/R} = e^{\mathbf{i}\theta^a_{L/R}\tau^a}$, the pion field transforms as

$$\pi^a \to \pi^a + \underbrace{\frac{F_\pi}{2}\left(\theta^a_L - \theta^a_R\right)}_{\text{nonlinear realization of } \mathsf{SU}(2)_{\text{axial}}} - \underbrace{\frac{1}{2} f^{abc}\left(\theta^a_L + \theta^a_R\right)\pi^c}_{\text{linear realiz'n (adj rep) of } \mathsf{SU}(2)_{\text{isospin}}} .$$

The fields $\pi^\pm, \pi^0$ create pions, they transform in the adjoint representation of the diagonal $\mathsf{SU}(2)_{\text{isospin}}$, and they *shift* under the broken symmetry. This shift symmetry forbids mass terms π^2. The radial excitation σ, on the other hand, is a fiction that we've introduced in (4.49), and which has no excuse to stick around at low energies

(and does not), *i.e.* we expect it to have a mass of order the cutoff. We can put it out of its misery by taking $m \to \infty, \lambda \to \infty$ fixing F_π. In the limit, the useful field to use is

$$U(x) \equiv \frac{\sqrt{2}}{V}\Sigma(x)|_{\sigma=0} = e^{\frac{2\mathbf{i}\pi^a\tau^a}{F_\pi}} \tag{4.50}$$

which is unitary $UU^\dagger = U^\dagger U = \mathbb{1}$. This last identity means that all terms in an action for U require derivatives, so (again) no mass for π. The most general Lagrangian for U can be written as an expansion in derivatives, and is called the *chiral Lagrangian*:

$$\mathcal{L}_\chi = \frac{F_\pi^2}{4}\mathrm{tr}D_\mu U\, D^\mu U^\dagger + L_1 \mathrm{tr}\left(D_\mu U\, D^\mu U^\dagger\right)^2 + L_2 \mathrm{tr}D_\mu U\, D_\nu U^\dagger \mathrm{tr}D^\nu U^\dagger D^\mu U + L_3 \mathrm{tr}D_\mu U\, D^\mu U^\dagger D_\nu U\, D^\nu U^\dagger + \cdots \tag{4.51}$$

In terms of π, the leading term expands into

$$L_\chi = \frac{1}{2}D_\mu\pi^a D^\mu\pi^a + \frac{1}{F_\pi^2}\left(-\frac{1}{3}\pi^0\pi^0 D_\mu\pi^+ D^\mu\pi^- + \cdots\right) + \frac{1}{F_\pi^4}\left(\frac{1}{18}\left(\pi^-\pi^+\right)^2 D_\mu\pi^0 D^\mu\pi^0 + \cdots\right)$$

This fixes the relative coefficients of many irrelevant interactions, all with two derivatives, suppressed by powers of F_π. The expansions of the L_i terms have four derivatives, and are therefore suppressed by further powers of E/F_π, the promised small parameter of this EFT.

Pion masses and the spurion method. The pions aren't actually massless: $m_{\pi^\pm} \sim 140$MeV. In terms of quarks, one source for such a thing is the quark mass term $\mathcal{L}_{QCD} \ni \bar{q}Mq$. This explicitly breaks the isospin symmetry if the eigenvalues of M aren't equal. But an *invariance* of $\mathcal{L}_{QCD}$ is

$$q_{L/R} \to g_{L/R}q_{L/R},\ M \to g_L M g_R^\dagger. \tag{4.52}$$

This is not a symmetry of QCD, because we are transforming a coupling constant.

But now consider a different theory where M is a field (such a fake field is sometimes called a *spurion*). In this other system, where M were an actual dynamical field, (4.52) is a symmetry. Consider integrating out all the horrors of QCD in that theory. In the effective action that summarizes all the drama of strong-coupling QCD in terms of pions, the field M must still be there, and if we transform it as in (4.52), it should still be an invariance.

Now notice that none of this requires actually doing the integral over M – maybe we're going to do the path integral over M later. So even if M is not dynamical, the efffective action, as a functional of M and the actual low-energy fields is still constrained by the invariance (4.52)! We just have to play the usual EFT game and write down all terms that respect the symmetry in a derivative expansion. Notice that it also does not require M to be small. This 'spurion' trick, an important application of procrastination, has applications all over physics.

So the chiral lagrangian $\mathcal{L}_\chi$ should depend on M and (4.52) should be an invariance. We can play the EFT game again, but now with both π_a and M as our dofs. We will also assume that M is small and only keep the smallest powers of M. This determines

$$\Delta\mathcal{L}_\chi = \frac{V^3}{2}\mathrm{tr}\left(MU^\dagger + MU^\dagger\right) + \cdots \overset{(4.50)}{=} V^3(m_u+m_d) - \frac{V^3}{2F_\pi^2}(m_u+m_d)\sum_a \pi_a^2 + \mathcal{O}(\pi^3).$$

The coefficient V^3 is chosen so that the first term matches $\langle\bar{q}Mq\rangle = V^3(m_u+m_d)$. The second term then gives

$$m_\pi^2 \simeq \frac{V^3}{F_\pi^2}(m_u+m_d)$$

which is called the Gell-Mann Oakes Renner relation.

SU(3) and baryons. A few more comments before we answer the previous question. The strange quark mass is also pretty small $m_s \sim 95$MeV, and $\langle\bar{s}s\rangle \sim V^3$. This means the approximate invariance and symmetry breaking pattern is actually $\mathsf{SU}(3)_L \times \mathsf{SU}(3)_R \to \mathsf{SU}(3)_{\text{diag}}$, meaning that there are $16-8=8$ pseudo NGBs. Besides $\pi^{\pm,0}$, the others are the kaons $K^{\pm,0,\bar{0}}$ (there are two neutral kaons, a basis of which is called long and short for their relative lifetimes, and another basis of which is called K^0 and $K^{\bar{0}}$ which have definite isospin) and η. It's still only the $\mathsf{SU}(2)_L$ that's gauged.

We can also include baryons $B = \epsilon_{\alpha\beta\gamma}q_\alpha q_\beta q_\gamma$. Since $q = (u,d,s) \in 3$ of the flavor $\mathsf{SU}(3)$, the baryons are in the representation

$$\begin{aligned} 3\otimes 3\otimes 3 &= (6\oplus\bar{3})\otimes 3 = 10\oplus 8\oplus 8\oplus 1 \\ \square\otimes\square\otimes\square &= (\square\square\oplus\begin{smallmatrix}\square\\\square\end{smallmatrix})\otimes\square = \square\square\square\oplus\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\oplus\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\oplus\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \end{aligned} \tag{4.53}$$

The proton and neutron are in one of the octets. This point of view brought some order (and some predictions) to the otherwise-bewildering zoo of hadrons.

Returning to the two-flavor $\mathsf{SU}(2)$ approximation, we can include the nucleons by introducing new (Weyl fermion) fields $N_{L/R} = \begin{pmatrix} p \\ n \end{pmatrix}_{L/R}$. Where do the (large compared to light quark and even pion masses) nucleon masses come from? The Dirac mass term $\bar{N}_L N_R$ is not invariant under the full $\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R$ and so is not allowed. But we can couple the nucleon field to the pions by the symmetric coupling

$$\mathcal{L} \ni \lambda_{NN\pi}\bar{N}_L \Sigma N_R. \tag{4.54}$$

The expectation value for Σ gives a nucleon mass: $m_N = \lambda_{NN\pi}F_\pi$, where $\lambda_{NN\pi}$ can be measured by scattering. This is a cheap version of the *Goldberger-Treiman relation*; for a better one see Peskin pp. 670-672.

Another consequence of the coupling (4.54) is an interaction between pions and nucleons. This was how the existence of the pion was predicted by Yukawa: a nucleus

is a bunch of protons and neutrons held together somehow in a small space – a strong attractive short-ranged force overcomes the Coulomb repulsion between the protons. Based on the separation between nucleons in the nucleus, Yukawa predicted a scalar particle of mass $m \sim 100MeV$, to get an attractive potential between nucleons of the form $V = \lambda^2 e^{-mr}/r$. The λ here is $\lambda_{NN\pi}$.

Electroweak interactions and charged pion decay. You may have noticed that I used covariant-looking Ds in (4.51). That's because the $\mathsf{SU}(2)_L$ symmetry we've been speaking about is actually gauged by W^a_μ. (The electroweak gauge boson kinetic terms are in the $\cdots$ of (4.51).) Recall that

$$\mathcal{L}_{\text{Weak}} \ni gW^a_\mu \left(\underbrace{J^a_\mu - J^{5a}_\mu}_{\text{`V' - `A'}} \right) = gW^a_\mu \left(V_{ij} \bar{Q}_i \gamma^\mu \frac{1-\gamma^5}{2} \tau^a Q_j + \bar{L}_i \gamma^\mu \tau^a \frac{1-\gamma^5}{2} L_i \right)$$

where $Q_1 = \begin{pmatrix} u \\ d \end{pmatrix}, L_1 = \begin{pmatrix} e \\ \nu_e \end{pmatrix}$ are doublets of $\mathsf{SU}(2)_L$.

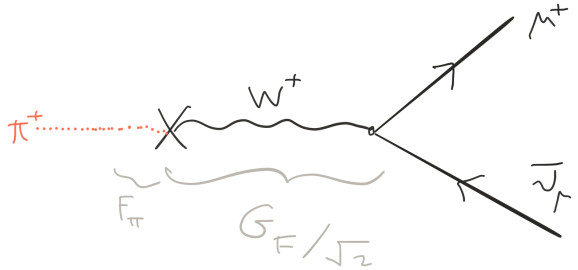
Now, in equations, the statement "a pion is a Goldstone boson for the axial $\mathsf{SU}(2)$" is:

$$\langle 0 | \, J^{5a}_\mu(x) \, | \pi^b(p) \rangle = \mathbf{i} p_\mu F_\pi e^{-\mathbf{i}p\cdot x} \delta^{ab}$$

where the state $|\pi^b(p)\rangle$ is a one-pion state of momentum p. If the vacuum were invariant under the symmetry transformation generated by J_μ, the BHS would vanish. The momentum dependence implements the fact that a global rotation ($p_\mu = 0$) does not change the energy. Contracting the BHS with ∂^μ_x and using current conservation (ignoring the explicit breaking just mentioned) would give $0 = p^2 F_\pi^2 = m_\pi^2 F_\pi^2$, a massless dispersion for the pions.

Combining the previous two paragraphs, we see that the following process can happen

$$\pi \overset{\text{Goldstone}}{\to} J^5_\mu \overset{\text{electroweak interaction}}{\to} \text{leptons}$$



(4.55)

and in fact is responsible for the dominant decay channel of charged pions. (Time goes from left to right in these diagrams, sorry.)

$$\mathcal{M}(\pi^+ \to \mu^+ \bar{\nu}_\mu) = \frac{G_F}{\sqrt{2}} F_\pi p^\mu \bar{v}_{\nu_\mu} \gamma^\mu (1-\gamma^5) u_\mu$$

where the Fermi constant $G_F \sim 10^{-5} GeV^{-2}$ (known from *e.g.* $\mu^- \to e^- \bar{\nu}_e \nu_\mu$) is a good way to parametrize the Weak interaction amplitude. Squaring this and integrating over two-body phase space gives the decay rate

$$\Gamma(\pi^+ \to \mu^+ \bar{\nu}_\mu) = \frac{G_F^2 F_\pi^2}{4\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 .$$

(You can see from the answer why the decay to muons is more important than the decay to electrons, since $m_\mu/m_e \sim 200$. This is called *helicity suppression* – the decay of the helicity-zero π^+ into back-to-back spin-half particles by the weak interaction (which only produces L particles and R antiparticles) can't happen if helicity is conserved – the mass term is required to flip the e_L into an e_R.) This contributes most of $\tau_{\pi^+} = \Gamma^{-1} = 2.6 \cdot 10^{-8} s$.

Knowing further the mass of the muon $m_\mu = 106\text{MeV}$ then determines $F_\pi = 92\text{MeV}$ which fixes the leading terms in the chiral Lagrangian. This is why F_π is called the *pion decay constant*. This gives a huge set of predictions for *e.g.* pion scattering $\pi^0\pi^0 \to \pi^+\pi^-$ cross sections. [End of Lecture 16]

Neutral pion decay. The symmetry current $J^{5,a=3}$ is both spontaneously broken and anomalous. Because of this, the neutral pion can decay by the anomaly into two photons:

$$q_\mu \langle p_1\epsilon_1; p_2\epsilon_2| J_\mu^{5,a=3}(q) |0\rangle = -c\frac{e^2}{4\pi^2}\epsilon^{\nu\lambda\alpha\beta} p_1^\nu \epsilon_1^\lambda p_2^\alpha \epsilon_2^\beta$$

where $\langle p_1\epsilon_1; p_2\epsilon_2|$ is a state with two photons of polarizations $\epsilon_{1,2}$. We know this because it is a matrix element of the $J_e J_e J_{\mathsf{SU}(2)-\text{axial}}$ anomaly,

$$\partial_\mu J^{\mu 5a} = -\frac{e^2}{16\pi^2}\epsilon^{\nu\lambda\alpha\beta} F_{\nu\lambda} F_{\alpha\beta} \text{tr}\left(\tau^a Q^2\right)$$

where $Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$ is the quark charge matrix. Comments: (1) the $\mathsf{U}(1)$ symmetry generated by $J^{\mu 5,a=3}$ acts by $u \to e^{\mathbf{i}\theta\gamma^5} u, d \to e^{-\mathbf{i}\theta\gamma^5}$, and is *not* the same as the anomalous $\mathsf{U}(1)_A$ (which does $q_i \to e^{\mathbf{i}\theta\gamma^5} q_i$ for every flavor), and it's also not the same as isospin $u \to e^{\mathbf{i}\theta} u, d \to e^{-\mathbf{i}\theta}$, which is not chiral, and not spontaneously broken. Confusing! (2) Since the trace involves a sum over colors, the rate of π^0 decay (known since the 1940s) gives a measurement of the number of colors of QCD! (3) This effect can and must be encoded in the Lagrangian for the pions by a term

$$L \ni N_c \frac{e^2}{16\pi^2} \pi^0 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{4.56}$$

where $N_c = 3$ is the number of colors. The effective field theory consistently realizes the anomalies of the microscopic theory. This is another example of *'t Hooft anomaly*

matching. How did we miss this term in our list of all terms that manifestly respect the symmetries?

WZW terms in the chiral Lagrangian. Finally, I would be remiss not to mention that the chiral Lagrangian must be supplemented by WZW terms to have the correct realization of symmetries (in order to encode all the effects of anomalies, and in order to violate $\pi \to -\pi$ which is not a symmetry of QCD). This is an important additional ingredient in the EFT recipe book: although we wrote all the local Lagrangian terms that were *manifestly* consistent with the symmetries, this actually did *not* account for all the symmetric terms that we can add to the action! The WZW term can only be written in a manifestly-symmetric way at the expense of introducing some extra dimension (or gauge redundancy).

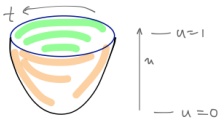
The chiral Lagrangian governs a *non-linear sigma model* (NLσM)– a QFT whose fields are maps from spacetime into some *target space*. In this case the target space is the coset space G/H, where G is the full symmetry group ($\mathsf{SU}(N_f)_L \times \mathsf{SU}(N_f)_R$) and H is the *unbroken* subgroup $\mathsf{SU}(N_f)_{\text{diagonal}}$. We can parametrize this space by $U = e^{\mathbf{i}\pi^a T^a \frac{2}{F_\pi}}$ where the T^a includes only generators of the broken part of the group, so the π^a are coordinates on G/H.

A WZW term is a term that we can sometimes add to a NLσM action; it is defined by the fact that it is symmetric under some group G, but *isn't* the integral of a symmetric local Lagrangian density in D dimensions. Making it manifestly symmetric requires the introduction of a fictitious extra dimension. This has the dramatic and surprising consequence that its coefficient is quantized.

To get the idea, consider a model in $D = 0 + 1$ where the field variable $\check{n}$ takes values on the unit sphere S^2, $1 = \sum_{a=1,2,3} \check{n}_a^2$. This is a special case of a nonlinear sigma model whose target space is a coset space with $\mathsf{G}/\mathsf{H} = \mathsf{SU}(2)/\mathsf{U}(1)$.

In order to write the WZW term in a manifestly symmetric way (under the $\mathsf{SO}(3)$ of rotations of the sphere, we have to extend the field into a (possibly fictitious) extra dimension whose coordinate is u.
We do this in such a way that the real system lives at $u = 1$:

$$\check{n}(t, u = 1) \equiv \check{n}(t), \quad \check{n}(t, u = 0) \equiv (0, 0, 1)$$



it goes to the north pole at the other end of the extra dimension for all t. Consider periodic boundary conditions in time $\check{n}(\beta) = \check{n}(0)$. Then this means that the full space is really a disk with the origin at $u = 0$, and the boundary at $u = 1$. Call this disk B, its boundary $\partial B = \mathcal{M}$ is the real spacetime (here a circle).

We can write the WZW term in terms of the S^2-valued field $\check{n}^{1,2,3}$ as

$$\mathcal{W}_0[\check{n}] = \frac{2\pi}{\Omega_2}\int_{B_2} \check{n}^a \mathrm{d}\check{n}^b \wedge \mathrm{d}\check{n}^c \epsilon_{abc} = \frac{1}{4\pi}\int_{\mathcal{M}} \mathrm{d}t\,(1-\cos\theta)\,\partial_t\phi.$$

The integrand here is the volume element of the image of a chunk of spacetime in the target S^2. If we integrate over the union of two balls with cancelling boundaries $B_2 \cup \bar{B}_2$, we get an integer multiple of 2π (the integer is the winding number of the map).

The coefficient k of $\mathcal{W}_1$ in the action $\Delta S[\check{n}] = k\mathcal{W}_0[\check{n}]$ thus must be an integer, since B_1 and $\bar{B}_1$ give equally good definitions of $\mathcal{W}_2$, which differ by $2\pi k$. So this ambiguity will not affect the path integral if $k \in \mathbb{Z}$.

Can you guess what familiar family of $\mathsf{SO}(3)$-invariant quantum mechanics systems parametrized by an integer is described by this path integral? Perhaps we will come back and derive this fact using coherent-state path integrals.

A simple generalization of this is a model in $D = d+1$ dimensions with a field variable $\check{n}$ taking values on S^{d+2}. Then we can write a WZW term as

$$\mathcal{W}_d[\check{n}] = \frac{2\pi}{\Omega_{d+2}}\int_{B_{d+2}} \check{n}^{a_0} d\check{n}^{a_1} \wedge \cdots \wedge dn^{a_{d+2}} \epsilon_{a_0\cdots a_{d+2}}. \tag{4.57}$$

The integrand is the volume element on the image of the chunk of spacetime. This term is manifestly $\mathsf{O}(d+3)$-symmetric. Again the EOM depend only on the fields at the boundary, and again the coefficient must be quantized.

The generalization to a group-valued variable U in any dimension is of the form

$$\mathcal{W}_{D-1} = c\int_{B_{D+1}} \mathrm{tr}\,\underbrace{U^{-1}dU \wedge U^{-1}dU \wedge \cdots \wedge U^{-1}dU}_{D+1 \text{ of these}}.$$

Such terms are interesting when $\pi_{D+1}(\mathcal{N})$ is nontrivial, where $\mathcal{N}$ is the space where the fields live (the target space, $\mathcal{N} = \mathsf{G}/\mathsf{H}$ when it arises by symmetry breaking), that is, there are maps from S^{D+1} to $\mathcal{N}$ that cannot be smoothly deformed to the trivial map where every point in the base space goes to the same point in the target. The

variation of $\mathcal{W}_{D-1}$ with respect to U is (for even D)[32]:

$$\delta\mathcal{W}_{D-1} = (D+1)c\int_{B_{D+1}} \mathrm{tr}\left\{\left(U^{-1}dU\right)^D \underbrace{\delta\left(U^{-1}dU\right)}_{=U^{-1}d(\delta UU^{-1})U}\right\} \tag{4.59}$$

$$= (D+1)c\int_{B_{D+1}} \mathrm{tr}\left\{\left(dUU^{-1}\right)^D d(\delta UU^{-1})\right\} \tag{4.60}$$

$$= (D+1)c\int_{B_{D+1}} d\mathrm{tr}\left\{\left(U^{-1}dU\right)^D U^{-1}\delta U\right\} \tag{4.61}$$

$$\overset{\text{Stokes}}{=} (D+1)c\int_{\mathcal{M}} \mathrm{tr}\left\{\left(U^{-1}dU\right)^D U^{-1}\delta U\right\}$$

which only depends on the field configuration on the actual spacetime $\mathcal{M}$, not on the extension to B_{D+1}. Again there can be topologically distinct ways to make the extension; demanding that they always give the same answer determines c in terms of volumes of spheres (so that $c\int_{S^{D+1}}\mathrm{tr}(U^{-1}dU)^{D+1}\in\mathbb{Z}$ is the winding number), and the coefficient must be an integer multiple of 2π. (In $D=4$, we have $c=\frac{\mathbf{i}}{240\pi^2}$.)

This WZW term is *less* topological than the theta term we discussed above, in the sense that it affects the equations of motion for the field $\check{n}(t)$ or $U(x)$. The variation of $\mathcal{W}$ *is* local in D dimensions. The following table gives a comparison between theta terms and WZW terms for a field theory in D spacetime dimensions, on a spacetime $\mathcal{M}_D$:

---

[32] Why do I restrict to even D?

$$\mathrm{tr}\left(U^{-1}dU\right)^{D+1} = \epsilon^{\mu_1\cdots\mu_{D+1}}\mathrm{tr}\left(U^{-1}\partial_{\mu_1}U\cdots U^{-1}\partial_{\mu_{D+1}}U\right)$$

but $\epsilon^{\mu_1\cdots\mu_{D+1}} = -(-1)^{D+1}\epsilon^{\mu_{D+1}\mu_1\cdots\mu_D}$ so $\mathcal{W}_{D-1} = (-1)^D\mathcal{W}_{D-1}$ vanishes for odd D. The step from (4.60) to (4.61) also relies on this fact. Using $1 = U^{-1}U$ and hence $0 = \delta(U^{-1}U) = d(U^{-1}U)$, so that

$$dU^{-1} = -U^{-1}dUU^{-1}, \tag{4.58}$$

the term by which (4.60) and (4.61) differ is

$$\mathrm{tr}\left\{\left(d\left(U^{-1}dU\right)^D\right)\delta UU^{-1}\right\}$$

$$\overset{\text{product rule}}{=}\mathrm{tr}\left\{\left(dU^{-1}\wedge dU\wedge\left(U^{-1}dU\right)^{D-1} - \left(U^{-1}dU\wedge dU^{-1}\wedge dU\wedge\left(U^{-1}dU\right)^{D-2}+\cdots\right)\delta UU^{-1}\right\}$$

$$\overset{(4.58)}{=} -\mathrm{tr}\left\{\left(U^{-1}dU\wedge U^{-1}dU\wedge\left(U^{-1}dU\right)^{D-1} - U^{-1}dUU^{-1}\wedge dUU^{-1}\wedge dU\wedge\left(U^{-1}dU\right)^{D-2}+\cdots\right)\delta UU^{-1}\right\}$$

$$= \mathrm{tr}\left\{\underbrace{(1-1+1-1...)}_{D-1\text{ of these}}\left(U^{-1}dU\right)^{D-1}\delta UU^{-1}\right\} \overset{D-1\text{ even}}{=} 0.$$

See Weinberg, vol 2, §23.4 for more.

theta term	WZW term
$\mathcal{H} = \int_{\mathcal{M}_D} h$	$\mathcal{W}_{D-1} = \int_{B_{D+1}} w, \quad \partial B_{D+1} = \mathcal{M}_D$
$h = \mathrm{d}q$	$\delta w = \mathrm{d}v$
Doesn't affect EOM	Affects EOM
Invisible in perturbation theory	Appears in perturbation theory, *e.g.* in beta functions
$\mathcal{H} \in \mathbb{Z}$ for $\mathcal{M}_D$ closed Coefficient of $\mathcal{H}$ is 2π-periodic.	Coefficient of $\mathcal{W} \in 2\pi\mathbb{Z}$ in order for path integral to be well-defined.

Pion physics is the context where these terms were first discovered, and where it was realized that their coefficients are quantized. In particular the coefficient of the WZW term $W_3[U]$ here is N_c, the number of colors, as Witten shows by explicitly coupling to electromagnetism, and finding the term (4.56) that encodes $\pi^0 \to \gamma\gamma$. Apparently Witten realized that such a term was required because without it the chiral Lagrangian had an extra symmetry under $\pi \to -\pi$ which is absent in QCD; the WZW term also produces a 5-pion amplitude that violates this symmetry.

[End of Lecture 17]

One dramatic consequence here is that the chiral Lagrangian (with some higher-derivative terms) has a topological soliton solution (the skyrmion) which is a *fermion* if the number of colors of QCD is odd. The field configuration $U(x,t)$ is constant in time and approaches the vacuum at infinity, so we can regard it as a map

$$U : (\text{space} \cup \infty \sim S^d) \to G/H, \tag{4.62}$$

where G is the full symmetry group and H is the unbroken subgroup, so G/H is the space of Goldstones (in the chiral Lagrangian, $G/H = \mathsf{SU}(3) \times \mathsf{SU}(3)/\mathsf{SU}(3)_{\text{preserved}} \simeq \mathsf{SU}(3)_{\text{broken}}$). The configuration is topological in the sense that as a map from $S^3 \to G/H$, it cannot be smoothly deformed to the trivial map – it represents a nontrivial element of $\pi_3(G/H)$. Its nontriviality is witnessed by a winding number, which can be written as the integral of a local density. In fact, the baryon number of this configuration comes from the anomalous (WZW) contribution to the baryon number current[33]

$$B_\mu = \frac{\epsilon_{\mu\nu\alpha\beta}}{24\pi^2} \mathrm{tr} U^{-1}\partial_\nu U U^{-1}\partial_\alpha U U^{-1}\partial_\beta U \tag{4.63}$$

[33] Witten gives two arguments for this. One is by including the couplings to the $\mathsf{SU}(2)_L$ electroweak gauge bosons, he shows that this term is related by a gauge transformation to terms responsible for the $\mathsf{U}(1)_B\mathsf{SU}(2)^2_{EW}$ anomaly. The second is an appeal to a generalization of the calculation of Goldstone and Wilczek described on the first homework.

whose conserved charge $\int_{\text{space}} B_0$ is exactly the winding number of the map from space (plus the point at infinity) to the space of goldstones. And finally this object a fermion because the WZW term evaluates to π on a spacetime trajectory where the soliton makes a 2π rotation. So this object is a fermionic particle which carries baryon number. It also carries isospin. It's a nucleon! Above we added nucleon fields to the chiral Lagrangian, but we actually didn't need to – they were already there as solitonic excitations. Note that the *size* of the soliton (the region of space over which the fields vary) is determined by the higher-derivative terms in the chiral lagrangian, so we shouldn't take too seriously the substructure of the proton predicted by this picture. But it doesn't do too badly.

Constraints on solitons in scalar field theories. The fact that the chiral Lagrangian has nontrivial, stable, static solitonic particle solutions merits some further comment. The irrelevant terms actually play an important role. Without them, we can show that no such stable solutions exist.

Derrick's argument: Consider a field theory of scalars with 0-derivative and 2-derivative terms. For purposes of finding static solutions, extremizing the action is the same as extremizing the energy:

$$E[\phi] = \int d^d x \left(g(\phi) \left(\vec{\nabla}\phi\right)^2 + V(\phi) \right) \equiv I_1 + I_2.$$

There could be multiple scalars, so for example, the argument applies to the leading term in the chiral lagrangian $L = \text{tr}\,(U^{-1}\partial U)^2$. We'll assume $I_1 > 0$, since otherwise there is an obvious gradient instability of the theory.

Suppose we have a solution $\underline{\phi}$ that extremizes E. To describe a particle excitation of the vacuum, it must approach the vacuum value far away, $\underline{\phi}(x) \overset{x\to\infty}{\to} \phi_0$.

Now consider a dilated configuration $\phi_\lambda(x) \equiv \underline{\phi}(\lambda x)$. Plugging in and changing integration variables gives

$$E[\phi_\lambda] = \frac{I_1}{\lambda^{d-2}} + \frac{I_2}{\lambda^d}.$$

Demanding that $\underline{\phi}$ is a stationary point implies

$$0 = \partial_\lambda E[\phi_\lambda]|_{\lambda=1} = (2-d)I_1 - dI_2 \quad \Longrightarrow \quad I_2 = \frac{2-d}{d} I_1$$

and then

$$\partial_\lambda^2 E[\phi_\lambda]|_{\lambda=1} = (2-d)(1-d)I_1 + d(d+1)I_2 = -2(d-2)I_1 < 0.$$

So the solution is unstable to dilations for $d > 2$.

If we add a term with more derivatives, like $I_3 = \frac{1}{M^4}\int(\vec{\nabla}\phi)^6$, it will contribute positively to $\partial_\lambda^2 E[\phi_\lambda]|_{\lambda=1}$ and the argument is no longer valid. The length scale $1/M$ in front of this higher-derivative term then determines the size of the soliton.

I should also mention that WZW terms are important in the study of interacting spin systems, for example in our understanding of the dependence on the s of Heisenberg spin-s chains (§4.1 here), and in phase transitions beyond the Landau-Ginzburg (symmetry-breaking) paradigm (*i.e.* deconfined quantum criticality, §5.5 here).

4.10 Coherent-state path integral for spin systems

[Wen §2.3.1, Fradkin, Sachdev, *QPT*, chapter 13 and §2.2 of cond-mat/0109419]

I was trying to resist saying something about this, but I feel compelled to explain the origin of the WZW term in the path integral for a spin system. I'll skip most of this in lecture.

Start with a spin one-half system, with

$$\mathcal{H}_{\frac{1}{2}} \equiv \text{span}\{|\uparrow\rangle, |\downarrow\rangle\}.$$

Define spin coherent states $|\vec{n}\rangle$ by[34]:

$$\vec{\boldsymbol{\sigma}} \cdot \vec{n}\,|\vec{n}\rangle = |\vec{n}\rangle \quad .$$

These states form another basis for $\mathcal{H}_{\frac{1}{2}}$; they are related to the basis where $\boldsymbol{\sigma}^z$ is diagonal by:

$$|\vec{n}\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle, \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\theta}{2}e^{i\psi/2} \\ e^{+i\varphi/2}\sin\frac{\theta}{2}e^{i\psi/2} \end{pmatrix} \tag{4.64}$$

as you can see by diagonalizing $\vec{n}\cdot\vec{\boldsymbol{\sigma}}$ in the $\boldsymbol{\sigma}^z$ basis. Notice that

$$\vec{n} = z^\dagger \vec{\boldsymbol{\sigma}} z, \quad |z_1|^2 + |z_2|^2 = 1$$

and the phase of z_α does not affect $\vec{n}$ (this is the Hopf fibration $S^3 \to S^2$). In (4.64) I chose a representative of the phase. The space of independent states is a two-sphere:

$$S^2 = \{(z_1, z_2)||z_1|^2 + |z_2|^2 = 1\}/(z_\alpha \simeq e^{i\chi} z_\alpha).$$

[34] For more general spin representation with spin $s > \frac{1}{2}$, and spin operator $\vec{\mathbf{S}}$, we would generalize this equation to

$$\vec{\mathbf{S}} \cdot \vec{n}\,|\vec{n}\rangle = s\,|\vec{n}\rangle\,.$$

It is just the ordinary Bloch sphere of pure states of a qbit.

These states are not orthogonal (there are infinitely many of them and the Hilbert space is only 2-dimensional!): the overlap between two of them is

$$\langle \check{n}|\check{n}'\rangle = z^\dagger z' = (z_1^\star, z_2^\star)\begin{pmatrix} z_1' \\ z_2' \end{pmatrix},$$

as you can see using the σ^z-basis representation (4.64). The (over-)completeness relation in this basis is:

$$\int \frac{\mathrm{d}^2\vec{n}}{2\pi}|\vec{n}\rangle\langle\vec{n}| = \mathbb{1}_{2\times 2}. \tag{4.65}$$

As always, we can construct a path integral representation of any amplitude by inserting many copies of $\mathbb{1}$ in between successive time steps. For example, using (4.65) many times, we can construct such a representation for the propagator:

$$\begin{aligned}\mathbf{i}G(\vec{n}_f,\vec{n}_0,t) &\equiv \langle\vec{n}_f|\, e^{-\mathbf{i}\mathbf{H}t}\,|\vec{n}_1\rangle \\ &= \int \prod_{i=1}^{M\equiv\frac{t}{\mathrm{d}t}} \frac{\mathrm{d}^2\vec{n}(t_i)}{2\pi} \lim_{\mathrm{d}t\to 0} \langle\vec{n}(t)|\vec{n}(t_M)\rangle \ldots \langle\vec{n}(t_2)|\vec{n}(t_1)\rangle\,\langle\vec{n}(t_1)|\vec{n}(0)\rangle\end{aligned} \tag{4.66}$$

with $\vec{n}_0 = \vec{n}(0), \vec{n}_f = \vec{n}(t)$. (Notice that $\mathbf{H} = 0$ here, so $\mathbf{U} \equiv e^{-\mathbf{i}\mathbf{H}t}$ is actually the identity.) The crucial ingredient is

$$\langle\vec{n}(t+\epsilon)|\vec{n}(t)\rangle = z^\dagger(\mathrm{d}t)z(0) \overset{0=1-1}{=} 1 - z^\dagger(\mathrm{d}t)\,(z(\mathrm{d}t) - z(0)) \approx e^{-z^\dagger\partial_t z\mathrm{d}t}.$$

$$\mathbf{i}G(\vec{n}_f,\vec{n}_0,t) = \int_{\vec{n}(0)=\vec{n}_0}^{\vec{n}(t)=\vec{n}_f} [D\vec{n}]\, e^{\mathbf{i}S_B[\vec{n}(t)]}, \qquad S_B[\vec{n}(t)] = \int_0^t \mathrm{d}t\mathbf{i}z^\dagger\dot{z}\ . \tag{4.67}$$

Even though the Hamiltonian of the spins was zero – whatever their state, they have no potential energy and no kinetic energy – the action in the path integral is not zero. This phase $e^{\mathbf{i}S_B}$ is a quantum phenomenon (again) called a Berry phase.

Starting from the action S_B and doing the Legendre transform to find the Hamiltonian you will get zero. The first-derivative action says that $z^\dagger$ is the canonical momentum conjugate to z: the space with coordinates $(z, z^\dagger)$ becomes the phase space (just like position and momentum)! But this phase space is curved. In fact it is the two-sphere

$$S^2 = \{(z_1,z_2)||z_1|^2 + |z_2|^2 = 1\}/(z_\alpha \simeq e^{\mathbf{i}\psi}z_\alpha).$$

In terms of the coordinates θ, φ above, we have

$$S_B[z] = S_B[\theta,\varphi] = \int dt\frac{1}{2}\left(\cos\theta\dot{\phi} + \dot{\psi}\right)|_{\psi=0} = 4\pi s W_0[\hat{n}]|_{s=\frac{1}{2}}. \tag{4.68}$$

At the last step we chose a gauge $\psi = 0$. BIG CONCLUSION: This is the 'area' term that we studied above, with $s = \frac{1}{2}$! So the expression in terms of z in (4.67) gives another way to write the area term which is manifestly SU(2) invariant; this time the price is introducing these auxiliary z variables, with their gauge redundancy $z(t) \to e^{\mathbf{i}\chi(t)} z(t)$.

Making different choices of the phase ψ at different times can shift the constant in front of the second ($\dot{\psi}$) term in (4.68); as we observed earlier, this term is a total derivative. Different choices of ψ change the overall phase of the wavefunction, which doesn't change physics (recall that this is why the space of normalized states of a qbit is a two-sphere and not a three-sphere). Notice that $\mathcal{A}_t = z^\dagger \partial_t z$ is like the time component of a gauge field. Adding a total derivative to the action (by changing $\psi(t)$) imparts a gauge transformation.

The Berry phase $S_B[n]$ is *geometric*, in the sense that it depends on the trajectory of the spin through time, but not on its parametrization, or speed or duration. It is called the Berry phase of the spin history because it is the phase acquired by a spin that follows the instantaneous groundstate (*i.e.* adiabatic evolution) $|\Psi_0(t)\rangle$ of $H(\check{n}(t), t) \equiv -h\check{n}(t) \cdot \mathbf{S}$, with $h > 0$. This is Berry's adiabatic phase, $S_B[\check{n}] = -\lim_{\partial_t h \to 0} \int dt \mathrm{Im} \langle \Psi_0(t)| \partial_t |\Psi_0(t)\rangle$.

Since S_B is geometric, like integrals of differential forms, let's take advantage of this to make it pretty and relate it to familiar objects. Introduce a vector potential (the Berry connection) on the sphere $A^a, a = x, y, z$ so that

$$S_B = \oint d\tau \dot{n}_a A^a = \oint_\gamma A \overset{\text{Stokes}}{=} \int_D F$$

where $\gamma = \partial D$ is the trajectory. ($F = \mathrm{d}A$ is the Berry curvature.) What is the correct form? We must have $(\nabla \times A) \cdot \check{n} = \epsilon^{abc} \partial_{n^a} A^b n^c = 1$ (for spin half). This is a monopole field. Two choices that work are

$$A^{(1)} = -\cos\theta d\varphi, \quad \text{and} \quad A^{(2)} = (1 - \cos\theta) d\varphi.$$

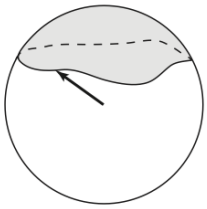
These two expressions differ by the gauge transformation $\mathrm{d}\varphi$, which is locally a total derivative. The first is singular at the N and S poles, $\check{n} = \pm\check{z}$. The second is singular only at the S pole. Considered as part of a 3d field configuration, this codimension two singularity is the 'Dirac string'. The demand of invisibility of the Dirac string quantizes the Berry flux. The gauge transformations that move around the singularities of A are accomplished by adding total derivatives to the action, *i.e.* by choosing $\psi(t)$. For example, by choosing $\psi(t) = \pm\varphi(t)$ we find the gauge that is nonsingular away from the north and south poles, respectively.

If we redo the above coherent-state quantization for a spin-s system we'll get the expression with general s (see below). Notice that this only makes sense when $2s \in \mathbb{Z}$.

We can add a nonzero Hamiltonian for our spin; for example, we can put it in an external Zeeman field $\vec{h}$, which adds $\mathbf{H} = -\vec{h} \cdot \vec{\mathbf{S}}$. This will pass innocently through the construction of the path integral, adding a term to the action $S = S_B + S_h$,

$$S_h = \int \mathrm{d}t \left(s\vec{h} \cdot \vec{n} \right)$$

where s is the spin.



We see that the system we get by 'geometric quantization' of the sphere is a quantum spin. The quantized coefficient of the area is $2s$: it determines the dimension of the spin space to be $2s + 1$. Here the quantization of the WZW term is just quantization of angular momentum. (In higher-dimensional field theories, it is something else.)

Deep statement: the purpose in life of the WZW term is to enforce the commutation relation of the SU(2) generators, $[\mathbf{S}^i, \mathbf{S}^j] = \mathbf{i}\epsilon^{ijk}\mathbf{S}^k$. It says that the different components of the spin don't commute, and it says precisely what they don't commute to.

Incidentally, another way to realize this system whose action is proportional to the area of the sphere is to take a particle on the sphere, put a magnetic monopole in the center, and take the limit that the mass of the particle goes to zero. In that context, the quantization of $2s$ is Dirac quantization of magnetic charge. And the degeneracy of $2s + 1$ states is the degeneracy of states in the lowest Landau level for a charged particle in a magnetic field; the $m \to 0$ limit gets rid of the higher Landau levels (which are separated from the lowest by the cylotron frequency, $\frac{eB}{mc}$).

In the crucial step, we assumed the path $z(t)$ was smooth enough in time that we could do calculus, $z(t + \epsilon) - z(t) = \epsilon \dot{z}(t) + \mathcal{O}(\epsilon^2)$. Is this true of the important contributions to the path integral? Sometimes not, and we'll come back to this later.

I've written the path integral for a single spin. The generalization to a many body spin system is simple in principle: just do the above for each site.

Digression on $s > \frac{1}{2}$. [Auerbach, *Interacting Electrons and Quantum Magnetism*] I want to say something about larger-spin representations of $\mathsf{SU}(2)$, partly to verify the claim above that it results in a factor of $2s$ in front of the Berry phase term. Also, large s allows us to approximate the integral by stationary phase.

In general, a useful way to think about the coherent state $|\check{n}\rangle$ is to start with the maximal-spin eigenstate $|s, s\rangle$ of $\mathbf{S}^z$ (the analog of spin up for general s), and *rotate it* by the rotation that takes $\mathbf{S}^z$ to $\mathbf{S} \cdot \check{n}$:

$$|\check{n}\rangle = \mathcal{R}(\chi, \theta, \varphi) \, |s, s\rangle \, .$$

The form of $\mathcal{R}$ involves Euler angles; let's find a better route than remembering about Euler angles.

Schwinger bosons. The following is a helpful device for spin matrix elements. Consider two copies of the harmonic oscillator algebra, with modes a, b satisfing $[a, a^\dagger] = 1 = [b, b^\dagger], [a, b] = [a, b^\dagger] = 0$. Then the objects

$$\mathbf{S}^+ = a^\dagger b,\ \mathbf{S}^- = b^\dagger a,\ \mathbf{S}^z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right)$$

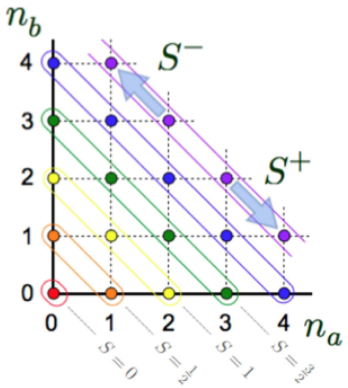
satisfy the $\mathsf{SU}(2)$ algebra. The no-boson state $|0\rangle$ is a singlet of this $\mathsf{SU}(2)$, and the one-boson states $\begin{pmatrix} a^\dagger|0\rangle \\ b^\dagger|0\rangle \end{pmatrix}$ form a spin-half doublet.

More generally, the states

$$\mathcal{H}_s \equiv \text{span}\{|n_a, n_b\rangle\, |a^\dagger a + b^\dagger b \equiv n_a + n_b = 2s\}$$

form a spin-s representation. Algebraic evidence for this is the fact that $\vec{S}^2 P_s = s(s+1)P_s$, where P_s is the projector onto $\mathcal{H}_s$. The spin-s eigenstates of $\mathbf{S}^z$ are

$$|s, m\rangle = \frac{(a^\dagger)^{s+m}}{\sqrt{(s+m)!}} \frac{(b^\dagger)^{s-m}}{\sqrt{(s-m)!}} |0\rangle\,.$$



[nice figure from Arovas and Auerbach, 0809.4836.]

The fact that $\begin{pmatrix} a^\dagger|0\rangle \\ b^\dagger|0\rangle \end{pmatrix} = \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}|0\rangle$ forms a doublet means that $\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}$ itself must be a doublet. But we know how a doublet transforms under a rotation, and this means we know how to write the coherent state:

$$|\check{n}\rangle = \mathcal{R}\,|s, s\rangle = \mathcal{R}\frac{(a^\dagger)^{2s}}{\sqrt{(2s)!}}|0\rangle = \mathcal{R}\frac{(a^\dagger)^{2s}}{\sqrt{(2s)!}}\mathcal{R}^{-1}\mathcal{R}\,|0\rangle = \frac{(a'^\dagger)^{2s}}{\sqrt{(2s)!}}|0\rangle = \frac{(z_1 a^\dagger + z_2 b^\dagger)^{2s}}{\sqrt{(2s)!}}|0\rangle\,.$$

Here $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{\mathbf{i}\varphi/2}\cos\frac{\theta}{2}e^{\mathbf{i}\psi/2} \\ e^{-\mathbf{i}\varphi/2}\sin\frac{\theta}{2}e^{\mathbf{i}\psi/2} \end{pmatrix}$ as above[35].

But now we can compute the crucial ingredient in the coherent state path integral, the overlap of successive coherent states:

$$\langle\check{n}|\check{n}'\rangle = \frac{e^{-\mathbf{i}s(\psi-\psi')}}{(2s)!}\underbrace{\langle 0|\,(z_1^\star a + z_2^\star b)^{2s}(z_1' a^\dagger + z_2' b^\dagger)^{2s}\,|0\rangle}_{\overset{\text{Wick}}{=}(2s)!\left([z_1^\star a + z_2^\star b, z_1' a^\dagger + z_2' b^\dagger]\right)^{2s}} = e^{-\mathbf{i}s(\psi-\psi')}(z_1^\star z_1' + z_2^\star z_2')^{2s} = \left(e^{-\mathbf{i}(\psi-\psi')/2} z^\dagger \cdot z'\right)^{2s}.$$

[35] Sometimes you may see the notation $z_1 \equiv u, z_2 \equiv v$.

Here's the point: this is the same as the spin-half answer, raised to the $2s$ power. This means that the Berry phase just gets multiplied by $2s$, $S_B^{(s)}[n] = 2sS_B^{\left(\frac{1}{2}\right)}[n] = 4\pi s W_0[n]$, as we claimed.

Semi-classical spectrum. Above we found a path integral representation for the Green's function of a spin as a function of time, $G(n_t, n_0; t)$. The information this contains about the spectrum of the hamiltonian can be extracted by Laplace transforming

$$G(n_t, n_0; E) \equiv -\mathbf{i}\int_0^\infty dt G(n_t, n_0; t) e^{\mathbf{i}(E+\mathbf{i}\epsilon)t}$$

and taking the trace

$$\Gamma(E) \equiv \int \frac{d^2 n_0}{2\pi} G(n_0, n_0; E) = \mathrm{Tr}\ \frac{1}{E - \mathbf{H} + \mathbf{i}\epsilon}.$$

This function has poles at the eigenvalues of $\mathbf{H}$. Its imaginary part is the spectral density, $\rho(E) = \frac{1}{\pi}\mathrm{Im}\Gamma(E) = \sum_\alpha \delta(E - E_\alpha)$.

Its path integral representation is then

$$\Gamma(E) = -\mathbf{i}\int dt \oint D\check{n}\ e^{\mathbf{i}((E+\mathbf{i}\epsilon)t + sS[n])}.$$

The $\oint$ indicates periodic boundary conditions, $\check{n}(0) = \check{n}(t)$, and $S[n] = S_B[n] - \int^t dt' H_{\rm cl}[n]/s$. Here $H_{\rm cl}[n] \equiv \langle \check{n} | \mathbf{H} | \check{n} \rangle$.

At large s, field configurations that vary too much in time are cancelled out by the rapidly oscillating phase, that is: we can try to do these integrals by stationary phase. The stationarity condition for the n integral is the equations of motion $0 = \dot{n} \times n - \partial_n H_{\rm cl}$. If $\mathbf{H} = \vec{h} \cdot \mathbf{S}$, this gives the Landau-Lifshitz equation for precession. We keep only solutions periodic with $t = nT$ an integer multiple of the period T. The stationarity condition for the t integral is

$$0 = E + \partial_t S[n] = E - H_{\rm cl}[n].$$

In the second equality we used the fact that the Berry phase is geometric, it depends only on the trajectory, not on t (how long it takes to get there). So the semiclassical trajectories are periodic solutions to the EOM with energy $E = H_{\rm cl}[n^E]$. The exponent evaluated on such a trajectory is then just the Berry term. Denoting by n_1^E such trajectories that traverse once ('prime' orbits),

$$\Gamma(E) \sim \sum_{n_1^E} \sum_{m=0}^\infty e^{\mathbf{i}msS_B[n]} = \sum_{n_1^E} \frac{e^{\mathbf{i}sS_B[n]}}{1 - e^{\mathbf{i}sS_B[n]}}.$$

This is an instance of the Gutzwiller trace formula. The locations of poles of this function approximate the eigenvalues of $\mathbf{H}$. They occur at $E = E^m_{sc}$ such that $S_B[\vec{n}^{E_m}] = \frac{2\pi m}{s}$, with $m \in \mathbb{Z}$. The actual eigenvalues are $E^m = E^m_{sc} + \mathcal{O}(1/s)$.

If the path integral in question were a 1d particle in a potential, with $S_B = \int p dx$, and $H_{\text{cl}} = p^2 + V(x)$, the semiclassical condition would reduce to

$$2\pi m = \oint_{x^{E_m}} p(x)dx = \int_{\text{turning points}} \sqrt{E_m - V(x)}$$

the Bohr-Sommerfeld condition.

4.11 Topological terms from integrating out fermions

[Abanov ch 7] Here is a much simpler example (than QCD) where WZW terms in an EFT remember a topological property of the microscopic theory. Consider a 0+1 dimensional model of *spinful* fermions $\mathbf{c}_\alpha, \alpha = \uparrow, \downarrow$ coupled to a single spin s, $\vec{\mathbf{S}}$. Let's couple them in an $\mathsf{SU}(2)$-invariant way:

$$H_K = M\left(\mathbf{c}^\dagger \vec{\sigma} \mathbf{c}\right) \cdot \vec{\mathbf{S}}$$

by coupling the spin of the fermion $\mathbf{c}^\dagger_\alpha \vec{\sigma}_{\alpha\beta} \mathbf{c}_\beta$ to the spin. 'K' is for 'Kondo'. Notice that M is an energy scale. $M > 0$ is an antiferromagnetic interaction between the spin of the fermion mode and the spin $\vec{\mathbf{S}}$. (Exercise: find the spectrum of H_K.)

Now apply both of the previous coherent state path integrals that we've learned to write the (say euclidean) partition sum as

$$Z = \int [D\psi D\bar{\psi} D\check{n}] e^{-S_0[n] - \int_0^T \mathrm{d}t \bar{\psi}(\partial_t - M\vec{n}\cdot\vec{\sigma})\psi}$$

where $\psi = (\psi_\uparrow, \psi_\downarrow)$ is a two-component Grassmann spinor, and $\vec{\sigma}$ are Pauli matrices acting on its spinor indices. $\check{n}^2 = 1$. Let $S_0[n] = \int K\dot{n}^2 + (2s)2\pi W_0[n]$, where I've added a second-order kinetic term for reasons we'll see below.

First of all, consider a fixed, slowly-varying configuration of $\check{n}$. What does this do to the propagation of the fermion? I claim that it gaps out the fermion excitations, in the sense that

$$\left\langle \mathcal{T} \mathbf{c}^\dagger_\alpha(t) \mathbf{c}_\beta(0) \right\rangle = \left\langle \bar{\psi}_\alpha(t) \psi_\beta(0) \right\rangle$$

will be short-ranged in time. Let's see this using the path integral.

We can do the (gaussian) integral over the fermion, to get:

$$Z = \int [D\vec{n}] e^{-S_{\text{eff}}[\vec{n}]}$$

with

$$S_{\text{eff}}[\vec{n}] = S_0[\vec{n}] - \log\det\left(\partial_t - M\vec{n}\cdot\vec{\sigma}\right) \equiv S_0 - \log\det D \equiv S_0 + S_1.$$

The variation of the new term in the effective action under a variation of $\vec{n}$ is:

$$\delta S_1 = -\mathrm{tr}\left(\delta D D^{-1}\right) = -\mathrm{tr}\left(\delta D D^\dagger \left(D D^\dagger\right)^{-1}\right)$$

where $D^\dagger \equiv -\partial_t - M\vec{n}\cdot\vec{\sigma}$. This is

$$\delta S_1 = M\mathrm{tr}\left(\delta\vec{n}\cdot\vec{\sigma}\left(\partial_t + M\vec{n}\cdot\vec{\sigma}\right)\left(\underbrace{-\partial_t^2 + M^2 - M\dot{\vec{n}}\cdot\vec{\sigma}}_{=DD^\dagger}\right)^{-1}\right). \tag{4.69}$$

We can expand the denominator in $\dot{\vec{n}}/M$ (and use $n^2 = 1$) to get

$$\delta S_1 = \int \mathrm{d}t \left(-\frac{M}{|M|}\frac{1}{2}\delta\vec{n} \cdot \left(\vec{n} \times \dot{\vec{n}}\right) + \frac{1}{4M}\delta\dot{\vec{n}}\dot{\vec{n}} + ... \right)$$

where ... is higher order in the expansion and we ignore it. But we know this is the variation of

$$S_1 = -2\pi\frac{M}{|M|}W_0 + \int_0^T \mathrm{d}t \left(\frac{1}{8M}\dot{\vec{n}}^2\right) + \mathcal{O}\left(\frac{\dot{n}}{M}\right)^3$$

where W_0 is the WZW term. Integrating out the fermions has shifted the coefficient of the WZW term from $s \to s \mp \frac{1}{2}$ depending on the sign of M. This is satisfying: we are adding angular momenta, $s \otimes \frac{1}{2} = \left(s - \frac{1}{2}\right) \oplus \left(s + \frac{1}{2}\right)$. If $M > 0$, it is an antiferromagnetic interaction whose groundstates will be the ones with smaller eigenvalue of $\vec{S}^2$. If $M < 0$, it is ferromagnetic, and the low-energy manifold grows. This agrees precisely with the coefficient of the WZW term in our effective action, which is $4\pi\left(s - \frac{1}{2}\mathrm{sign}(M)\right)$.

Here is a more direct (?) calculation of the fermion determinant S_1 (also from Abanov).

$$S_1 = -\ln\det D = -\mathrm{Tr}\ \ln D \stackrel{?}{=} -\mathrm{Tr}\ \ln\tilde{D} \tag{4.70}$$

where $\tilde{D} \equiv U^\dagger D U = \partial_t - \mathbf{i}a - M\sigma^3$ where we've defined the unitary transformation U so that

$$\sigma^3 \stackrel{!}{=} U^\dagger\vec{n}\cdot\vec{\sigma}U, \qquad \text{and} \qquad a \equiv U^\dagger\mathbf{i}\partial_t U.$$

In terms of the free propagator $G_0^{-1} \equiv \partial_t - M\sigma^3$, we can write

$$\tilde{D} = G_0^{-1}(1 - G_0\mathbf{i}a).$$

Then we can expand in powers of a

$$S_1 = -\mathrm{Tr}\ \ln\tilde{D} = \mathrm{Tr}\ \left(\ln G_0 + G_0\mathbf{i}a + \frac{1}{2}\left(G_0\mathbf{i}a\right)^2 + \cdots\right) \equiv S_{(0)} + S_{(1)} + \cdots .$$

The first term is some constant which we ignore. The term linear in a is

$$S_{(1)} = \mathrm{tr}G_0\mathbf{i}a = \quad = \mathrm{tr}_\sigma \int dsdtG_0(s-t)a(t)\delta(t-s) \tag{4.71}$$

$$= \mathrm{tr}_\sigma \underbrace{\int \text{đ}\omega \frac{e^{\mathbf{i}\omega dt}}{-\mathbf{i}\omega - M\sigma^3}}_{=\theta(M\sigma^3)} \mathbf{i}a_{\omega=0} = -\mathrm{sign}(M)\mathbf{i}\int dta^3(t). \tag{4.72}$$

Here $a^3 \equiv \frac{1}{2}\mathrm{tr}_\sigma a\sigma^3 = \frac{1}{2}\cos\theta\dot{\varphi}$. In evaluating $G(t=0)$, I used a point-splitting regularization motivated by the derivation of the path integral. From this we conclude

$$S_{(1)} = -2\pi\mathrm{sign}(M)W_0[n].$$

Similarly, the next term is

$$S_{(2)} = \frac{1}{2}\mathrm{tr}(G_0 \mathbf{i}a)^2 = \quad (4.73)$$

$$= \frac{1}{2}\int đ\omega_1 \int đ\omega_2 \mathrm{tr}_\sigma \left(\frac{1}{-\mathbf{i}\omega_1 - M\sigma^3} \mathbf{i}a_{-\omega_2} \frac{1}{-\mathbf{i}(\omega_1+\omega_2) - M\sigma^3} \mathbf{i}a_{\omega_2} \right) \quad (4.74)$$

$$= \frac{1}{8M}\int đ\omega \mathrm{tr}_\sigma \left(a_{-\omega}a_\omega - \sigma^3 a_{-\omega}\sigma^3 a_\omega \right) \left(1 + \mathcal{O}\left(\frac{1}{M}\right)\right) \quad (4.75)$$

$$= \frac{1}{2M}\int dt \left(a_1^2 + a_2^2\right)\left(1+\mathcal{O}\left(\frac{1}{M}\right)\right) = \frac{1}{8M}\int dt \left(\partial_t \vec{n}\right)^2 \left(1+\mathcal{O}\left(\frac{1}{M}\right)\right). \quad (4.76)$$

To see (4.75), note that unless there is a σ^1 or σ^2 in between the two propagators, their poles are on the same side of the frequency contour, and so we get zero by closing the contour on the opposite side:

$$\int_{-\infty}^{\infty} đ\omega \frac{1}{-\mathbf{i}\omega_1 - s_1 M} \frac{1}{-\mathbf{i}(\omega_1+\omega) - s_2 M} = \begin{cases} 0, & \text{if } s_1 = s_2 \\ \frac{1}{2M - \mathbf{i}\omega s_1} = \frac{1}{2M}\left(1 + \mathcal{O}\left(\frac{\omega}{M}\right)\right), & \text{if } s_1 = -s_2 \end{cases} \quad (4.77)$$

We could also do the integral by the methods we used for fermion loops in QED, like Feynman parameters.

The second term in S_1 is a shift of K. Higher-order terms are suppressed by more powers of $\frac{\dot{n}}{M}$, so for $\dot{n} \ll M$, this is a local action. That means that the coupling to n must have gapped out the fermions. That the term proportional to M is a funny mass term for the fermions is clear from the expression for $DD^\dagger$ in (4.69): when n is static, $DD^\dagger = -\partial_t^2 + M^2$, so that the fermion propagator is

$$\left\langle \bar{\psi}_\alpha(t)\psi_\beta(0)\right\rangle = \left(\frac{1}{D}\right)_t = \left(\frac{D^\dagger}{DD^\dagger}\right)_t = \int đ\omega \frac{e^{\mathbf{i}\omega t}\left(\delta_{\alpha\beta}\omega + \mathbf{i}M\vec{n}\cdot\sigma_{\alpha\beta}\right)}{\omega^2 + M^2} \sim e^{-Mt}$$

which is short-ranged in time. So indeed the fermions are fast modes in the presence of the coupling to the n-field.

Why did I put a question mark in (4.70)? If we redefine U by $U \to Ue^{\mathbf{i}\sigma^3\psi(t)}$, $a \to e^{-\mathbf{i}\sigma^3\psi}(a - \mathbf{i}\partial_t)e^{\mathbf{i}\sigma^3\psi}$ transforms like a gauge field, and the action S_1 changes by $\int dt\dot{\psi}$, a total derivative.

Such topological terms are one way in which some (topological) information from short distances can persist in the low energy effective action. Being quantized, they can't change under the continuous RG evolution. The WZW term manages to be

independent of M, the mass scale of the fermions. Here the information is that the system is made of fermions (or at least a half-integer spin representation of $\mathsf{SU}(2)$).

The above calculation generalizes well to higher dimensions. The general idea is that integrating out fermions with Yukawa terms involving bosons ϕ produces WZW terms for ϕ. This is how the theory of ϕ remembers that the system is made of fermions. For many examples of its application, see this paper. (For more context for this paper see §5.5 here).

4.12 Heavy quarks and non-relativistic fermions

Earlier, we found a description of a non-relativistic field by starting with a relativistic scalar field and focusing on its slow-moving excitations. We could ask the analogous question about fermions. One good motivation to do so is that b and c quarks are much heavier than u, d, s, and quite a bit about the hadrons containing them can be understood using an EFT that takes advantage of this fact.

[Manohar and Wise, *Heavy Quark Physics*, especially §2.6 and §4.1] Let $p^\mu = mv^\mu + k^\mu$, where $v^\mu v_\mu = 1$, so k^μ describes some (small) deviation from an on-shell momentum. The quark propagator is $\mathbf{i}$ times

$$\frac{\not{p} + m}{p^2 - m^2 + \mathbf{i}\epsilon} = \frac{m\not{v} + m + \not{k}}{2mv\cdot k + k^2 + \mathbf{i}\epsilon} \overset{k \text{ small}}{\simeq} \frac{1+\not{v}}{2}\frac{1}{v\cdot k + \mathbf{i}\epsilon}. \tag{4.78}$$

Here

$$P_v^+ = \frac{1+\not{v}}{2} \overset{v^\mu = (1,\vec{0})^\mu}{\longrightarrow} \frac{1+\gamma^0}{2} \tag{4.79}$$

is the projector onto the particle (as opposed to antiparticle) component of the 4-component spinor. So we can decompose the (heavy) quark field into

$$Q(x) \equiv e^{-\mathbf{i}mv\cdot x}\left(Q_v^+(x) + Q_v^-(x)\right) \tag{4.80}$$

where

$$Q_v^\pm(x) \equiv e^{\mathbf{i}mv\cdot x} P_v^\pm Q(x) = e^{\mathbf{i}mv\cdot x}\frac{1\pm\not{v}}{2}Q(x). \tag{4.81}$$

The field Q_v^- creates antiparticles and its effects will be suppressed by $1/m$.

First, just ignoring Q_v^-, the quark Lagrangian becomes

$$\bar{Q}\left(\mathbf{i}\not{D} - m\right)Q|_{Q^-=0} = \bar{Q}_v^+ \mathbf{i}\not{D} Q_v = \bar{Q}_v^+ \mathbf{i}v\cdot D Q_v^+ \tag{4.82}$$

where in the last step we inserted P_v^+ next to both fields; to see the final (m-independent) expression evaluate it in the rest frame and then boost. The propagator coming from (4.82) is exactly ($\mathbf{i}$ times) the final expression in (4.78). In the rest frame, $v^\mu = (1,\vec{0})^\mu$, this is just $\bar{Q}^+\mathbf{i}\partial_t Q^+$, with no spatial derivatives.

This Lagrangian on the RHS of (4.82) has some emergent symmetries not present in the full quark Lagrangian. In particular, it has heavy quark flavor symmetry, rotating different species of heavy quarks amongst each other (since it does not depend on their mass). Also, spin-orbit couplings are gone, so the spin rotations of the heavy quarks decouples from the rotations and becomes an independent symmetry.

If we, more properly, keep Q_v^-, the Lagrangian is

$$\bar{Q}\left(\mathbf{i}\not{D} - m\right)Q = \bar{Q}_v^+ \mathbf{i}v\cdot D Q_v^+ + \bar{Q}_v^-\left(\mathbf{i}v\cdot D + 2m\right)Q_v^- + \bar{Q}_v^+\mathbf{i}\not{D}Q_v^- + \bar{Q}_v^-\mathbf{i}\not{D}Q_v^+. \tag{4.83}$$

For a general 4-vector, let $X^\mu_\perp = X^\mu - (x\cdot v)v^\mu$ be the component of X^μ transverse to v^μ. Since $\bar{Q}^+_v \not{v} Q^-_v = 0$, $\bar{Q}^+_v \mathbf{i}\not{D} Q^-_v = \bar{Q}^+_v \mathbf{i}\not{D}_\perp Q^-_v$. Note also that $\bar{Q}^+_v Q^-_v = 0$ as you can see using $Q^\pm_v = P_\pm Q^\pm_v$. You can see from this expression that Q^-_v creates an excitation with minimum energy $2m$, so we can integrate it out. The leading order contribution (the only one if we ignore gauge field interactions) is at tree level: $(\mathbf{i}v\cdot D + 2m)\, Q^-_v = \mathbf{i}\not{D}_\perp Q^+_v$, and plugging this back into the Lagrangian gives

$$L_{\text{eff}} = \bar{Q}^+_v \left(\mathbf{i}v\cdot D + \mathbf{i}\not{D}_\perp \frac{1}{2m + \mathbf{i}v\cdot D} \mathbf{i}\not{D}_\perp \right) Q^+_v \tag{4.84}$$

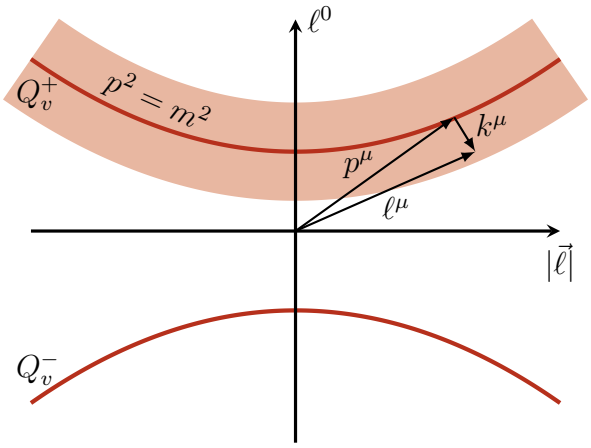
$$= \bar{Q}^+_v \left(\mathbf{i}v\cdot D - \frac{1}{2m}\not{D}_\perp \not{D}_\perp \right) Q^+_v + \mathcal{O}(m^{-2}). \tag{4.85}$$

The new object is

$$\not{D}_\perp \not{D}_\perp = D^2_\perp + \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}. \tag{4.86}$$

Since they depend on m, both terms break the heavy-quark flavor symmetry, and the second term also violates the heavy-quark spin rotation symmetry.

This is what I meant about 'integrating out antiparticles', which was not necessary in the scalar case. At right is a nice diagram from 2505.03566, about which I'll say more in the next section, where we'll see that this system has an addition symmetry of a new kind, a one-form symmetry.



5 Generalized symmetries

Maybe it is obvious that it is a good idea to try to look for more possible symmetries of QFT and quantum many-body systems. Symmetries imply selection rules. Symmetries provide RG invariants, for example in the form of 't Hooft anomalies.

One perhaps-less-obvious motivation for wanting to generalize our notion of symmetry is the following. Landau told us that phases of matter are classified by how they represent their symmetries, this is called the *Landau paradigm*. For example, we can distinguish a magnet from a paramagnet by whether or not the groundstate spontaneously breaks a (spin-rotation) symmetry. Let me attempt to paraphrase the Landau Paradigm:

1. Phases of matter should be labelled by how they represent their symmetries, in particular whether they are spontaneously broken or not.

2. A further belief that comes with this point of view is that gapless degrees of freedom, or groundstate degeneracy, in a phase, should be swept out by a symmetry. That is, they should arise as Goldstone modes for some spontaneously broken symmetry.

3. The degrees of freedom at a critical point are the fluctuations of the order parameter.

Beyond its conceptual utility, this perspective has a weaponization, in the form of Landau-Ginzburg theory, in terms of which we may find representative states, understand gross phase structure, and, when suitably augmented by the renormalization group (RG), even quantitatively describe phase transitions.

In the past few decades, there's been an enormous effort in condensed matter physics of discovering phases of matter that are distinguished by properties that apparently have nothing to do with symmetry, they are called *topological phases*. In the introductions to talks on such subjects Landau has really taken a beating. Indeed there are many apparent exceptions to the Landau Paradigm. Let me list some apparent exceptions to item 1:

- **Topologically-ordered states.** These are phases of matter distinguished from the trivial phase by something other than a local order parameter. Symptoms include a groundstate degeneracy that depends on the topology of space, and *anyons*, excitations that cannot be created by any local operator. Real examples found so far include fractional quantum Hall states, as well as gapped spin liquids.

- **Other deconfined states of gauge theory.** This category includes gapless spin liquids such as spinon Fermi surface or Dirac spin liquids (most candidate spin liquid materials are gapless). Another very visible manifestation of such a state is the photon phase of quantum electrodynamics in which our vacuum lives.

- **Fracton phases.** Gapped fracton phases are a special case of topological order, where there are excitations that not only cannot be *created* by any local operator, but cannot be *moved* by any local operator.

- **Topological insulators.** Here we can include both free-fermion states with topologically non-trivial bandstructure, as well as interacting symmetry-protected topological (SPT) phases.

- **Landau Fermi liquid.**

But it turns out that Landau was more right than we thought. Many of these topological phases actually can be understood in terms of symmetry, but a more general notion of symmetry.

[End of Lecture 18]

5.1 What is a symmetry of a quantum many-body system?

Let's begin by considering the familiar case of a continuous (soon-to-be-called 0-form) symmetry. Noether's theorem guarantees a conserved current J_μ satisfying $\partial^\mu J_\mu = 0$. In the useful language of differential forms, this is $d \star J = 0$, where $\star$ is the Hodge duality operation[36]. This continuity equation has the consequence that the *charge* $Q_\Sigma = \int_{\Sigma_{D-1}} \star J = \int n^\mu_\Sigma J_\mu$ is independent of the choice of time-slice Σ. (Σ here is a closed d-dimensional surface, of codimension one in spacetime, and n^μ_Σ is a normal vector. If Σ is the surface of fixed t, then $Q_\Sigma = \int d^d x J_0(x,t)$.) That is, if we deform Σ to some other Σ' through some region R, with $\partial R = \Sigma' \cup \bar{\Sigma}$, then the change in Q is

$$Q_{\Sigma'} - Q_\Sigma = \int_{\Sigma'} \star J - \int_\Sigma \star J \overset{\text{Stokes}}{=} \int_R d \star J = 0 \tag{5.1}$$

Notice that this is a topological condition. In particular, since Q_Σ is independent of shifts of the time coordinate, $0 = \dot{Q}_\Sigma = \mathbf{i}[H, Q_\Sigma]$, so Q_Σ commutes with the Hamiltonian, the generator of time translations, and therefore so does the unitary operator

[36] The Hodge dual of a p-form ω_p on a D-dimensional space with metric $g_{\mu\nu}$ has components $(\star\omega_p)_{\mu_1 \cdots \mu_{D-p}} = \sqrt{\det g} \epsilon_{\mu_1 \cdots \mu_D} \omega_p^{\mu_{D-p+1} \cdots \mu_D}$, where indices are raised with the inverse metric $g^{\mu\nu}$ and $\epsilon_{\mu_1 \cdots \mu_D}$ is the antisymmetric Levi-Civita symbol.

$U_\alpha = e^{i\alpha Q}$, which we call the *symmetry operator* or symmetry defect operator or topological defect operator[37].

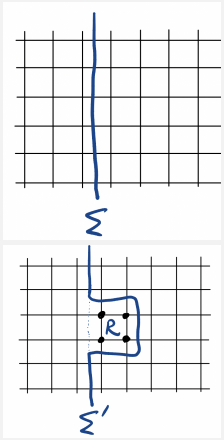
If the charge is carried by particles, Q_Σ counts the number of particle worldlines piercing the surface Σ (as in Fig. 9, left), and the conservation law $\dot{Q} = 0$ says that charged particle worldlines cannot end except on charged operators. If instead of a $\mathsf{U}(1)$ symmetry, we only had a discrete $\mathbb{Z}_p$ symmetry we could simply restrict $\alpha \in \{0, 2\pi/p, 4\pi/p ... (p-1)2\pi/p\}$ in the symmetry operator U_α. In that case, particles can disappear in groups of p.

Objects charged under a 0-form symmetry are created by local operators. Local operators transform under the symmetry by $\mathcal{O}(x) \to U_\alpha \mathcal{O}(x) U_\alpha^\dagger = e^{\mathbf{i}q\alpha}\mathcal{O}(x), d\alpha = 0$, where q is the charge of the operator. The infinitesimal version is: $\delta\mathcal{O}(x) = \mathbf{i}[Q, \mathcal{O}(x)] = \mathbf{i}q\mathcal{O}(x)$.

An old-fashioned symmetry is an action on the degrees of freedom that preserves the action functional S. Noether's theorem relates symmetries to **topological defect operators** $U_g(\Sigma)$ (or symmetry operators). The fact that the symmetry actions form a group implies that these operators enjoy the "fusion rule" $U_g(\Sigma)U_{g'}(\Sigma) = U_{gg'}(\Sigma)$ (up to possible interesting phases). Since the discussion of topological defect operators can be pretty abstract, let's give a very concrete example.

A very concrete example of a topological defect operator: Let's think about the nearest-neighbor Ising model on a Euclidean lattice of any dimension, $Z = \sum_{\{\sigma\}} e^{-S[\sigma]}$, with $S[\sigma] = \sum_{\langle xy \rangle} J_{xy}\sigma_x\sigma_y$. The topological defect operator $U_{-1}(\Sigma)$ is an instruction to flip the sign of J for any bond crossing Σ. (The -1 is the nontrivial element of the group $\mathbb{Z}_2 = \{1, -1\}$ in multiplicative notation.) That is:

$$\langle \cdots U_{-1}(\Sigma)\rangle = Z^{-1} \sum_{\{\sigma\}} (\cdots)\, e^{-S}|_{J_\ell \to -J_\ell \text{ if } \Sigma \text{ crosses } \ell} \qquad (5.2)$$



where $\cdots$ is any collection of operators made of the spins. If $\Sigma' - \Sigma = \partial R$, $U_{-1}(\Sigma)$ and $U_{-1}(\Sigma')$ are related by redefining $\sigma_x \to -\sigma_x$ for $x \in R$, so U is topological. This implies that σ_x is charged:

$$\langle \cdots \sigma_x U_{-1}(\Sigma)\rangle = -\langle \cdots \sigma_x U_{-1}(\Sigma')\rangle \qquad \text{if } x \in R. \qquad (5.3)$$

[37] Throughout I will assume that the normalization is such that $Q \in \mathbb{Z}$, so that $\alpha \equiv \alpha + 2\pi$.

The interesting realization of recent years is that it's useful to reverse our perspective on Noether's theorem. Topological defect operators are a sufficient condition for symmetry, and for most of our uses of symmetry we don't really care about the action on the degrees of freedom. This allows us to treat continuous and discrete symmetries on equal footing. Further, in your studies of field theory you may have noticed an awkward asymmetry between quantities that are conserved by Noether's theorem and quantities that are conserved because of topology of field space (like soliton numbers). This is awkward because we know that this distinction is not invariant under dualities, which can exchange field quanta and solitons. The new perspective is clearly better because it treats Noether symmetries and topological symmetries on equal footing. And most importantly it allows generalizations.

So from the old-fashioned point of view, a symmetry implies a collection of operators $\{U_g\}$ with the following properties

1. $[H, U_g] = 0$.
2. U_g respects locality: if $\mathcal{O}(x)$ is a local operator, so is $U_g \mathcal{O}(x) U_g^\dagger$.
3. U_g is supported on a whole constant-time slice.
4. U_g are fully topological.
5. $\{U_g\}$ form a group $U_{g_1} U_{g_2} = U_{g_1 g_2}$.

Of these properties, the only ones we really can't give up are the first and second[38]. Let's start by giving up the third condition.

5.2 Higher-form symmetry

The concept of higher-form symmetry that we review here was explained here. It is easiest to introduce using a relativistic notation. Indices μ, ν run over space and time.

Now let us consider a continuous 1-form symmetry. This means that there is a conserved current that has two indices, and is completely antisymmetric:

$$J_{\mu\nu} = -J_{\nu\mu} \text{ with } \partial^\mu J_{\mu\nu} = 0. \tag{5.4}$$

We can regard J as a 2-form and write the conservation law (5.4) as $d \star J = 0$. As a consequence, for any closed codimension-*two* locus in spacetime Σ_{D-2}, the quantity

[38] I especially require the second condition because it rules out projectors onto eigenstates of H.

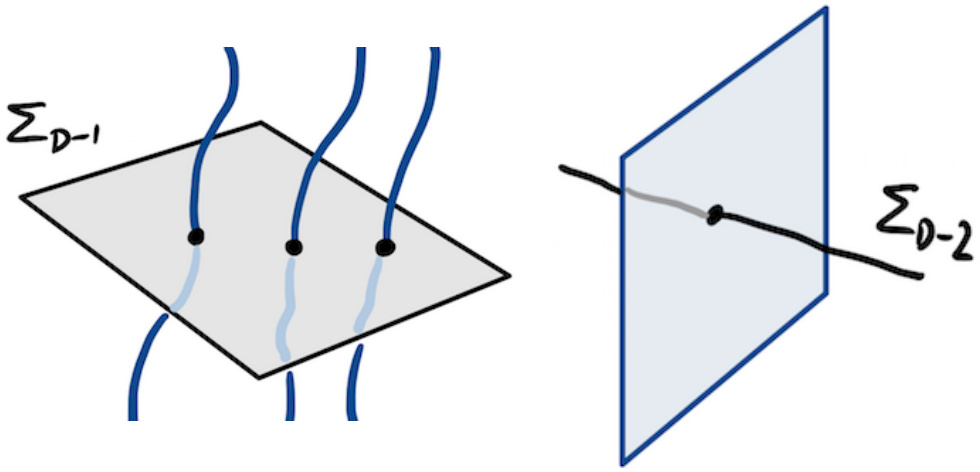


Figure 9: Left: In the case of an ordinary 0-form symmetry, the charge is integrated over a codimension-one slice of spacetime Σ_{D-1}, often a slice of constant time. All the particle worldlines (blue curves) must pass through this hypersurface. Right: The charge of a 1-form symmetry is integrated over a codimension-two locus of spacetime Σ_{D-2} (a string in the case of $D = 2+1$). This surface intersects the worldsheets of strings (blue sheet).

$Q_\Sigma = \int_{\Sigma_{D-2}} \star J$ depends only on the topological class of Σ. The analog of the symmetry operator is the unitary operator

$$U_\alpha(\Sigma) = e^{\mathbf{i}\alpha Q_\Sigma}. \tag{5.5}$$

Notice that reversing the orientation of Σ produces the adjoint of U: $U_\alpha(-\Sigma) = U_\alpha^\dagger(\Sigma)$.

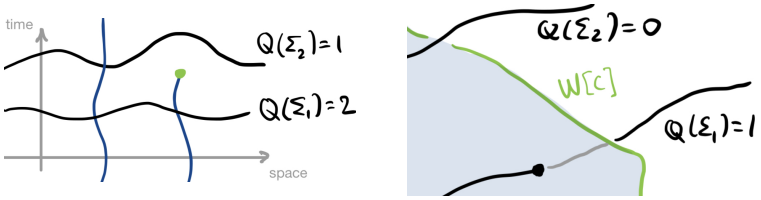
The charge Q_Σ in the 1-form case counts the number of charged string worldsheets intersecting the surface Σ (as in Fig. 9, right). The conservation law (5.4) then says that charged string worldsheets cannot end except on charged operators. The objects charged under a 1-form symmetry are loop operators, $W(C)$. Fixing a constant-time slice M_{D-1}, such a loop operator transforms as

$$W(C) \to U_\alpha(\Sigma) W(C) U_\alpha^\dagger(\Sigma) = e^{\mathbf{i}\alpha \oint_C \Gamma_\Sigma} W(C), \qquad d\Gamma_\Sigma = 0. \tag{5.6}$$

Here $\Sigma_{D-2} \subset M_{D-1}$ is any closed $(D-2)$-manifold, and Γ_Σ is its Poincaré dual in M_{D-1}, in the sense that $\int_{M_{D-1}} \eta^{(D-2)} \wedge \Gamma_\Sigma = \int_{\Sigma_{D-2}} \eta^{(D-2)}$ for all η; $d\Gamma_\Sigma = 0$ because Σ has no boundary. The infinitesimal version of this transformation law is

$$\delta W(C) = \mathbf{i}[Q_\Sigma, W(C)] = \mathbf{i} q \#(\Sigma, C)\, W(C), \tag{5.7}$$

where $\#(\Sigma, C)$ is the intersection number in M.

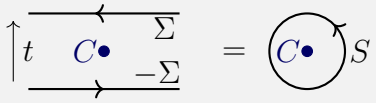


In the case of a discrete 1-form symmetry, there is no current, but the symmetry operator $U_\alpha(\Sigma)$ is still topological. If the 1-form symmetry group is $\mathbb{Z}_p$, strings can disappear or end only in groups of p.

For general integer $p \geq -1$, a p-form symmetry means the existence of topological operators $U_\alpha(\Sigma_{D-p-1})$ labelled by a group element α and a closed codimension-$(p+1)$ submanifold of spacetime[39]. For coincident submanifolds, these operators satisfy the "fusion rule" $U_\alpha(\Sigma)U_\beta(\Sigma) = U_{\alpha+\beta}(\Sigma)$. The operators charged under a p-form symmetry are supported on p-dimensional loci, and create p-brane excitations. The conservation law asserts that the $(p+1)$-dimensional worldvolume of these excitations will not have boundaries.

For $p \geq 1$, the symmetry operators commute with each other – higher-form symmetries are abelian. To see this, consider a path integral representation of an expectation value with two symmetry operators $U(\Sigma_1)U(\Sigma_2)$ inserted on the same time slice t. The ordering of the operators can be specified in the path integral by shifting the left one to a slightly later time $t+\epsilon$. If $p \geq 1$, then $\Sigma_{1,2}$ have codimension larger than one, and their locations can be continuously deformed to reverse their order.

Action of higher form symmetry operators in Hamiltonian description. The relation between the spacetime point of view on higher-form symmetries and the Hamiltonian point of view common in the condensed matter literature can be confusing. Above I have written the expression for the transformation as $U(\Sigma)W(C)U^\dagger(\Sigma)$. This operator ordering is obtained by placing the support of these operators on successive time slices. Since U is topological, from a spacetime point of view, the same result obtains if instead we deform the surfaces Σ and $-\Sigma$ to a single surface S in spacetime that *surrounds* the locus C, as illustrated here in cross-section:



(5.8)

The variation of the operator then depends on the *linking number* of S and C in spacetime.

Here is a recipe for thinking about this:

Choose a constant-time slice $\mathcal{M}_{D-1}$. For each $\Sigma_{D-p-1} \subset \mathcal{M}_{D-1}$,

$$W(C_p) \mapsto U_\alpha(\Sigma_{D-p-1})W(C_p)U_\alpha^\dagger(\Sigma_{D-p-1}) = e^{\mathbf{i}\alpha q \oint_{C_p} \delta_\Sigma} W(C_p) \tag{5.9}$$

where δ_Σ is the Poincaré dual of Σ_{D-p-1} in $\mathcal{M}_{D-1}$: $\int_{\mathcal{M}_{D-1}} \eta^{D-p-1} \wedge \delta_\Sigma = \int_{\Sigma_{D-p-1}} \eta^{D-p-1}$, $\forall \eta_{D-p-1}$. ($d\delta_\Sigma = 0$ since $\partial\Sigma = 0$.)

[39] $p = -1$ is a bit funny. You can read about it here.

- $U_\alpha(-\Sigma) = U_{-\alpha}(\Sigma) = U_\alpha^\dagger(\Sigma)$.
- Infinitesimal version: $\delta W(C) = \mathbf{i}[Q_\Sigma, W(C)] = \mathbf{i}q\#_{\mathcal{M}_{D-1}}(\Sigma, C)W(C) = \mathbf{i}q\ell_{\mathcal{M}_{D-1}}(S, C)W(C)$
- If we assume Lorentz symmetry:

$$\mathcal{O}(C) \to U_\alpha(\Sigma)\mathcal{O}(C)U_\alpha^\dagger(\Sigma) = U_\alpha(S)\mathcal{O}(C) \tag{5.10}$$

in the euclidean path integral.

Physics examples of higher-form symmetries. A higher-form symmetry can be exact:

- Maxwell theory in $D = 3 + 1$ with electric charges but no magnetic charges has a continuous 1-form symmetry with current $J^{\mu\nu}_{(m)} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \equiv \frac{1}{2\pi}(d\tilde{A})^{\mu\nu}$. The statement that this current is conserved $\nabla_\mu J^{\mu\nu}_{(m)} = 0$ is the Bianchi identity expressing the absence of magnetic charge. The symmetry operator is $U_\alpha^{(m)}(\Sigma) = e^{\frac{\mathbf{i}\alpha}{2\pi}\int_\Sigma F}$. The fact that the charge operator $\int_\Sigma F$ depends only on the topological class of Σ is the magnetic gauss law – when Σ is contractible, it counts the number of magnetic monopoles inside. This symmetry shifts the dual gauge field $\tilde{A}$ by a flat connection; the charged line operator is the 't Hooft line, $W^{(m)}[C] = e^{\mathbf{i}\oint_C \tilde{A}}$.

  In free Maxwell theory without electric charges, there is a second 1-form current, $J_{(e)} = F$ whose charged operator is the Wegner-Wilson line $W^{(e)}[C] = e^{\mathbf{i}\oint_C A}$. The symmetry operator for this 'electric' 1-form symmetry is $U_\alpha^{(e)}(\Sigma_2) = e^{\mathbf{i}\frac{2\alpha}{g^2}\int_{\Sigma_2}\star F}$, which (by canonical commutators) shifts the gauge field A by a flat connection.

  As we'll discuss below, this symmetry survives the presence of charged matter as long as there is no particle production, as for example in the heavy-quark EFT in (4.84).

- Pure $\mathsf{SU}(N)$ gauge theory or $\mathbb{Z}_N$ gauge theory or $\mathsf{U}(1)$ gauge theory with charge-N matter has a $\mathbb{Z}_N$ 1-form symmetry, called the 'center symmetry'. The charged line operator is the Wegner-Wilson line in the minimal irrep, $W[C] = \mathrm{tr}Pe^{\mathbf{i}\oint_C A}$.

... or it can be emergent.

- When we spontaneously break a 0-form U(1) symmetry in $d = 2$, there is an emergent 1-form U(1) symmetry whose charge counts the winding number of the phase variable φ around an arbitrary closed loop C, $Q[C] = \oint_C \text{đ}\varphi$. It is conserved because $d(d\varphi) = 0$ if φ is single-valued. In d spatial dimensions, this produces a $(d-1)$-form symmetry. The charged operator creates a vortex (in $d = 2$, or

a vortex line or sheet in $d > 2$), which makes a codimension-two locus where φ is not single-valued, so $d^2\varphi \neq 0$. Unlike the examples above, this symmetry is generally not an exact symmetry of a microscopic Hamiltonian for a superfluid; it is explicitly broken by the presence of vortex configurations.

An analog of this 'dual symmetry' emerges any time we spontaneously break any symmetry.

- Spin liquids, FQHE.

[End of Lecture 19]

5.3 Spontaneous symmetry breaking

Anything we can do with ordinary (0-form) symmetries, we can do with higher-form symmetries. In particular, they can be spontaneously broken. A symmetry is spontaneously broken if a ground state (more generally an equilibrium state) is not invariant.

SSB $\Leftrightarrow$ LRO.
There exists a charged operator O with $\langle\psi| O |\psi\rangle \neq 0$ (long-range order) if and only if $|\psi\rangle$ is not invariant under the symmetry (SSB).
Proof:
$\boxed{\Rightarrow}$ We'll prove the contrapositive (the state is symmetric means that no charged operator has an expectation value). Assume by way of contradiction that the state is a stationary state of the symmetry operator, $S|\psi\rangle = e^{i\alpha}|\psi\rangle$. Then for any charged operator $SO = e^{i\gamma}OS$, $\gamma \notin 2\pi\mathbb{Z}$, *i.e.* $O = S^\dagger O S e^{i\gamma}$,

$$\langle\psi| O |\psi\rangle = e^{i\gamma}\langle\psi| S^\dagger O S |\psi\rangle = e^{i\gamma}\langle\psi| O |\psi\rangle \tag{5.11}$$

which says $\langle\psi| O |\psi\rangle = 0$.
$\boxed{\Leftarrow}$ [a] For any region X, we can write its reduced density matrix as

$$\rho_X = \mathrm{tr}_{\bar{X}}|\psi\rangle\langle\psi| = \sum_I \langle O_I\rangle O_I \tag{5.12}$$

where $\bar{X}$ is the complement of X and $\{O_I\}$ is a basis of hermitian operators on X orthonormal under the Hilbert-Schmidt norm $\mathrm{tr}O_I O_J = \delta_{IJ}$. If no charged operator has a nonzero expectation value, then the sum only contains neutral operators. But then $S\rho_X S^\dagger = \rho_X$, meaning that the state is invariant.

[a] This argument was explained to me by Tarun Grover.

A comment about the preceding result: This argument makes no assumptions about the support of the symmetry operators. The argument for LRO $\implies$ SSB also says nothing about the support of the charged operators – they actually needn't be operators of the correct dimension indicated above[40].

To appreciate the consequences of SSB for higher-form symmetries, let's spend a little time reviewing the story for 0-form symmetries. One way to characterize the unbroken phase of a 0-form symmetry is that correlations of charged operators are short-ranged, meaning that they decay exponentially with the separation between the operators

$$\left\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \right\rangle \overset{x\to\infty}{\sim} e^{-m|x|}. \tag{5.13}$$

A language that will generalize is to regard the two points at which we insert a charged operator and its conjugate as an S^0, a zero-dimensional sphere, and the separation between the points as the size of the sphere. The broken phase for 0-form symmetry can be diagnosed by long-range correlations:

$$\left\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \right\rangle \overset{x\to\infty}{\sim} \left\langle \mathcal{O}^\dagger(x) \right\rangle \left\langle \mathcal{O}(0) \right\rangle + \cdots , \tag{5.14}$$

independent of the size of the S^0.

For a p-form symmetry, the unbroken phase is also when correlations of charged operators are short-ranged, and decay when the charged object grows. For a 1-form symmetry, this is when the charged loop operator exhibits an area law:

$$\langle W(C) \rangle \sim e^{-T_{p+1}\mathrm{Area}(C)}, \tag{5.15}$$

where $\mathrm{Area}(C)$ is the area of the minimal surface bounded by the curve C. In the case of electricity and magnetism, an area law for $\left\langle W^E(C) \right\rangle$ is the superconducting phase.

The broken phase for a p-form symmetry is signalled by a failure of the expectation value of the charged operator to decay with size. For a 1-form symmetry, this is when the charged loop operator exhibits a perimeter law:

$$\langle W(C) \rangle = e^{-T_p \mathrm{Perimeter}(C)} + \cdots . \tag{5.16}$$

The coefficient T_p can be set to 0 by modifying the definition of $W(C)$ by counterterms local to C (for $p = 1$, this is mass renormalization of the probe particle), so (5.16) says that a large loop has an expectation value.

SSB of higher-form symmetry has been a fruitful idea. The fact that charged operators have long-range correlations means that the generators of the symmetry act nontrivially on the groundstate – the argument in the box above was not special to 0-form symmetry. In the next two subsections, I'll illustrate the consequences in the case of continuous and discrete higher-form symmetries, respectively.

[40] Thanks to Sal Pace for raising this question.

5.4 Photon as Goldstone boson

What protects the masslessness of the photon? The case of quantum electrodynamics (QED) is the most visible version of this question; the same question arises in condensed matter as: why are there $\mathsf{U}(1)$ spin liquid phases, with an emergent photon mode?

Higher-form symmetries provide a satisfying answer to this question (unlike appeals to gauge invariance, which is an artifact of a particular description): the gaplessness of the photon can be understood as required by spontaneously-broken $\mathsf{U}(1)$ 1-form symmetry, as a generalization of the Goldstone phenomenon.

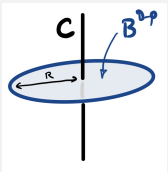
Here is the proof of the p-form Goldstone theorem from here. A continuous p-form symmetry implies a $p+1$-form current J such that

$$U_\alpha(\Sigma_{D-p-1}) = e^{\mathbf{i}\alpha \int_{\Sigma_{D-p-1}} \star J}. \tag{5.17}$$

A covariant form of the statement that the $p+1$-form current J is conserved and that $W[C]$ is a charged operator is the Ward identity

$$(d \star J(x))W[C] = \mathbf{i}q\delta_C(x)W[C] \tag{5.18}$$

where the $D-p$-form delta function (Poincaré dual) $\delta_C(x)$ satisfies $\int b_p \wedge \delta_C(x) = \int_C b_p$ for any p-form b_p, and J is the $p+1$-form current. Let's consider the broken phase and choose $W[C]$ to have the multiplicative normalization where $\langle W[C]\rangle = c$, so no perimeter law.
Take C to be an infinite flat p-plane and integrate the BHS of (5.18) with respect to x over a $(D-p)$-ball B^{D-p} of radius R that intersects C at a single point:



$$W[C] \int_{\partial B^{D-p}} \star J = \mathbf{i}qW[C]\ . \tag{5.19}$$

The boundary of the ball ∂B^{D-p} is a $D-p-1$ sphere linked with C in spacetime, a Gaussian surface. Taking expectation values of the BHS we have

$$\langle J(R)W[C]\rangle \sim \frac{\mathbf{i}qc}{R^{D-p-1}}, \tag{5.20}$$

a power-law correlation, implying the presence of a gapless mode.

Here is a perspective on the zero-form version of the Goldstone theorem. Given a continuous zero-form symmetry with current j_μ, we can couple to a background gauge

field $\mathcal{A}$ by adding to the Lagrangian $\Delta L \ni j_\mu \mathcal{A}^\mu$. If the symmetry is spontaneously broken, the effective Lagrangian will contain a Meissner term proportional to $\mathcal{A}^2$. But the effective action must be gauge invariant, and this requires the presence of a field that transforms nonlinearly under the $\mathsf{U}(1)$ symmetry: $\varphi \to \varphi + \lambda, \mathcal{A} \to \mathcal{A} - d\lambda$; this is a global symmetry if $d\lambda = 0$. Altogether, the effective Lagrangian must contain a term of the form

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4\pi g} (d\varphi + \mathcal{A})^2 \tag{5.21}$$

(where by $(\omega)^2$ I mean $\omega_p \wedge \star\omega_p = \frac{1}{p!}\omega_{\mu_1 \cdots \mu_p}\omega^{\mu_1 \cdots \mu_p}$). The coefficient $\frac{1}{4\pi g}$ is the superfluid stiffness. Any L that is a function only of $\partial_\mu \varphi$ has this symmetry.

The analog for a continuous 1-form symmetry works as follows. The current is now a two-form, so the background field must be a two-form gauge field $\mathcal{B}_{\mu\nu}$ and the coupling is $\Delta L \ni J_{\mu\nu}\mathcal{B}^{\mu\nu}$. The same logic implies that the effective action for the broken phase must contain a term

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2g^2} (da + \mathcal{B})^2 \tag{5.22}$$

where the Goldstone mode a is a 1-form that transforms nonlinearly $a \to a + \lambda, \mathcal{B} \to \mathcal{B} - d\lambda$; this is a global symmetry if $d\lambda = 0$. Setting the background field $\mathcal{B} = 0$, we recognize this as a Maxwell term for a. The coupling strength g is determined by the analog of the superfluid stiffness. Any L that is a function only of $F_{\mu\nu}$ has this symmetry.

For p-form $\mathsf{U}(1)$ symmetry, we conclude by the same logic that there is a massless p-form field a with canonical kinetic term

$$S_{\text{Max}}[a] = -\frac{1}{2g^2} \int da \wedge \star da. \tag{5.23}$$

Returning to QED, we see that the familiar Coulomb phase is the SSB phase for the $\mathsf{U}(1)$ 1-form symmetry. The unbroken phase is the superconducting phase, where the photon has short-ranged correlations. (In an ordinary superconductor, where the Cooper pair has charge two, a $\mathbb{Z}_2$ subgroup of the 1-form symmetry remains broken.)

As in the case of 0-form SSB, the broken phase can be understood via the condensation of charged objects; in this case the charged objects are the strings of electric flux. Notice that the presence of charged matter, on which these strings can end, and which therefore explicitly breaks this symmetry, does not necessarily destroy the phase. We'll comment on this robustness more in §5.7. In fact, because of electromagnetic duality, the Coulomb phase is the broken phase for *either* the electric 1-form symmetry or the magnetic 1-form symmetry.

Soft theorems from higher-form symmetry. [2505.03566]
Let's return to the effective theory of heavy quarks, (4.84), where the covariant derivatives are with respect to the photon field:

$$\mathcal{L}_{\text{eff}} = \mathbf{i}\varphi_v^\star v\cdot D\varphi_v + \varphi_v^\star \frac{D_\perp^2}{2M}\varphi_v + \alpha\varphi_v^\star \frac{F_{\mu\nu}\sigma^{\mu\nu}}{4M}\varphi_v + \cdots \tag{5.24}$$

Here $\varphi_v^\star$ could be Q_v^+ or a non-relativistic bosonic field with charge Q under an abelian gauge group with gauge coupling g. I claim that this theory has a one-form symmetry, $A_\mu \to A_\mu + \lambda_\mu$, $d\lambda = 0$. The basic idea is that the one-form symmetry in QED is explicitly broken by the possibility of pair production – the electric flux lines can break by ending on charged particles. But in this theory, we have integrated out the antiparticles, and so there is no pair production and so flux strings cannot break. To make this explicit, consider the change of variables

$$\varphi_v \equiv W_Q[C]\tilde{\varphi}_v \equiv e^{\mathbf{i}Q\int_0^\infty ds v^\mu A_\mu(sv)}\tilde{\varphi}_v. \tag{5.25}$$

I claim that under this change of variables,

$$\mathcal{L}_{\text{eff}} = \mathbf{i}\tilde{\varphi}_v^\star v\cdot D\tilde{\varphi}_v + \tilde{\varphi}_v^\star \frac{1}{2M}\left(\partial_\perp + \frac{1}{v\cdot\partial}F_{\mu\nu}v^\mu\right)^2\varphi_v + \alpha\tilde{\varphi}_v^\star\frac{F_{\mu\nu}\sigma^{\mu\nu}}{4M}\tilde{\varphi}_v + \cdots \tag{5.26}$$

where we used the identity

$$D_\mu\varphi_v = W_Q(C)\left(\partial_\mu + \int_0^\infty ds F_{\mu\nu}(x+sv)v^\nu\right)\tilde{\varphi}_v = W_Q(C)\left(\partial_\mu + \frac{1}{v\cdot\partial}F_{\mu\nu}v^\nu\right)\tilde{\varphi}_v.$$

The important point about (5.26) is that it only depends on $F_{\mu\nu}$ and not directly on A_μ, and therefore it manifestly has the one-form symmetry. This is true to all orders in the derivative expansion in k/M.
A consequence of this symmetry is the soft photon theorem for the amplitude to scatter a photon of (soft) momentum q^μ and polarization ϵ^μ off of n particles with charge Q_i:

$$\mathcal{A}_{n+\gamma(q)} \overset{q\to 0}{\simeq} g\sum_i Q_i\frac{\epsilon\cdot p_i}{q\cdot p_i}\mathcal{A}_n \tag{5.27}$$

where $\mathcal{A}_n$ is the amplitude without the photon. A special case is if there are no charged particles, in which case the RHS is zero.
As a warmup let's prove the 0-form analog which is called the pion soft theorem or 'Adler zero'. The pion interpolation relation

$$\left\langle \pi^b(p)\right| J_\mu^a(x)\left|0\right\rangle = \mathbf{i}f_\pi p_\mu\delta^{ab}e^{\mathbf{i}px} \tag{5.28}$$

implies that the axial current has the form $J^a_\mu(x) = f_\pi \partial_\mu \pi^a(x) + \mathcal{O}(\pi^3)$, which means that the pion shifts under the (broken) symmetry: $\pi^a \to \pi^a + c^a$, with $dc^a = 0$ (*i.e.*c^a is a constant). The charged operator is $U(x) = e^{\frac{2\mathbf{i}\pi^a t^a}{f_\pi}}$.
Let $|\alpha\rangle$ be a state of n pions. Then

$$\langle\alpha| J^a_\mu(q) |0\rangle = -\frac{f_\pi q_\mu}{q^2} \underbrace{\langle \alpha + \pi^a(q)|0\rangle}_{=\mathbf{i}\mathcal{A}_{n+\pi(q)}} + \langle\alpha| J^a_{H\mu} * (q) |0\rangle \tag{5.29}$$

where J_H is the 'hard' part of the current that does things other than make a pion. Current conservation implies the Ward identity[a] $\partial_\mu J^a_\mu(x) = \sum_i Q_i \delta(x - x_i)$, where x_i are the positions of charged operators in the correlation function. But there are no other charged operators in this correlation function, so

$$\mathcal{A}_{n+\pi(q)} = -\mathbf{i}\frac{1}{f_\pi} q^\mu \langle\alpha| J^a_{H\mu}(q) |0\rangle \overset{q\to 0}{\simeq} 0 + \mathcal{O}(q) \tag{5.32}$$

where the crucial input about the hard part is that it is regular at $q \to 0$.
Now let's think about SSB of a $\mathsf{U}(1)$ one-form symmetry with symmetry current $J^{\mu\nu}$. The analog of the pion interpolation relation is

$$\langle\gamma_\epsilon(p)| J^{\mu\nu}(x) |0\rangle = \frac{\mathbf{i}}{g} \left(p^\mu \epsilon^{\nu\star} - p^\nu \epsilon^{\mu\star}\right) e^{\mathbf{i}px} \tag{5.33}$$

where the LHS is a one-photon state with polarization ϵ and momentum p, and the gauge coupling g is the analog of the pion decay constant. This implies that in terms of the photon field the current takes the form $J^{\mu\nu}(x) = \frac{1}{g^2}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \cdots$, which means that $A \to A + \lambda$, $d\lambda = 0$ under the symmetry. A charged operator is made out of the photon field by

$$W_Q(C) = e^{\mathbf{i}Q\int_C A} = e^{\mathbf{i}Q\int_C A_\mu dx^\mu}. \tag{5.34}$$

As an extreme simple case, let's think about the Euler-Heisenberg EFT of just photons that results from integrating out the electron. In this theory since there's just photons, $|\alpha\rangle$ is a state of n photons, and

$$\langle\alpha| J^{\mu\nu}(q) |0\rangle = \frac{1}{g} \sum_{\epsilon_{1,2}} \frac{1}{q^2} \left(q^\mu \epsilon^{\nu\star} - q^\nu \epsilon^{\mu\star}\right) \underbrace{\langle \alpha + \gamma_\epsilon(q)|0\rangle}_{=\mathbf{i}\mathcal{A}_{n+\gamma_\epsilon(q)}} + \langle\alpha| J^{\mu\nu}_H |0\rangle . \tag{5.35}$$

Again the Ward identity says $\partial_\mu J^{\mu\nu} = \sum \delta\,(\text{charged operators})$ but here there are no charged operators around, so

$$\mathcal{A}_{n+\gamma_\epsilon(q)} = -\mathbf{i}g q_\mu \epsilon_\nu \langle\alpha| J^{\mu\nu}_H(q) |0\rangle \overset{q\to 0}{\to} 0 \tag{5.36}$$

since the hard part can't be too singular in q.

Now let's try to do with with charges. Using the change of variables (5.25), a correlation function of charged operators can be written (to leading order in the $1/M$ expansion) essentially as a product of Wilson lines:

$$\mathbf{i}\mathcal{A}_n = \left\langle \prod_{i=1}^{n} W_{Q_i}(C_i)\mathcal{O}_H \right\rangle + \mathcal{O}\left(\frac{1}{M}\right) \tag{5.37}$$

where $C_i = \{y_i^\mu(s) = sv_i^\mu \equiv s\frac{p_i^\mu}{M_i}, s \in [0,\infty)\}$, and $\mathcal{O}_H$ is a 'hard operator' at $x = 0$. For this particular correlation function, a more explicit form of the one-form symmetry Ward identity is

$$\partial_\mu \left\langle J^{\mu\nu}(x) \prod_i W_{Q_i}(C_i)\mathcal{O}_H \right\rangle = \left(\sum_i Q_i \int_0^\infty ds \frac{dy_i^\nu}{ds} \delta^{(4)}(x - y_i(s)) \right) \left\langle \prod_{i=1}^{n} W_{Q_i}(C_i)\mathcal{O}_H \right\rangle . \tag{5.38}$$

Now multiply the BHS by $\epsilon^\nu(q)$ and Fourier transform. Using (5.33), the LHS is

$$\epsilon_\nu \int d^4x e^{-\mathbf{i}qx} \partial_\mu \left\langle J^{\mu\nu} \prod_i W_i \mathcal{O}_H \right\rangle = -\frac{\mathbf{i}}{g} \underbrace{\langle \gamma_\epsilon(q)| \prod_i W_{Q_i}(C_i)\mathcal{O}_H |0\rangle}_{=\mathbf{i}\mathcal{A}_{n+\gamma_\epsilon(q)}} + \mathbf{i}q_\mu \epsilon_\nu \langle 0| J_H^{\mu\nu} \prod_i W_i \mathcal{O}_H |0\rangle . \tag{5.39}$$

The RHS involves

$$\int d^4x e^{-\mathbf{i}qx} Q_i \int_0^\infty ds \epsilon \cdot v \delta^4(x - vs) = Q_i \epsilon \cdot v_i \int_0^\infty ds e^{\mathbf{i}sq\cdot v} = \mathbf{i}Q_i \frac{\epsilon \cdot v_i}{q \cdot v_i}. \tag{5.40}$$

Putting these together, we find

$$\mathcal{A}_{n+\gamma_\epsilon(q)} = g \sum_i Q_i \frac{\epsilon \cdot p_i}{v \cdot p_i} \mathcal{A}_n + \mathbf{i}g q_\mu \epsilon_\nu \langle 0| J_H^{\mu\nu}(q) \prod_i W_{Q_i}(C_i)\mathcal{O}_H |0\rangle \tag{5.41}$$

where the crucial point again is that the hard part of the current can't be too singular at $q \to 0$.

---

[a] Let's remember the proof of this statement. Consider an arbitrary time-ordered Green's function

$$G \equiv \langle \Omega | \mathcal{T} \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | \Omega \rangle = \int D\Psi e^{\mathbf{i}S} \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)$$

where the operators $\mathcal{O}_1(x_1) \mapsto e^{-\mathbf{i}Q_i\alpha}\mathcal{O}_1(x_1)$ have charge Q_i under a global $\mathsf{U}(1)$ symmetry. For example the $\mathcal{O}(x)$ could be just the elementary field $\Psi(x)$

Now change variables in the path integral so that $\mathcal{O}_i(x_i) \mapsto e^{-\mathbf{i}Q_i\alpha(x_i)}\mathcal{O}_i(x_i)$; the action will shift by $S \mapsto S - \int \alpha \partial_\mu j^\mu$ where j^μ is the Noether current. The path integral doesn't change

at all, so its infinitesimal variation is

$$0 = \delta G = \int D\Psi \left(- \int \mathbf{i}\alpha \partial^\mu j_\mu e^{\mathbf{i}S} \mathcal{O}_1 \cdots \mathcal{O}_n - \mathbf{i} \sum_i Q_i \alpha(x_i) e^{\mathbf{i}S} \mathcal{O}_1 \cdots \mathcal{O}_n \right) \tag{5.30}$$

$$= \int d^d x \alpha(x) \left[\mathbf{i}\partial_\mu \langle j^\mu(x) \mathcal{O}_1 \cdots \mathcal{O}_n \rangle - \sum_i Q_i \delta(x - x_i) G \right]. \tag{5.31}$$

Since this is true for any $\alpha(x)$, we learn that the thing in square brackets is zero: $\partial_\mu j^\mu = 0$ up to contact terms. This is called the Ward-Takahashi identity or Ward identity.

5.5 Topological order as SSB

Suppose we spontaneously break a discrete higher-form symmetry. The generators of the broken part of the higher-form symmetry commute with the Hamiltonian and take a groundstate to a different groundstate. These groundstates are therefore related to each other by the action of an extended operator, rather than by a local operator. But this is precisely a definition of topological order: the presence of a groundstate subspace of *locally indistinguishable* states.

Let's think about the example of $\mathbb{Z}_p$ gauge theory (whose solvable limit is the toric code) in D spacetime dimensions. This is a system with $\mathbb{Z}_p$ 1-form symmetry with symmetry operators $U(M_{D-2})$, supported on a $(D-2)$-dimensional manifold, and charged operators $V(C_1)$, supported on a curve. In terms of the toric code variables, we can be completely explicit. On each link we have a p-state system on which act the Pauli operators $Z = \sum_{k=0}^p \omega^k |k\rangle\langle k|$ and $X = \sum_{k=0}^p |k+1\rangle\langle k|$ (where $\omega \equiv e^{2\pi \mathbf{i}/p}$ and the arguments of the kets are understood mod p). Then $V(C) = \prod_{\ell \in C} Z_\ell$, $U(M) = \prod_{\ell \perp M} X_\ell$, where we regard M as a surface in the dual lattice, and $\ell \perp M$ indicates all links crossed by the surface M. Their algebra is

$$U^m(M) V^n(C) = e^{2\pi \mathbf{i} \frac{mn}{p} \#(C,M)} V^n(C) U^m(M) \tag{5.42}$$

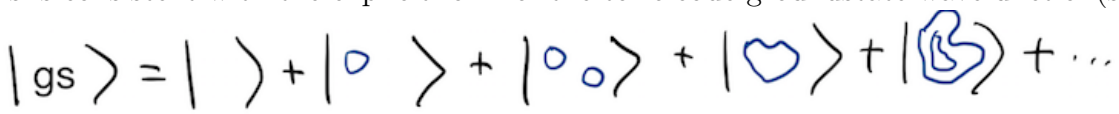
where $\#(C, M)$ is the intersection number of the curve C with the surface M. This is the algebra of electric strings and magnetic flux surfaces in $\mathbb{Z}_p$ gauge theory. Deep in this gapped phase, $H = 0$, and there is a description in terms of topological field theory. A simple realization is BF theory of a 1-form potential a and $(D-2)$-form potential b with action

$$S[b, a] = \frac{p}{2\pi} \int_D b \wedge da = \frac{p}{2\pi} \int d^D x \epsilon^{\mu_1 \cdots \mu_D} b_{\mu_1 \cdots \mu_{D-2}} \partial_{\mu_{D-1}} a_{\mu_D} \tag{5.43}$$

in terms of which

$$U^n(M) = e^{\mathbf{i}n \oint_M b_{D-2}}, \quad V^m(C) = e^{\mathbf{i}m \oint_C a}. \tag{5.44}$$

The algebra (5.42) follows from canonical commutation relations in this gaussian theory. Since $V(C)$ has a perimeter law in the deconfined phase, the charged objects whose condensation breaks the 1-form symmetry are the loops of electric flux. (Recall that an excitation is condensed if the operator that creates it has an expectation value.) This is consistent with the explicit form of the toric code groundstate wavefunction(s)



Another example is the Laughlin fractional quantum Hall states. So far in our discussion the symmetry operators for a 1-form symmetry with group A form a representation of A on the 1-cycles of space, Z, *i.e.* a linear map $U : Z \to U(1)$, where the representation operators commute $U(M)U(M') = U(M')U(M)$. This relation can be generalized to allow for phases – *i.e.* a projective representation. Consider a system in $D = 2 + 1$ with a $\mathbb{Z}_k$ 1-form symmetry that is realized projectively in the following sense:

$$U^m(C)U^n(C') = e^{\frac{2\pi \mathbf{i} mn \#(C,C')}{k}} U^n(C')U^m(C) \tag{5.45}$$

where $\#(C, C')$ is the intersection number of the two curves C, C' in space. Regarding $U(C)$ as the holonomy of a charged particle along the loop C, this is the statement that flux carries charge. Representing this algebra nontrivially gives k groundstates on T^2. This algebra, too, has a simple realization via abelian Chern-Simons theory, $S[a] = \frac{k}{4\pi} \int a \wedge da$, with $U^m(C) = e^{\mathbf{i}m \oint_C a}$.

The algebra in (5.45) is a further generalization of 1-form symmetry, in that the group law is only satisfied up to a phase. This is an example of a 1-form symmetry anomaly.

The preceding discussion applies to *abelian* topological orders. In this context, abelian means that the algebra of the line operators transporting the anyons forms a group, which must be abelian by the argument above. In §5.9 we discuss the further generalization that incorporates non-abelian topological orders.

Loophole. It is not true that SSB of higher-form symmetry implies topological order. In fact an additional assumption is required, namely that the higher-form symmetry is *anomalous*. This is visible in both the examples above where we see nontrivial commutation relations between the symmetry generators. To see that this is a real loophole, I refer you to [1] or the appendix of [2].

5.6 Effects of IR fluctuations

The analog of the Hohenberg-Coleman-Mermin-Wagner (HCMW) theorem for higher-form symmetries [3, 4, 5, 6] is interesting. As in the proof of the HCMW theorem, we suppose that a p-form $\mathsf{U}(1)$ symmetry in D spacetime dimensions is spontaneously broken and that there is therefore a Goldstone mode, a massless p-form field a. Then we ask if indeed the symmetry is broken by evaluating the expectation value of a charged operator W_C, including the fluctuations of the would-be-Goldstone mode a. We can choose C to be a copy of $\mathbb{R}^p \subset \mathbb{R}^D$ so that we can do the integrals, and the result is (see [6] for a discussion of a convenient gauge choice)

$$\langle W_C \rangle = Z^{-1} \int [Da] e^{-S_{\rm Max}[a] + \mathbf{i} \int_C a} \simeq \exp\left(-\frac{1}{2} g^2 L^p \int \frac{\text{đ}^{D-p} k}{k_\perp^2} \right) \tag{5.46}$$

where $\text{đ}k \equiv \frac{dk}{2\pi}$ and $k_\perp$ is the momentum transverse to C. The integral in the exponent of (5.46) is IR divergent when $D - p \leq 2$. As in the $p = 0$ case, we interpret this as the statement that the long-wavelength fluctuations of the would-be-Goldstone mode necessarily destroy the order. (For $D - p \geq 2$, the integral is UV divergent. This divergence can be absorbed in a counterterm locally redefining the operator $W_C \to W_C e^{-\delta T \int_C d^p x}$, which can be interpreted as a renormalization of the tension T of the charged brane.) In the marginal case of $p = D - 2$, the long-range order is destroyed, but $\langle W_C \rangle$ decays as a power-law in the loop size, rather than an exponential; this is a higher-form analog of algebraic long-range order in $D = 2$.

The calculation above is independent of compactness properties of the Goldstone form field, in the sense that in (5.46) we just did a Gaussian integral over the topologically-trivial fluctuations of a. In the marginal case $D = 2+1, p = 1$, if we treat a as a compact $\mathsf{U}(1)$ gauge field, SSB of the 1-form symmetry is avoided instead because monopole instantons generate a potential for the dual photon $d\sigma = \star da/2\pi$ [7]. This mechanism generalizes to any case with $D - p = 2$ [6].

5.7 Robustness of higher-form symmetries

We are used to the idea that consequences of emergent (aka accidental) symmetries are only approximate: explicitly breaking a spontaneously-broken continuous 0-form symmetry gives a mass to the Goldstone boson.

This raises a natural question. The existence of magnetic monopoles with $m = M_{\rm monopole}$ explicitly breaks the 1-form symmetry of electrodynamics:

$$\partial^\mu J^E_{\mu\nu} = j^{\rm monopole}_\nu \ .$$

If the photon is a Goldstone for this symmetry, does this mean the photon gets a mass? Perhaps surprisingly, the answer is 'no' (early discussions of the robustness of broken higher-form symmetries using different words include [8, 9, 10]). This is a way in which zero-form and higher-form symmetries are quite distinct.

A cheap way to see that 'no' is the right answer is by dimensional analysis. How does the mass of the photon m_γ depend on the mass of the magnetic monopole, M_{monopole}? Suppose all the electrically charged matter (such as the electron) is very heavy or massless. We must have $m_\gamma \to 0$ when $M_{\text{monopole}} \to \infty$. But there is no other mass in the problem to make up the dimensions.

A slightly less cheap way to arrive at this answer is by dimensional reduction. If we put quantum electrodynamics (QED) on a circle of radius R, we arrive at low energies at abelian gauge theory in $D = 2{+}1$, which is confined by monopole instantons [7]. The monopole instantons arise from euclidean worldlines of magnetic monopoles wrapping the circle, and so their contribution to the mass of the (2+1)d photon is

$$m_\gamma(R) \sim e^{-RM_{\text{monopole}}}. \tag{5.47}$$

The polarization of the photon along the circle gets a mass from euclidean worldlines of charged matter (like the electron) wrapping the circle, so its mass is

$$m_4(R) \sim e^{-Rm_e}. \tag{5.48}$$

But now the point is simply that when $R \to \infty$, both of these effects go away and the (3+1)d photon is massless.

A third argument is that operators charged under a 1-form symmetry are loop operators – they are not local. We can't add non-local operators to the action at all. This argument is not entirely satisfying, since on the lattice even the action for pure gauge theory is a sum over (small) loop operators. The question is whether the dominant contributors in this ensemble of charged loop operators grow under the RG. [11] describes a toy calculation to address this question: begin in a phase with a perimeter law $\langle W[C]\rangle \sim t^{\text{length}[C]}$ and consider adding to the action $g\int [dC]W[C] + h.c.$ in perturbation theory in g. Regularizing on the lattice and neglecting collisions of loops, the result is the same as integrating out a charged particle whose mass is determined by the parameter t:

$$\left\langle \sum_C W[C] \right\rangle = -\frac{L^D}{2}\int \text{đ}^D q \log\left(1 - 2t\sum_{\mu=1}^{D}\cos q_\mu\right) . \tag{5.49}$$

Thus, for small enough $t \leq \frac{1}{2D}$ there is an IR divergence indicating a transition to a phase where the charged particle is condensed. Until that happens, the SSB phase survives. A useful slogan extracted from this calculation is that a loop operator becoming relevant (changing the IR physics) indicates the onset of a Higgs transition.

The discrete analog of this phenomenon is instructive. In the solvable toric code model, the discrete 1-form symmetries are exact. But in the rest of the deconfined (spontaneously broken) phase, they are *emergent*, but still spontaneously broken, and still imply a topology-dependent groundstate degeneracy that becomes exact in the thermodynamic limit. A rigorous proof of this [10] constructs (slightly thickened) string operators by quasi-adiabatic continuation.

Known forms of topological order in $D \leq 3+1$ have the property that at any $T > 0$ they are smoothly connected to $T = \infty$ (a trivial product state). If the 1-form symmetry is emergent, then as soon as $T > 0$, a mass *is* generated for the photon (by the argument above, with the circle regarded as the thermal circle, so that $R = 1/T$), and the state is smoothly connected to $T = \infty$.

We do know an example of a topologically ordered phase that is stable at $T > 0$, namely the two-form toric code in $D = 4+1$ [12]. In the $\mathsf{U}(1)$ version of this theory, the masslessness of the two-form gauge field should survive explicit short-distance breaking of the $\mathsf{U}(1)$ two-form symmetry, even at finite temperature. The reason is that a theory with a two-form symmetry on a circle still has a 1-form symmetry.

We conclude that the consequences of higher-form symmetries are more robust to explicit breaking than zero-form symmetries [11, 2, 13, 14]. This is a double-edged sword. One the one hand, it means that even though higher-form symmetries are rarely microscopically exact, they can be generic. On the other hand, it means that the Generalized Landau Paradigm is not as simple as the old-fashioned one. In classifying phases of matter, we can't just worry about the exact symmetries of the microscopic Hamiltonian. We also have to worry about symmetries that may emerge.

5.8 Subsystem symmetries and fracton phases

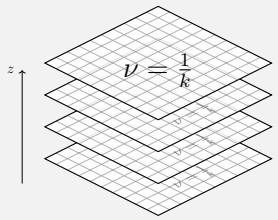
Above we have discussed p-form symmetries, described by symmetry operators acting on codimension-$(p+1)$ submanifolds of spacetime. These operators were *flexible*, in the sense that their correlations only depend on their deformation class in spacetime (avoiding any charged operator insertions).

A distinct generalization of the notion of symmetry arises by defining symmetry operators acting independently on *rigid* subspaces of the space on which the system is defined. That is, we can imagine that there is a different symmetry operator for each subspace, even in the same homology class, so that the symmetry operators are not topological, but still commute with the Hamiltonian. This is sometimes called a "faithful" symmetry [15] or *subsystem symmetry*. This generalization is not compatible with Lorentz invariance, since the operators are still topological in time.

Properties of symmetry operator	Ordinary symmetry	Higher-form symmetry	Subsystem symmetry	Non-invertible symmetry
Codimension in spacetime	1	> 1	> 1	≥ 1
How topological is it?	fully	fully	not completely	fully
Fusion rule	group $g_1 \cdot g_2 = g_3$	group $g_1 \cdot g_2 = g_3$	group $g_1 \cdot g_2 = g_3$	category $\mathcal{D} \cdot \mathcal{D}^\dagger \neq 1$

Table 2: This table (from Shu-Heng Shao) gives a nice overview of further possible generalizations of the notion of symmetry.

An object charged under such a subsystem symmetry cannot leave the locus on which the symmetry is defined. This sort of restricted mobility of excitations is a defining property of fracton phases [16, 17]. A fracton phase with topological order can be identified as one that spontaneously breaks such a "faithful" higher-form symmetry [15, 18, 19]. Foliated fracton phases [20] like the X-cube model [21] spontaneously break a 'foliated 1-form symmetry' acting independently on each plane of a lattice [15].



A trivial example of a fracton phase can be made by stacking 2+1d topological states. For example, let's stack a bunch of copies of abelian quantum Hall states each extended in the xy plane, but separated in the z direction at locations $z = Ia$, $I = 1..L$. Each layer is described at low energies by abelian CS theory and the whole thing has the action

$$S[a_I] = \sum_I \int_{x,y} \frac{k}{4\pi} a_I da_I. \tag{5.50}$$

The anyons in each layer are fractons in the sense that they cannot escape their layer (they are specific kind of fracton called 'planeons'). Notice that with pe-

riodic boundary conditions, the GSD goes like k^L: the log GSD is linear in the system size. This is characteristic of 3+1D fracton phases.
We can make a more interesting class of fracton models simply by coupling the layers to each other. That is, consider instead the following action [22]:

$$S[a_I] = \sum_{IJ} \frac{K_{IJ}}{4\pi} \int_{x,y} a_I da_J. \tag{5.51}$$

If K is a quasi-diagonal integer matrix, this can arise as an effective description of coupled layers of quantum Hall states, and sometimes is gapped. Now the log GSD as a function of L is more interesting, but still has a linear envelope. The excitations still move in planes, but they can have interesting braiding statistics (encoded in inverse of the matrix K) that approach irrational numbers as $L \to \infty$ and are not ultralocal in $I - J$. Actually this construction may even have a realization in experiments on quantum Hall layers [23].

Fracton phases are interesting for many reasons. One is that gapped fracton phases are a huge class of counterexamples to the lore that the low energy physics of gapped phases is always described by TQFT. These phases can arise from ordinary-looking lattice models, like the layered quantum Hall system described above, or the X-cube model, but even the GSD depends on the geometry of the lattice and therefore they are definitely not described by ordinary TQFT in any regime. A second reason is the bad news I mentioned in §5.7 about the lack of robustness of known topological order in 3+1d to finite temperature. One of the routes [24] to the discovery of fractons was the quest for finite-temperature passive quantum memory. (They have not quite provided such a thing as of yet.) A third reason is that they problematize our notions of what is a phase of matter. A phase of matter is a sharp notion in the thermodynamic limit, $L \to \infty$. In fracton phases, the GSD depends on L and so makes it difficult even to define such a limit. A good idea for how to get around this is to strengthen the equivalence relation defining a phase to allow the addition not only of decoupled qubits, but also decoupled 2d layers [25].

A closely-related concept to subsystem symmetries is that of multipole symmetries (*e.g.* [26, 27, 28, 29, 30, 31]). A multipole symmetry is one where the continuity equation involves extra derivatives, like $\partial_0 J^0 + \partial_i \partial_j J^{ij} = 0$ (a dipole symmetry). Such a conservation law produces conserved charges that need not be integrated over all of space, and act independently of each other. (For example [28], consider the continuity equation $\partial_0 J^0 + \partial_x \partial_y J = 0$ in $D = 2 + 1$; then $Q_x(x) = \int dy J^0(x, y)$ is conserved for each x.) The simplest example is that conservation of dipole moment implies that charges are immobile [26].

Models with such symmetries have been studied for a long time in the condensed matter literature [32]. Efforts to understand how the rules of ordinary field theory must be relaxed to accommodate such systems and their symmetries have been vigorous (see *e.g.* [33, 20, 34, 35, 36, 28, 29, 30, 37] and references therein and thereto). Attempts have been made to classify subsystem-symmetry-protected topological phases [38] and their anomalies [39], and to understand subsystem-symmetry-enriched topological order [40]. A subsystem-symmetry-based understanding of Haah's code [24], the most interesting gapped fracton model, appears in [41].

An important issue is the robustness of such phases, especially in the gapless case, upon breaking the large symmetry group. At least in examples, the scaling dimensions of operators charged under the subsystem symmetry is large, and in fact diverges in the continuum limit [32, 36, 28, 29, 30] (see in particular Eq. (121) of the first reference). This shows that there is at least a small open set in the space of subsystem-symmetry-breaking couplings in which such phases persist.

Fractal symmetry. The subsystem on which a symmetry acts can be more interesting than just a line or a plane. For example, it can be a fractal [42, 43]. The Newman-Moore model [44] is a simple example of a model with a symmetry operator supported on a fractal subset of space. Put qubits on the sites i of the triangular lattice and consider

$$H = \sum_{ijk\in\Delta} Z_i Z_j Z_k + g \sum_i X_i, \tag{5.52}$$

where the sum is only over up-pointing triangles. To see that this has a fractal symmetry, pick a spin to flip, say the circled spin in Fig. 10. Moving outward from that starting point and demanding that each up-triangle contains an even number of flipped spins, there are many possible self-similar subsets of the lattice we can choose to flip. In fact, there is an extensive number.

This transverse-field Newman-Moore model (5.52) has a number of interesting properties. It has a self-duality mapping $g \to 1/g$, obtained by defining dual spins $\tilde{X}_\Delta \equiv \prod_{i\in\Delta} Z_i Z_j Z_k$ on a new lattice with sites corresponding to the up-pointing triangles. A phase transition at $g = 1$ separates a gapped paramagnetic phase from a gapless phase in which the fractal $\mathbb{Z}_2$ symmetry is spontaneously broken. There is some controversy about the nature of the critical point in the literature: though [45] sees evidence of interesting critical behavior, earlier work [42, 46] found indications of a first-order transition, which seems to be confirmed in more recent simulations [47, 48]. Perhaps some deformation of this lattice model does have a critical point. Such critical points were claimed [45] to be 'beyond renormalization'; rather, what is broken is the connection between short distances and high energies [37]. Other models with such fractal symmetry have been studied in [49].

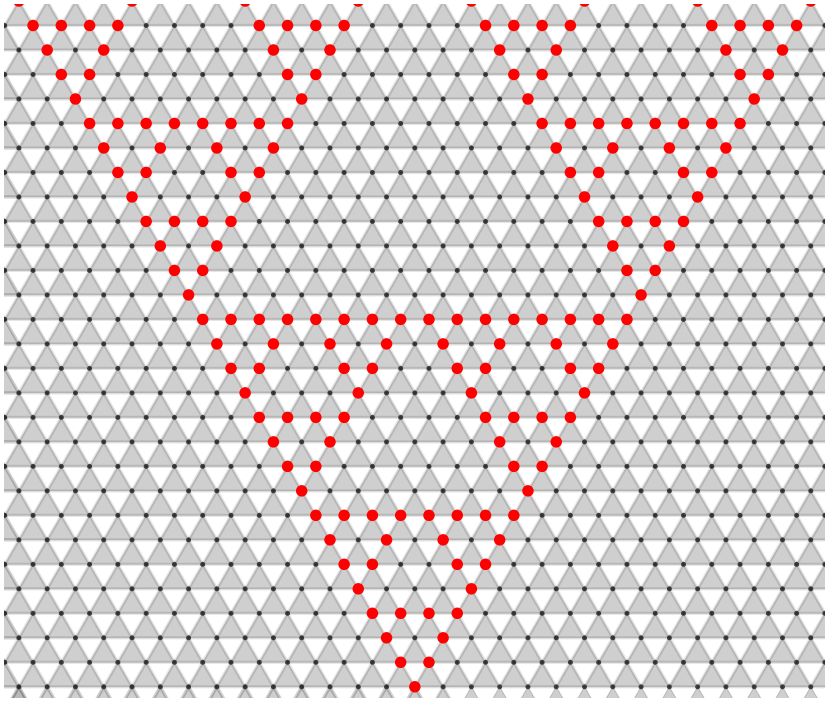


Figure 10: An example of the support of a fractal symmetry operator in the Newman-Moore model. If we flip only the red spins, it preserves the Hamiltonian (5.52). That is, every up-triangle has an even number of red dots. There are extensively many other ways to accomplish this goal.

In fact, there are translation-invariant lattice models with symmetries on subspaces with essentially no structure, like a random dust of points.

5.9 Non-invertible symmetries

The preceding discussion suggests a further generalization, which we will need in order to describe *non-abelian* topological order as SSB: if the worldlines of abelian anyons are generalized symmetry operators, what about the worldlines of non-abelian anyons? This is a dramatic step because the algebra of topological operators T_a that transport non-abelian anyons is no longer a group. Rather, they satisfy the fusion algebra:

$$T_a T_b = \sum_c N_{ab}^c T_c. \tag{5.53}$$

By definition, a topological order is non-abelian if there is more than one term on the RHS of this equation for some choice of a, b. Whereas multiplication of two elements of a group always produces a unique third element, here we produce a superposition of elements, weighted by *fusion multiplicities* N_{ab}^c. Further, there is some tension between the fusion algebra (5.53) and unitarity of the operators T_c. The trivial anyon

corresponds to the identity operator, $T_1 = 1$. Each type of anyon a has an antiparticle $\bar{a}$. Since $T_{\bar{a}}$ corresponds to transporting a in the opposite direction, we expect that $T_{\bar{a}} = T_a^\dagger$, and therefore (5.53) says in particular

$$T_a T_a^\dagger = \sum_c N_{a\bar{a}}^c T_c. \tag{5.54}$$

If the RHS here has a term other than $N_{a\bar{a}}^1$, then T_a is not unitary. As an example, consider the Ising topological order, with three anyon types $\{1, \psi, \sigma\}$ and the fusion rules

$$T_\sigma T_\sigma = 1 + T_\psi, \quad T_\sigma T_\psi = T_\psi T_\sigma = T_\sigma, \quad T_\psi T_\psi = T_1. \tag{5.55}$$

Note that σ is its own antiparticle. (5.55) implies that the topological line operator T_σ cannot be unitary, and moreover does not have an inverse. Such symmetries are called *non-invertible symmetries* (or sometimes *categorical symmetries* or *fusion category symmetries*).

More generally, any algebra of topological operators acting on a physical system can be regarded as encoding some kind of generalized symmetry.

At the moment, condensed matter applications of the idea of fusion category symmetries remain in the realm of relatively formal developments, as opposed to active phenomenology of real materials. But one application is to understand non-abelian topological order as spontaneous symmetry breaking[41]. A concrete example of a (2+1)d model with non-invertible symmetries is G_k Chern-Simons (CS) theory, with non-Abelian gauge group G at level $k > 1$. The non-invertible symmetry operators are the Wegner-Wilson lines. The specific example of $\mathsf{SU}(2)_2$ CS theory can describe the Ising topological order, and is possibly realized as part of the effective low-energy description of $\nu = \frac{5}{2}$ quantum Hall states.

More generally, any topological field theory for non-Abelian topological order enjoys such a non-invertible symmetry. A nice example of the application of this perspective on anyon worldlines as symmetry operators is [52] which provides a condition on the anyon data required for a general 2+1D topological order to admit a gapped boundary condition, beyond vanishing chiral central charge.

Part of the reason for the nomenclature 'categorical symmetry' is that such a collection of symmetry operators comes with some additional data. Besides putting two symmetry operators right on top of each other, we can also consider symmetry operators associated with branched manifolds, as in Fig. 11a. Once we allow such objects, we must also consider more complicated objects related to the associativity of the product,

[41] A related perspective appears via the 'pulling-though' operators in the tensor network description of topological orders reviewed in [50]. For a study of categorical symmetries realized as matrix product operators, see [51].

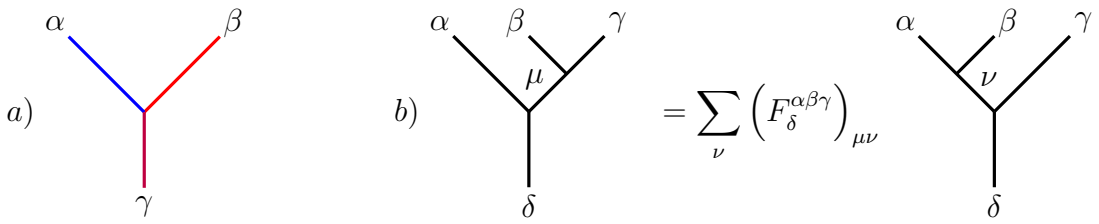


Figure 11: a) Fusion of symmetry operators: this junction is allowed if $N^\gamma_{\alpha\beta} \neq 0$. b) Associativity data of fusion of symmetry operators (in the simpler case where the fusion coefficients $N^\gamma_{\alpha\beta}$ are only 0 or 1).

as in Fig. 11b, which relates the two ways of resolving a 4-valent junction of topological operators into two 3-valent junctions. This associativity information (creatively called F-symbols) is part of the specification of the categorical symmetry, and must satisfy the pentagon identities. In the case of 1-form symmetry in (2+1)-D, there is further information associated with braiding.

5.10 How general is the Generalized Landau Paradigm?

Above we have captured some previously beyond-Landau phases in terms of how they represent their symmetries. Some frontiers worth mentioning are:

- Landau's own Fermi liquid is a gapless phase, that is in some sense protected by the size of the Fermi surface. Recent work [53] describes a large emergent symmetry of a certain class of states generalizing the Fermi liquid, and its anomalies. Perhaps this is a good starting point.

- I said that SPTs are distinguished by the anomaly of the edge theory. What about invertible phases which don't have topological order, but don't need symmetry to distinguish them from the trivial phase. An example is the integer quantum Hall state, which above I described as merely an SPT for particle number symmetry. Such states do have an anomaly, but it is an anomaly involving the coupling to spacetime curvature and I don't know how to think about this as involving a symmetry (diffeomorphisms are a redundancy, not a symmetry).

- Crystalline solids are distinguished from liquid and gas by the fact that they spontaneously break translation symmetry down to the symmetry of the lattice. What about amorphous solids? Well, one thing that distinguishes them from fluids is that the particles are frozen in place, rather than mobile. This means

that the Edwards-Anderson order parameter will be nonzero. This is obtained by measuring the density excess $\rho(x,0) - \rho_0$ and the density at the same point in space later $\rho(x,t) - \rho_0$ and averaging over space and/or configurations. It is zero in a fluid. It is a sign of a breakdown of ergodicity – that there is more than one equilibrium state. How do we think of this as symmetry breaking? One way to detect it is to consider n copies of the system. There is a S_n symmetry that permutes the copies, called replica symmetry. If ergodicity is broken, the different copies can go into different equilibria and spontaneously break the replica symmetry.

[End of Lecture 20]

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6 Conformal field theory

Conformal field theories are special QFTs that have conformal invariance. Why should they receive extra attention? Well, the idea is that any other continuum QFT arises by starting with a CFT in the far UV, and perturbing it by some relevant operator(s). In the far IR it flows to some other (possibly trivial) CFT. So a general continuum QFT is merely a flow between CFTs, and therefore if we understand the CFTs well enough, we understand all of QFT. (One small loophole in this grand picture is that often we are confronted with QFTs that arise from lattice models in the UV, rather than a continuum theory in the UV. Possibly this is more general.)

[$D > 2$: Simmons-Duffin, Rychkov, $D = 2$: Ginsparg, Tong]

In a CFT, there is a preferred basis of local operators, which are eigenvectors under the scale transformation, and their OPE reduces to a collection of numbers c_{abc}. This gives a way to think about CFT non-perturbatively: A CFT is a list of operators with definite scaling dimensions $\{\mathcal{O}_a, \Delta_a\}$ and their OPE structure constants, c_{abc}, appearing in (3.24). From this information you can compute any correlation function of local operators by successive uses of the OPE. So the 1+1d Ising CFT is:

$$\{\underbrace{\mathbb{1}}_{(0,0)}, \underbrace{\varepsilon \equiv \mathbf{i}\chi_L\chi_R}_{(\frac{1}{2},\frac{1}{2})}, \underbrace{\sigma}_{(\frac{1}{16},\frac{1}{16})}, \underbrace{\mu}_{(\frac{1}{16},\frac{1}{16})}, \underbrace{T \equiv \chi_L\partial\chi_L}_{(2,0)}, \underbrace{\bar{T} \equiv \chi_R\bar{\partial}\chi_R}_{(0,2)}, \cdots\}$$

$$\sigma \times \sigma = \mathbb{1} + c_{\sigma\sigma\varepsilon}\varepsilon, \varepsilon \times \varepsilon = \mathbb{1} + \underbrace{c_{\varepsilon\varepsilon\varepsilon}}_{=0}\varepsilon, \sigma \times \varepsilon = c_{\sigma\sigma\varepsilon}\sigma.$$

Notice that I include the identity operator in this list; it has dimension zero for sure. In this way of enumerating the operators the list goes on forever: *e.g.* we can keep appending more derivatives. We'll learn to how do better soon by listing only the *primary operators* at the top of each representation of the conformal group. But in fact, as in effective field theory, one can often get far just knowing about the few lowest-dimension operators[42]. A special role is played by the operator $T(z)$ in this list, the *stress tensor*.

6.1 The stress tensor and conformal invariance (abstract CFT)

Let's think a bit about relativistic field theory in the continuum. Any continuum QFT has a stress tensor[43]; there are two useful perspectives on this operator. The simpler,

[42] Naturally, this strategy is generally called 'effective conformal field theory'. A recent victory in this direction can be found in this paper, which solves QCD in $D = 2$ by diagonalizing a 5×5 matrix (!).

[43] Maybe this is not true. The following is a very technical comment which you should ignore if you want. It is certainly true if the field theory has a definition in terms of a lagrangian or a local

but worse, one is to regard it as the Nöether current for spacetime translations. The stress tensor $T_{\mu\nu}$ is conserved if the action is translation-invariant: that is, invariant under the replacement $\phi(x^\mu) \to \tilde{\phi}(\tilde{x}) \equiv \phi(x^\mu + a^\mu)$. This is $d+1$ symmetries, so we have $d+1$ conserved currents: $T^\mu_\nu, \partial_\mu T^\mu_\nu = 0$. The associated charges are the energy and momentum

$$\mathbf{H} = \int_{\text{space}} T_{00}, \quad \mathbf{P}_i = \int_{\text{space}} T_{0i} \ .$$

These generate translations in time and space by commutators (or classically by Poisson brackets):

$$\delta_a \phi(x) = \tilde{\phi}(\tilde{x}) - \phi(x) = a^\mu [\mathbf{P}_\mu, \phi(x)].$$

The finite solution of this equation is

$$\phi(x+a) = e^{\mathbf{i}\mathbf{P}_\mu a^\mu} \phi(x) e^{-\mathbf{i}\mathbf{P}_\mu a^\mu}.$$

The current we get from the Nöether method is not symmetric in its indices.

The better way to think about the stress tensor is as the (linear) response of the system to a local, small perturbation of the metric of spacetime:

$$\delta S = \int \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} \ , \quad \text{aka} \quad T^{\mu\nu}(x) = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}(x)}. \tag{6.1}$$

Here $\sqrt{g} \equiv \sqrt{\det g}$ is the covariant measure.[44] Consider in particular making a *scale transformation*, which we can accomplish by changing the metric by:

$$\delta g_{\mu\nu} = 2\lambda g_{\mu\nu} \quad \Longrightarrow \quad \delta S = \int d^D x \delta g_{\mu\nu}(x) \frac{\delta S}{\delta g_{\mu\nu}(x)} = \int T^\mu_\mu 2\lambda \tag{6.2}$$

where λ is a constant. Therefore we see that if $T^\mu_\mu = 0$ then the theory is scale invariant. In the other direction, scale invariance (vanishing of (6.2)) actually only implies that $T^\mu_\mu = \partial^\mu K_\mu$ is a total derivative. But the object K is a vector operator whose scaling dimension must be $D-1$ (since that of the stress tensor is D). On general grounds, a vector operator of dimension $D-1$ is a conserved current (at least this is the only

hamiltonian. Possible exceptions come from: (1) field theory in a fixed AdS geometry; the absence of dynamical gravity means no stress tensor. This is called 'generalized free field theory'. (An attempt to take such a thing seriously as a CFT is in this paper by Heemskerk et al.)

(2) In his CFT notes, Rychkov discusses a long-range Ising system with a fixed point with no stress tensor; he claims without further discussion that this is the same as the first example. (3) Weird things 'defined' by scaling limits of string theory. Does a 2d CFT without a stress tensor have a central charge?

[44] The $T^{\mu\nu}$ from the metric variation can be related to the Nöether current for translations by "improvement," which means adding boundary terms to the action; this modifies the Nöether current.

way I know to protect its dimension). And a conserved current has $\partial^\mu K_\mu = 0$, so that $T^\mu_\mu = 0$ anyway.

If $T^\mu_\mu = 0$ then nothing we said above depended on the fact that λ is a constant, and we should also consider the following transformation:

$$\delta g_{\mu\nu} = 2\delta\Omega(x) g_{\mu\nu} \quad \to \quad \delta S = \int T^\mu_\mu 2\delta\Omega(x) \tag{6.3}$$

Such a transformation, which rescales the whole metric (and therefore preserves angles between vectors at the same point) in a position-dependent way, is a *Weyl transformation*, closely related to a conformal transformation. The difference is that for arbitrary $\Omega(x)$, the new metric will be curved ($\mathcal{R} \propto \nabla^2 \log\Omega$). If we are not interested in QFT in curved spacetime, we should restrict ourselves to choices of Ω that preserve the initial choice of curvature; this means that they can be undone by a coordinate transformation. Infinitesimally, such a transformation is $x^\mu \to (x')^\mu = x^\mu + \xi^\mu(x)$, and the metric change is

$$\delta g_{\mu\nu}(x) = \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) \stackrel{!}{=} 2\delta\Omega g_{\mu\nu}. \tag{6.4}$$

This a set of PDEs for ξ_μ and $\delta\Omega$.

For the case of $g_{\mu\nu} = \eta_{\mu\nu}$, Minkowski spacetime, the stage of special relativity, the constraint (6.4) is solved by the following formulae[45] (and by translations and rotations and boosts, which don't change the Minkowski metric at all). The conserved currents and charges of the transformations above (in flat spacetime) are:

$$S_\mu = x^\nu T_{\mu\nu} \qquad \to \quad D \equiv \int S_0 d^d x = \int x^\mu \hat{p}_\mu \tag{6.5}$$

$$C_{\mu\nu} = (2x_\mu x_\lambda - x^2 g_{\mu\lambda}) T^\lambda_\nu \qquad \to \quad C_\mu \equiv \int C_{0\mu} d^d x \tag{6.6}$$

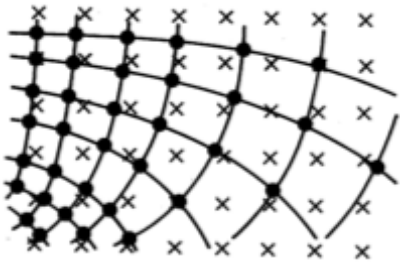
since both $\partial^\mu S_\mu$ and $\partial^\mu C_{\mu\nu}$ are proportional to T^μ_μ. In the last equation of the first line, $\hat{p}_\mu$ is the momentum *density*; the action of the integrand on fields is via $x^\mu\partial_\mu$, a rescaling.

Here is the right way to think about this condition on finite transformations. We are demanding a change of coordinates that accomplishes the following:

$$\eta_{\mu\nu}(dx')^\mu (dx')^\nu \stackrel{!}{=} \Omega^2(x) dx^\mu dx^\nu \eta_{\mu\nu}.$$

The Jacobian for this change of coordinates must therefore satisfy

$$J^\mu_\nu \equiv \frac{\partial (x')^\mu}{\partial x^\nu} = \Omega(x) \mathsf{O}^\mu_\nu(x)$$



[Fig credit: Rychkov]

[45] The details are at the beginning of the Ginsparg notes, or on page 3 here.

where $\mathsf{O}^\mu_\nu(x)$ is a spacetime-dependent Lorentz transformation ($\mathsf{O}\eta\mathsf{O} = \eta$). This is a spacetime-dependent rescaling and rotation; you should think of it as a position-dependent RG transformation.

We conclude from the above discussion that, at least classically, if $T^\mu_\mu = 0$ (and $T^{\mu\nu}$ is symmetric) then the theory has both scale invariance and conformal invariance. The precise logical relation between scale invariance and conformal invariance is the subject of a lot of discussion. There is no example of an interacting relativistic unitary fixed point without conformal invariance.

The resulting set of transformations forms an extension of the Poincare group. There are $d+2$ extra generators $\mathbf{C}_\mu$ and $\mathbf{D}$. In fact the algebra is $\mathsf{so}(d+1,2)$.

Conformal algebra. Here is how to think about this algebra. Most of the commutators just say that vectors (like $\mathbf{C}_\mu$ and $\mathbf{P}_\mu$) transform as vectors and D is a scalar. The important ones are:

$$[\mathbf{D}, \mathbf{P}_\mu] = \mathbf{i}\mathbf{P}_\mu, \quad \mathbf{P}_\mu \text{ is a raising operator for } \mathbf{D}$$

$$[\mathbf{D}, \mathbf{C}_\mu] = -\mathbf{i}\mathbf{C}_\mu, \quad \mathbf{C}_\mu \text{ is a lowering operator for } \mathbf{D}$$

$$[\mathbf{C}_\mu, \mathbf{P}_\nu] = 2\mathbf{i}\left(\eta_{\mu\nu}\mathbf{D} - \mathbf{M}_{\mu\nu}\right)$$

which says you can recover $\mathbf{D}$ and the spin from $\mathbf{C}_\mu$.

The right way to think about the transformation $\mathbf{C}$ generates is:

$$\text{inversion} \circ \text{translation} \circ \text{inversion:} \qquad \frac{x'^\mu}{\vec{x'}.\vec{x'}} = \frac{x^\mu}{\vec{x}.\vec{x}} + b^\mu \quad . \tag{6.7}$$

Inversion is $x^\mu \to -\frac{x^\mu}{x^\nu x_\nu}$; Poincaré plus inversion implies conformal.

So if we want to study representations of this algebra, we can diagonalize $\mathbf{D}$ and some of the rotation generators. (Notice that the familiar Poincaré casimir $\mathbf{P}^\mu\mathbf{P}_\mu$ (the mass-squared) is *not* a casimir of the conformal group.) Then we can build representations starting with states that have been lowered as much as possible, so $\mathbf{C}\,|\text{primary}\rangle = 0$; the rest of the representation is obtained by acting with derivatives, *i.e.* $\mathbf{P}_\mu$ (and $\mathsf{SO}(d)$ rotations).

Fields also form representations of the spacetime symmetry group. A primary field is one that is local enough that its transformation rule under a conformal transformation is the same as if it were just a scale transformation:

$$\Phi(x) \to \Phi'(x') = \Omega(x)^{-\Delta}\Phi(x)$$

for a scalar field, or

$$\Phi(x) \to \Phi'(x') = \Omega(x)^{-\Delta}\mathcal{R}[\mathsf{O}(x)^\mu_\nu]\Phi(x)$$

for a field with spin. $\mathcal{R}$ is the representation matrix for the rotation O in the Φ representation; for a vector field V^μ, this is just $\mathsf{O}(x)^\mu_\nu V^\nu(x)$. This condition on the transformation law on the field is equivalent to the demand that $[C_\mu, \Phi(0)] = 0$ and $[D, \Phi(0)] = \mathbf{i}\Delta\Phi(0)$. (See Rychkov's notes for the demonstration of this claim.)

Fields that are not primaries (for example, the derivative of a primary) are called descendants and have more complicated transformation rules. Fields can be organized into eigenstates of $\mathbf{D}$, of definite scaling; every such field is a primary or a descendant. The algebra determines the behavior of descendants from that of primaries.

6.1.1 Geometric interpretation of conformal group

Here is a geometric interpretation of the conformal group, called the *projective null cone* construction (due to Dirac (!)). The conformal group in $\mathbb{R}^{d,1}$ is isomorphic to $\mathsf{SO}(d+1,2)$, the Lorentz group of a space with two extra dimensions. This space $\mathbb{R}^{d+1,2}$ has metric

$$\eta_{ab} = \text{diag}(-++...+-)_{ab} \tag{6.8}$$

where the last two dimensions are the 'extra' ones. So we can find linear representations of the conformal group by temporarily adding some extra coordinates. A light ray in this space can be parameterized by $d+1$ dimensional coordinates x^μ in the following way:

$$\zeta^a = \kappa(x^\mu, \frac{1}{2}(1-x^2), \frac{1}{2}(1+x^2))^a \tag{6.9}$$

where κ is some arbitrary constant. The group $\mathsf{SO}(d+1,2)$ moves these light rays around. We can interpet these transformations as maps on the x^μ. In fact these transformations (combined by rescalings to get back to the original slice) are precisely the conformal transformations.

Regard the expression (6.9) as an embedding of $\mathbb{R}^{d,1} \subset \mathbb{R}^{d+1,2}$; this is an *isometric embedding*, *i.e.* the induced metric is again $ds^2 = -dt^2 + d\vec{x}^2$. To see that Lorentz boosts in the embedding space are conformal transformations on the null slice, note that on the slice, $\xi^a\xi_a = 0 \implies \xi^a d\xi_a = 0$, and so the transformation $\xi^a \to \lambda(x)\xi^a$ (which preserves a null subspace) takes $d\xi^a d\xi_a \to \lambda(x)^2 d\xi^a d\xi_a$, a conformal transformation.

Invariants in $\mathbb{R}^{d+1,2}$ should therefore be conformal invariants. Consider the object:

$$\zeta_1 \cdot \zeta_2 = \eta_{ab}\zeta_1^a\zeta_1^b = \frac{1}{2}\kappa_1\kappa_2(x_1 - x_2)^2. \tag{6.10}$$

ζ^a and $\lambda\zeta^a$ are identified with the same x^μ, so κ is a redundant variable. So conformal invariants actually are cross ratios of invariants in $\mathbb{R}^{d+1,2}$, for example

$$\frac{\zeta_1 \cdot \zeta_2 \zeta_3 \cdot \zeta_4}{\zeta_1 \cdot \zeta_3 \zeta_2 \cdot \zeta_4}. \tag{6.11}$$

An extremely useful consequence of this is the statement that $r_{12}^2 \equiv (x_1 - x_2)^\mu (x_1 - x_2)_\mu$ transforms under a conformal transformation by a rescaling

$$x_{12}^2 \to \frac{x_{12}^2}{\Omega(x_1)\Omega(x_2)}$$

where $\Omega(x_i) \equiv 1 + 2b \cdot x_i + b^2 x_i^2$ is the rescaling of the metric at x_i under the special conformal transformation $x^\mu \to \frac{x^\mu + b^\mu}{1 + 2b\cdot x + b^2 x^2}$.

6.1.2 Infinite conformal algebra in $D = 2$.

Here there is an important division between $D = 1 + 1$ and $D > 1 + 1$. In $D = 2$, in lightcone or holomorphic coordinates where $ds^2 = dzd\bar{z}$, tracelessness of T says

$$T^\mu_\mu \propto T_{z\bar{z}} = 0 \ .$$

Conservation of the stress tensor $0 = \partial^\mu T^\nu_\mu$ then says

$$\bar{\partial}_z T_{zz} = 0, \quad \partial_z T_{\bar{z}\bar{z}} = 0.$$

That is $T(z) \equiv T_{zz}$ is holomorphic (and $\tilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$ is antiholomorphic). In the quantum theory, these statements are true as operator equations; that is: they are exactly true away from other operator insertions in the path integral (the lingo for this is 'up to contact terms'). This holomorphic factorization has the following dramatic consequence: Given an arbitrary holomorphic function[46] $\xi(z)$, the current

$$j^{(\xi)}_\mu = (j_z, j_{\bar{z}})_\mu \equiv (\xi(z)T(z), 0)_\mu$$

is also conserved (!):

$$\partial^\mu j^{(\xi)}_\mu = \bar{\partial}_z j_z - \partial j_{\bar{z}} = 0.$$

This is infinitely many conserved currents! Basically, just from scale invariance.

What are these transformations? Recall that the current $T_{\mu\nu}$ generates translations, which by abuse of notation[47] we can write as $x^\mu \to x^\mu + a^\mu$. Accordingly, the current

[46] More precisely, since there is one for z and one for $\bar{z}$, $\xi^z, \xi^{\bar{z}}$ are components of a holomorphic vector field.

[47] This abuse of notation is both very tempting and very confusing. We are not merely relabeling our coordinates; that doesn't do anything – physics is coordinate-independent. We are transforming our fields by

$$\phi \to \phi'(x') = \phi(x) \ .$$

If you promise to keep this in mind, then we can use the less cumbersome expressions below.

$T(z)$ generates 'holomorphic translations': $z \to z + \xi$ with ξ constant. So it's not too shocking that $j_\mu^{(\xi)}$ generates the *local* transformation $z \to z + \xi(z)$. The finite version of this transformation is just an arbitrary holomorphic map:

$$z \to z'(z), \quad \bar{z} \to \bar{z}'(\bar{z}).$$

(The important thing here is that z' does not depend on $\bar{z}$!) What does this transformation do to the metric? The flat metric in holomorphic coordinates is

$$ds^2 = d\tau^2 + dx^2 = dzd\bar{z} \to \frac{\partial z}{\partial z'}dz' \frac{\partial \bar{z}}{\partial \bar{z}'}d\bar{z}' = f(z,\bar{z})dz'd\bar{z}'.$$

The metric has only changed by an overall function. This means that the *angle* between any two vectors has not changed. This is the definition of a *conformal transformation*. The conformal group is infinite dimensional in $D = 2$.

For the example of the free massless scalar in $D = 2$ (with curved-worldsheet action $S[\phi] = \frac{1}{2\pi K}\int dxdt\sqrt{g}g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi$) the stress tensor is

$$T_{\mu\nu} = \frac{1}{2\pi K}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\partial\phi\right)^2\right).$$

Notice that it is traceless: $g^{\mu\nu}T_{\mu\nu} \equiv T^\mu_\mu = 0$. In holomorphic coordinates, $ds^2 = dzd\bar{z}$, the nonzero components are

$$\begin{aligned} T_{zz} \equiv T(z) &= \frac{1}{2\pi K}\partial_z\phi\partial_z\phi = \frac{1}{2\pi K} :\partial_z\phi_L\partial_z\phi_L:, \\ \bar{T}_{\bar{z}\bar{z}} \equiv T(\bar{z}) &= \frac{1}{2\pi K}\bar{\partial}_z\phi\bar{\partial}_z\phi = \frac{1}{2\pi K} :\bar{\partial}_z\phi_R\bar{\partial}_z\phi_R: \end{aligned} \tag{6.12}$$

In the last step I've emphasized the factorization into L and R parts, and the fact that quantumly we must define this composite operator somehow, and we are doing it by normal ordering. This involves a choice of additive constant, about which there is a bit more to say.

Conformal invariance constrains the operator algebra of a CFT, and since (various moments of) the stress tensor components are generators of conformal transformations, their OPEs are highly constrained.

6.2 Radial quantization

Here is an important example of a conformal transformation: Consider a cylinder with complex coordinate $w = x + \mathbf{i}\tau$; I call it a cylinder because $x \simeq x + L$. Let's set $L = 2\pi$ for convenience. Consider the map

$$w \mapsto z = e^{-\mathbf{i}w} = e^{\tau - \mathbf{i}x}.$$

Since it's holomorphic ($z(w,\bar{w})$ doesn't depend on $\bar{w}$), this is a conformal transformation. The image is the complex z-plane. Equal-τ surfaces are circles. The spatial momentum operator was $\partial_\sigma = z\partial_z - \bar{z}\bar{\partial}_z$, which acts by $z \to e^{\mathbf{i}b}z$ – rotations about the origin. The hamiltonian on the cylinder was $\partial_\tau = z\partial_z + \bar{z}\bar{\partial}_z$. So time translations become rescaling about the origin of the z-plane: the dilatation operator (the operator $\mathbf{S}$ that generates scale transformations) is the hamiltonian in radial quantization. Time-ordered correlators on the cylinder are *radially-ordered*. One reason this is nice is that it provides an IR cutoff even when studying infinite-volume physics. (Notice that the operators $z\partial_z \pm \bar{z}\bar{\partial}_z$ commute and so we can label states by their scaling dimension Δ and *spin*.

In fact, so far, this all goes through for CFT in any D: Foliate $\mathbb{R}^D$ by round spheres about some (arbitrary) point, and regard these as equal-time slices; the CFT Hamiltonian $\int_{S^d} T_{00}$ which generates motion from one radial slice to the next is just the dilatation operator. To see that this map from the plane to the cylinder is still a conformal transformation, choose polar coordinates on $\mathbb{R}^D$ so the metric is $ds^2_{\text{flat}} = dr^2 + r^2 d\hat{n}^2$; the metric on the cylinder is $d\tau^2 + d\hat{n}^2 = \frac{1}{r^2} ds^2_{\text{flat}}$ with $\tau = \log r$. In $D > 2$, the equal-radial-time-slices are $D-1$-spheres, and therefore the single quantum number for spin would be replaced by a representation of $SO(D)$.

State-operator correspondence. (any D) The conclusion is that there is a one-to-one correspondence between eigenvectors $|\Delta\rangle$ of the CFT Hamiltonian on the sphere S^d, and local scaling operators on the plane $\mathcal{O}_\Delta$.

To get a CFT state $|\Delta\rangle$ from a local operator $\Phi_\Delta(z)$ just insert that operator at the origin acting on the CFT vacuum in radial quantization about the origin. It's easier in equations:

$$\Phi_\Delta(0)\,|0\rangle \equiv |\Delta\rangle\,.$$

The ground state $|0\rangle$ is then the image of the identity operator under this map. If Φ_Δ has dimension Δ in the sense that $[\Delta, \Phi_\Delta(0)] = \mathbf{i}\Delta\Phi_\Delta(0)$ then

$$D\,|\Delta\rangle = D\Phi_\Delta(0)\,|0\rangle = [D,\Phi_\Delta]\,|0\rangle + \Phi_\Delta D\,|0\rangle = \mathbf{i}\Delta\,|\Delta\rangle\,.$$

What happens if we don't insert it at the center? Then

$$\Phi_\Delta(x)\,|0\rangle = e^{\mathbf{i}Px}\Phi_\Delta(0)e^{-\mathbf{i}Px}\,|0\rangle = e^{\mathbf{i}Px}\,|\Delta\rangle = \sum_n \frac{1}{n!}(\mathbf{i}Px)^n\,|\Delta\rangle$$

is the primary plus a bunch of descendants in the same multiplet, since $P\,|\Delta\rangle = |\Delta+1\rangle$ is a state with dimension one larger.

To get an operator from a state $|\Delta\rangle$, define it by its correlation functions:

$$\langle\phi_1(x_1)\phi_2(x_2)\cdots\mathcal{O}_\Delta(0)| \equiv \langle 0|\,\phi_1(x_1)\phi_2(x_2)\cdots|\Delta\rangle\,.$$

To understand this further, let's think about the path integral on the ball Σ. Think of it as a functional of the boundary conditions on the fields (which I'll call ϕ; θ is a coordinate on $\partial\Sigma$):

$$\int_{\phi|_{\partial\Sigma}(\theta)=\phi_0(\theta)} [D\phi]e^{-S} = \Psi_0[\phi_0] = \langle\phi_0|0\rangle$$

This is an integral representation of the groundstate wavefunctional. If instead we consider the path integral with a local operator insertion, we get a wavefunctional for a different state:

$$\int_{\phi|_{\partial\Sigma}(\theta)\phi_0(\theta)} [D\phi]e^{-S}\Phi(0) = \Psi_\Phi[\phi_0] = \langle\phi_0|\Phi\rangle \ .$$

In a CFT this is related by a conformal transformation $z = e^{-\mathbf{i}w}$ to the path integral on the cylinder with the state $|\Phi\rangle$ inserted in the far past. So to get the operator corresponding to an arbitrary state just glue this path integral around where you want to put the operator. To move it to a different place, just act with translation generators: $\Phi(x) = e^{-\mathbf{i}\mathbf{P}^\mu x_\mu}\Phi(0)e^{\mathbf{i}\mathbf{P}^\mu x_\mu}$.

The state operator is a beautiful thing: it says that putting a CFT on S^d is a preferred IR regulator that respects conformal invariance – the spectrum of the Hamiltonian on S^d (up to an overall multiplicative factor and, in even spacetime dimensions, an additive constant) is the list of scaling dimensions. So if we have a nice UV regulator, too, it becomes a numerically practical thing. Such a nice UV regulator is the fuzzy sphere. For more on the beauty of the state-operator correspondence in D-dimensional CFT see the notes here by Rychkov.

Convergence of OPE. With this realization in mind, it is clear that the OPE in CFT is a *convergent* expansion. Draw a round sphere containing exactly two operator insertions, and on this time slice, insert the resolution of the identity on the Hilbert space in the particular basis of eigenstates of the dilatation operator. Each state in this sum correspond to a single scaling operator inserted at the center of the sphere, which is a term in the OPE of the two operators we started with.

Adjoint in radial quantization. The adjoint is a bit weird: it is just the adjoint on the cylinder, but two things. First, the state in the far future on the cylinder gets mapped to $z = \infty$. Second, we must remember that the map from the cylinder to the sphere produces a Jacobian. Since the adjoint operation involves an *inversion* (a particular element of the conformal group), the primary Φ acquires an extra factor from the Jacobian. This produces an extra factor in the state:

$$|\Phi\rangle^\dagger = \lim_{z\to\infty} \langle\Phi| \, z^{-h} \ .$$

This is sometimes called the BPZ adjoint after its discoverers.

Primaries and quasiprimaries. Def: A primary operator (or field) $\Phi_{h,\bar{h}}(z,\bar{z})$ of weight $(h,\bar{h})$ transforms under the conformal transformation[48]

$$(z,\bar{z}) \to (f(z),\bar{f}(\bar{z})) \qquad \text{by} \qquad \Phi_{h,\bar{h}}(z,\bar{z}) \to (\partial_z f)^h \left(\bar{\partial}_z \bar{f}\right)^{\bar{h}} \Phi(f(z),\bar{f}(\bar{z})).$$

(The way to remember this is that $\Phi(z) dz^h d\bar{z}^{\bar{h}}$ is a scalar.) For example,

$$\begin{aligned} \text{scaling:} \quad & z \to e^{\lambda} z \implies \Phi \to e^{\lambda\Delta}\Phi, \quad \Delta = h + \bar{h} \ \text{ (scaling dimension)} \\ \text{rotations:} \quad & z \to e^{\mathbf{i}\theta} z \implies \Phi \to e^{\mathbf{i}s\theta}\Phi, \quad s = h - \bar{h} \ \text{ (spin)} \end{aligned} \tag{6.13}$$

The infinitesimal transformation $f(z) = z + \xi(z)$ results in

$$\delta_\xi \Phi(z) = (\xi\partial_z + h\partial_z\xi)\,\Phi + \text{antiholomorphic bits.}$$

This transformation is *generated* by $\int \frac{dz}{2\pi\mathbf{i}} \xi(z) T(z) \equiv \mathbf{L}[\xi]$ (as in 6.1.2) in the sense that $\delta_\xi \Phi = \mathbf{i}[\mathbf{L}[\xi], \Phi]$.

Consider for a while a holomorphic operator, with $\bar{h} = 0$. It has a mode expansion

$$\Phi_h(z) = \sum_{n\in\mathbb{Z}} \Phi_n z^{-n-h}.$$

The shift by h in the moding on the plane comes from the conformal transformation from the cylinder:

$$\Phi_h(w = -\mathbf{i}\ln z) = \sum_n \Phi_n e^{-\mathbf{i}wn}$$

where this is just fourier expansion, and the conformal factor is $\left(\frac{\partial z}{\partial w}\right)^h = z^h$. Note that $n < 0$ is positive energy.

The modes of the stress tensor are called Virasoro operators

$$T(z) = \sum_n \mathbf{L}_n z^{-n-2}, \quad \mathbf{L}_n = \oint_{C_0} \frac{dz}{2\pi\mathbf{i}} z^{n+1} T(z).$$

The definition of primary implies that

$$[\mathbf{L}_0, \Phi_{h,\bar{h}}(0)] = h\Phi_{h,\bar{h}}(0), \qquad [\mathbf{L}_n, \Phi_{h,\bar{h}}(0)] = 0, \ \forall n > 0.$$

[48] In general dimension $D \geq 2$, the transformation of a primary operator of dimension Δ is

$$\Phi_\Delta(x) \to \left|\frac{\partial x'}{\partial x}\right|^\Delta \Phi'_\Delta(x')$$

where $\left|\frac{\partial x'}{\partial x}\right|$ is the Jacobian of the conformal map.

This in turn implies that the state

$$\left|\Phi_{h,\bar{h}}\right\rangle \equiv \Phi_{h,\bar{h}}(0)\left|0\right\rangle$$

is a *highest weight state of the Virasoro operators*, in the sense that

$$\mathbf{L}_0\left|\Phi_{h,\bar{h}}\right\rangle = h\left|\Phi_{h,\bar{h}}\right\rangle, \qquad \mathbf{L}_n\left|\Phi_{h,\bar{h}}\right\rangle = 0,\ \forall n > 0.$$

Note that the modes with $n > 0$ *raise* the value of $\mathbf{L}_0$. They include the ordinary special conformal generators $C_\mu = (\mathbf{L}_1, \bar{\mathbf{L}}_1)_\mu$.

In $D = 1+1$ it is important to distinguish between Vir primary and ordinary conformal primary, which is just killed by $\mathbf{L}_1$ and not $\mathbf{L}_{n\geq 2}$.

Contours and commutators. You may be bothered by the connection between the algebra in terms of OPEs on the complex plane

$$(\text{like} \quad \partial\phi(z)\partial\phi(w) \sim \frac{1}{(z-w)^2} + \cdots \quad)$$

and the perhaps-more-familiar algebra of mode operators.

$$(\text{like} \quad [\boldsymbol{\rho}_n, \boldsymbol{\rho}_m] = n\delta_{n+m} \quad).$$

The very direct connection between the two comes from radial quantization. Recall that the path integral on the plane produces *radially ordered* correlators:

$$\int [D\phi] \quad \underbrace{e^{-S[\phi]}...A(z)B(w).....}_{\text{these are numbers, order doesn't matter}} \quad = \langle T(....\mathbf{A}(z)\mathbf{B}(w)...)\rangle$$

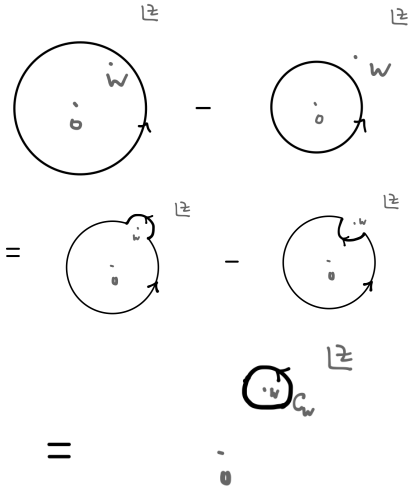
with

$$T(\mathbf{A}(z)\mathbf{B}(w)) = \begin{cases} \mathbf{A}(z)\mathbf{B}(w), & |z| > |w| \\ \mathbf{B}(w)\mathbf{A}(z), & |z| < |w| \end{cases}$$

So consider, for example, the commutator

$$[L[\xi], \Phi_h(w)] = \left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right)\frac{dz}{2\pi\mathbf{i}}\xi(z)T(z)\Phi_h(w)$$

$$= \oint_{C_w}\frac{dz}{2\pi\mathbf{i}}\xi(z)T(z)\Phi_h(w).$$



This is a general rule: the commutator of modes of two fields is given by the contour integral of one about the other.

The previous expression is of interest since it describes the conformal transformation of the primary Φ_h parameterized by the holomorphic vector field ξ:

$$\mathbf{i}[\mathbf{L}[\xi], \Phi_h(w)] = \delta_\xi \Phi_h(w) = (\xi\partial_z + h\partial_z\xi)\Phi.$$

Compare this expression to the general OPE

$$T(z)\Phi_h(0) = \sum_n \frac{1}{z^{n+1}}\mathcal{O}^{(n)}(0)$$

and we determine

$$\begin{cases} \mathcal{O}^{(1)} = & h\Phi_h \\ \mathcal{O}^{(0)} = & \partial\Phi_h \\ \mathcal{O}^{(\text{else})} = & 0 \end{cases}.$$

That is, we have shown that the OPE of the stress tensor with a *primary operator* of dimension Δ is

$$T(z)\mathcal{O}(0) \sim \frac{\Delta\mathcal{O}}{z^2} + \frac{\partial\mathcal{O}}{z}. \tag{6.14}$$

Notice that L_0 is scaling, L_{-1} is translations, and L_1 is special conformal.

Some examples to check: (1) $T(z)\partial\phi(0)$ for the free boson theory. (2) $T(z)T(0)$ for the free boson theory (this one is a trick question as we'll see in 6.2.1).

Not all operators of definite scaling dimension behave this way, and this can be taken as a definition of a primary operator. It implies that under a conformal transformation $(z, \bar{z}) \to (w(z), \bar{w}(\bar{z}))$,

$$\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{O}_{h,\bar{h}}(w(z), \bar{w}(\bar{z}))$$

6.2.1 The Virasoro central charge in 2d CFT

Cardy on the many facets of c.

The OPE of the stress tensor with itself is

$$T(z)T(0) \sim \frac{c_L/2}{z^4} + \frac{2T}{z^2} + \frac{\partial T}{z}. \tag{6.15}$$

(A good way to discover this is to evaluate it for the free scalar or the free fermion case.) The quantity c appearing here is called the *Virasoro central charge*. The word

'central' is because it is a c-number, not an operator. It is a crucial piece of data about the CFT. It can be extracted from

$$\langle T(z)T(0)\rangle = \frac{c_L/2}{z^4} \ .$$

There is also a right-moving central charge which appears in the OPE $\bar{T}\bar{T}$.

For free bosons, with $T(z) = \frac{1}{4\pi K}(\partial\phi)^2$ it is equal to $c_L = 1$ (note that the Ks all cancel out). Notice that it is additive: with N free bosons, the answer is $c_L = N$.

For a majorana fermion, with $T(z) = \chi_L \partial\chi_L$, it is equal to $c_L = \frac{1}{2}$. A basic check of bosonization is that a complex fermion has central charge $\frac{1}{2} + \frac{1}{2} = 1$.

The associated algebra of modes is called the Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}.$$

Notice that $\{L_{-1}, L_0, L_1\} form a closed subalgebra (from which c drops out); including both left and right, \{L_{-1}$ $\{P^\mu, D, M_{\mu\nu}, C^\mu\}$ are the six generators of the global conformal algebra that generalizes to higher dimensions.

It is monotonic under Lorentz-invariant RG flows between CFTs: $c_{UV} > c_{IR}$. Therefore it is a useful measure of the number of degrees of freedom. In fact, using the definition

$$c = \lim_{z\to 0} 2z^4 \langle T(z)T(0)\rangle$$

it can be extended away from RG fixed points; Zamolodchikov proved that this quantity is monotonic along any Lorentz-preserving RG flow. See David Tong's notes for a proof by demanding that the speed of sound is less than the speed of light.

Comparing (6.14) and (6.15) we see that nonzero c means that the stress tensor itself is not a primary. It transforms weirdly under conformal transformations, in a very definite way. This means, for example, that under the map from the cylinder to the plane, the additive normalization of the stress tensor changes. For an application of this to Casimir energy of CFT on a sphere, see the homework.

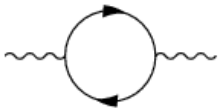
Relatedly, c can be interpreted as a *conformal anomaly* or *Weyl anomaly*.

$$T^\mu_\mu = \frac{c}{24\pi}\mathcal{R}.$$

The relation between this statement and the stress tensor OPE is the same as that between the chiral anomaly in the form $\partial^\mu j^A_\mu = k\frac{F}{2\pi}$ the current-current OPE

$$j(z)j(0) \sim \frac{k}{z^2} \ .$$

In the realization by free fermions, both of these effects come from the 'diangle' diagram:



This singularity in the OPE implies the anomalous nonconservation when coupling to a background gauge field. In the presence of a source coupling to the current (A for j, or curvature for $T_{\mu\nu}$), the statement about the two-point function implies a statement about the one-point function. (See David Tong's notes for a derivation of the Weyl anomaly in 1+1d.)

6.3 Back to general dimensions.

The conformal group in $D > 2$ is finite dimensional, but it still produces powerful constraints.

So we can make our list of operators specifying the CFT much shorter by simply enumerating the primaries (and their dimensions, spins, and structure constants). All the structure constants for the descendants are determined from this data by conformal invariance.

6.3.1 Constraints on correlation functions from CFT

(any D) We consider correlators of primaries; to get correlators of descendants just take derivatives of those of their primaries. We'll also focus for simplicity on scalar operators. A correlation function of N primaries transforms under a conformal transformation as

$$\left\langle \prod_i \Phi_i'(x_i') \right\rangle = \prod_i \Omega(x_i)^{-\Delta_i} \left\langle \prod_i \Phi_i(x_i) \right\rangle .$$

You should regard the operators Φ' on the LHS as the images of Φ under a spacetime-dependent RG transformation.

Suppose we have in our possession a conformal *invariant* I which depends on N spacetime positions.

- Translations imply that I depends only on differences $x_i - x_j$ (of which there are $D(N-1)$).
- Rotations imply that I depends only on distances $r_{ij} \equiv |x_i - x_j|$ of which there are $\frac{N(N-1)}{2}$.

- Scale invariance implies that I depends only on ratios of differences $\frac{r_{ij}}{r_{kl}}$.
- Special conformal transformations act on these distances by

$$(r'_{12})^2 = \frac{r_{12}^2}{\Omega_1 \Omega_2}$$

where

$$\Omega_i \equiv 1 + 2b \cdot x_i + b^2 x_i^2$$

is the factor by which the metric at x_i rescales under the associated transformation. (The easiest way to see this is by doing it for an inversion first, and then using the construction of special conformal transformations as $I \circ T \circ I$. Alternatively, this is where the projective null cone comes into its own.) Only *cross-ratios*

$$\frac{r_{ij} r_{kl}}{r_{ik} r_{jl}}$$

are invariant under this. There are $\frac{N(N-3)}{2}$ of these (and none at all for $N < 4$).

This discussion of invariants has the following implications.

1. One-point functions of primaries of nonzero dimension vanish if there is a conformally invariant vacuum, $\mathbf{D} |0\rangle = 0$. That this is so can be seen as follows. The infinitesimal scale transformation of a primary is

$$\delta\Phi(x) = \left(\Omega^{-\Delta}\Phi(x') - \Phi(x)\right)|_{\Omega = e^{-\lambda}, x' = e^{-\lambda} x} = \lambda \left(\Delta + x^\mu \partial_\mu\right) \Phi(x) \ = \mathbf{i}\lambda[\mathbf{D}, \Phi(x)].$$

   The last step is the statement that the dilatation operator $\mathbf{D}$ generates scale transformations. Therefore:

$$\langle 0| \Phi_\Delta(0) |0\rangle = \frac{\mathbf{i}}{\Delta} \langle 0| [\mathbf{D}, \Phi_\Delta(0)] |0\rangle = 0.$$

2. In CFT, two point functions of primaries are only nonzero if the primaries have the same dimension. This requires conformal invariance; the rest is determined by scale, translation and rotation:

$$\langle 0| \phi_1(z) \phi_2(w) |0\rangle = \frac{\delta_{h_1, h_2}}{r_{12}^{2h_1}}.$$

3.

$$\left\langle \prod_{i=1}^{3} \phi_i(z_i) \right\rangle = C_{123} r_{12}^{2(-h_1 - h_2 + h_3)} r_{23}^{2(-h_2 - h_3 + h_1)} r_{31}^{2(-h_3 - h_1 + h_2)}$$

   C_{123} is the OPE coefficient between the three operators. Notice that this is more than scaling, which just says that the sum of the powers should be $h_1 + h_2 + h_3$.

4.

$$\left\langle \prod_{i=1}^{4} \phi_i(z_i) \right\rangle = f(x_1, x_2) \prod_{i<j} r_{ij}^{-2(h_i+h_j+\sum_k h_k/3)}$$

where x_{12} are the two cross-ratios. In two dimensions, these are

$$x = \frac{z_{12} z_{34}}{z_{13} z_{24}}$$

and $\bar{x}$.

It seems like $N \geq 4$-point functions are underconstrained. However, we have see that the successive use of the OPE relates them (via *known* functions) to sums of 3-point functions. The unknown functions of cross-ratios are actually determined by this process! The general name for such objects is *conformal blocks*.

6.3.2 Thermodynamics of a CFT

Thermodynamics of scale-invariant theories is very constrained. The partition function is defined as

$$Z_{CFT} = \mathrm{Tr}_{CFT} \left(\exp(-H/T) \right).$$

In the thermodynamic limit, $\ln Z$ is *extensive*, *i.e.* proportional to the volume of the space. But $\ln Z$ is a dimensionless quantity. Hence, we must have $\ln Z \sim V T^d$ (d is the number of spatial dimensions) in the absence of any other energy scales (such as a chemical potential for some conserved charge). The free energy then will be

$$F = -T \ln Z = c V T^{d+1}.$$

where this c should be regarded as a rough measure of the number of degrees of freedom of the CFT. In $d = 2$ it is proportional to the Virasoro central charge. In other cases, this statement is less sharp.

As with any conservation law, $T^\mu_\mu = 0$ is an operator equation in the full quantum theory. What happens if we put it inside $\mathrm{Tr}(e^{-H/T})$, with no other insertions? The operator equation then translates into the following equation

$$0 = \mathrm{Tr}(T^\mu_\mu e^{-H/T}) = \langle T_{00} \rangle - \langle T_{ii} \rangle = \mathcal{E} - dP.$$

This last relation gives the speed of the sound of a CFT:

$$c_s = \sqrt{\left(\frac{\partial P}{\partial \mathcal{E}} \right)_S} = \sqrt{\frac{1}{d}} \tag{6.16}$$

6.3.3 High-energy density of states of CFT

Here is a consequence of conformal invariance for the spectrum of a CFT in $D = 1+1$.

Consider the thermal partition function of a $D = 1 + 1$ CFT on a circle of radius L. This can be computed by a path integral on $S^1 \times S^1$ (a 2-torus) where $x \equiv x + L$ and $\tau \equiv \tau + \beta$.

1) Conformal invariance implies that $Z(\beta, L)$ actually only depends on the ratio: $Z = Z(\beta/L)$.[49]

2) The path integral doesn't know which direction you think is time and which one you think is space. Therefore

$$Z(\beta/L) = Z(L/\beta). \tag{6.17}$$

Notice that this requires a rescaling to get back to the same volume, so isn't true in a QFT without scale invariance. Also, we used the Euclidean rotation invariance, *i.e.* Lorentz invariance, so that time and space are equivalent.

This condition of *modular invariance* relates the high-temperature ($T \gg L^{-1}$) behavior to the low-temperature ($T \ll L^{-1}$) behavior.

A brief comment on the fancy name: we can define a more general partition sum where we include a chemical potential for angular momentum:

$$Z(\tau, \bar{\tau}) \equiv \mathrm{tr} e^{-\beta(L_0 + \bar{L}_0 - \frac{\alpha c}{L})} e^{\mu(L_0 - \bar{L}_0)} \equiv \mathrm{tr} q^{L_0 - a} \bar{q}^{\bar{L}_0 - \bar{a}}$$

where $q \equiv e^{2\pi \mathbf{i} \tau}$. This complex parameter τ specifies the shape of a more general torus, where the complex spacetime coordinate is identified by

$$z \equiv z + 1, z \equiv z + \tau.$$

You get the same torus if τ is replaced according to

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in \mathsf{SL}(2, \mathbb{Z}).$$

[49] There is actually some danger in this statement. The conformal anomaly $T^\mu_\mu = \frac{c}{24\pi}\mathcal{R}$ implies that on a general Riemann surface, the partition sum *does* depend on the overall volume, in a way determined by the curvature. More generally, the partition sum on a surface Σ of genus g and overall scale R (this is the radius in the case where $\Sigma = S^2$) satisfies

$$R\partial_R Z_\Sigma(R) = \left\langle \int_\Sigma T^\mu_\mu \right\rangle = \left\langle \int_\Sigma \frac{c}{24\pi}\mathcal{R} \right\rangle \overset{\text{Gauss-Bonnet}}{=} \frac{c}{24\pi}(2 - 2g).$$

which we can integrate to find the volume dependence. But in the case of a torus ($g = 1$), the curvature integrates to zero and the answer does not depend on the volume.

3) On the homework, you showed that the funny transformation law of the stress tensor under a conformal transformation determines the vacuum energy on a circle in terms of the central charge:

$$H_{\text{CFT on } S^1} = L_0 + \bar{L}_0 - \alpha \frac{c}{L},$$

where the constant $\alpha = \frac{\pi^2}{3}$. The groundstate is the state $|0\rangle = |\mathbb{1}\rangle$ corresponding to the identity, which has $L_0 |\mathbb{1}\rangle = \bar{L}_0 |\mathbb{1}\rangle = 0$, so the groundstate energy is $E_0 = -\alpha \frac{c}{L}$.

This means that

$$Z(x = \beta/L) \overset{x\to\infty}{\simeq} e^{-\beta E_0} + \cdots = e^{+\frac{\beta\alpha c}{L}} + \cdots$$

Using (6.17), we learn that the free energy at high temperature is then

$$Z \overset{\beta/L\to 0}{\simeq} e^{\frac{+L\alpha c}{\beta}}.$$

That is, the free energy at high temperature is

$$F(T) = -T \log Z \overset{T\gg L}{\simeq} \alpha c L T^2. \tag{6.18}$$

The fact that it goes like T^2 is determined by dimensional analysis as we saw above, but here the exact coefficient is determined in terms of the central charge, c.

The result is usually stated in terms of the microcanonical entropy, $S(E)$, related to (6.18) by a Legendre transformation. If $F(T) = \gamma L^d T^{d+1}$, then

$$E = F + TS|_{S=-\partial_T F} = d\gamma L^d T^d \implies T = \left(\frac{E}{d\gamma L^d}\right)^{1/d}$$

which says

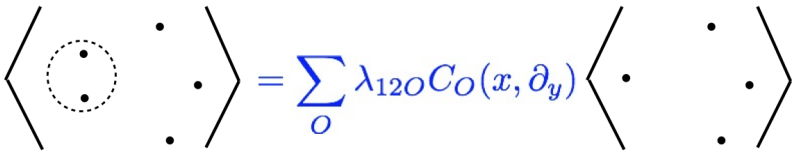
$$S(E) = -\partial_T F = \frac{(d+1)\gamma L^d}{(d\gamma L^d)^{1/d}} E^{\frac{d}{d+1}}.$$

In $D = 2$ γ is determined by the central charge.

This result is due to Cardy, and was more recently made rigorous using techniques from analytic number theory.

6.3.4 A few words about the Conformal Bootstrap

Earlier I said that given a 'solution' to a CFT in the form of a list of primaries and their dimensions and OPE structure constants, you could compute any n-point correlator by repeated use of the OPE to reduce it to a two-point function, by the following kind of operation:



[from Rychkov's CFT notes]

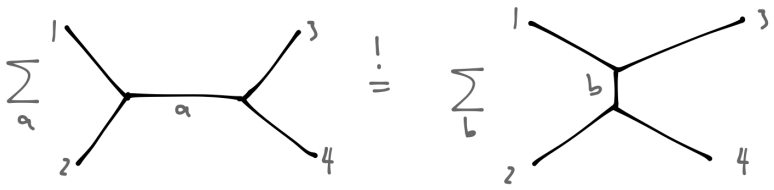
Some interesting questions and previews of the answers:

(1) Which sets of scaling dimensions arise from CFTs which actually exist?

The general answer is not known. One set of constraints follows from unitarity in the form of positivity of the inner product on the CFT Hilbert space. For example, the dimension of a scalar operator (other than $\mathbb{1}$) must be larger than the free field dimension $(\frac{D-2}{2})$[50].

(2) In implementing the reduction to three-point functions and structure constants, you have a choice about the order in which you group the operators. Do you get the same answer independent of the order?

The condition that you do – the associativity of the OPE – imposes significant constraints on the structure constants and dimensions.



(3) Are there more constraints from higher-order diagrams?

Claim: no. The analog of the 'pentagon identity' is automatic.

To make use of the (many!) associativity constraints, a further ingredient is required, which is called *conformal blocks*. For more in this direction, a good starting point is Rychkov and the longer review by Simmons-Duffin.

In $D = 2$ it really works [BPZ]. This approach has led to complete solutions of all unitary CFTs with $c < 1$ (specifically, unitary representations of Virasoro with $c < 1$

[50] This follows from the fact that $\mathbf{D}$ appears on the RHS of $[\mathbf{C}_\mu, \mathbf{P}_\nu]$. A crucial ingredient comes from the fact that the dagger of a lowering operator is a raising operator (obvious, right?). In radial quantization this implies the weird-looking formula:

$$\mathbf{P}_\mu^\dagger = \mathbf{C}_\mu \ .$$

For more details, I recommend the discussion leading up to eqn (5.57) of Jared Kaplan's AdS/CFT notes.

requires $c = 1 - \frac{6}{m(m+1)}|_{m=3,4,5\ldots} = \frac{1}{2}, \frac{4}{5}, \frac{7}{8}\cdots$, and very specific values of the scaling dimensions of primaries, $m = 3$ is the Ising model; this is explained in Ginsparg's notes), and a number of interesting examples (*e.g.* Liouville theory) with $c > 1$. More recently, this program has had success in $2 + 1$ dimensions: the 3d Ising model has been cornered.

[End of Lecture 21]

7 Holographic duality

Quantum gravity is different. For three quarters now we've been talking about field theories, which you'll recall I defined as systems with extensive degrees of freedom. In a system with extensive degrees of freedom:

(total # of possible states) is proportional to (# possible states of each site)$^{(\#\text{ of sites})} \sim 2^V$

The number of sites V (the volume of space in units of the lattice spacing) is something bigger than Avogadro's number 10^{23}, so these are some catastrophically big numbers. To make the numbers a bit more manageable let's take the logarithm of both sides:

$$\log(\#\text{ of states}) \sim V \log 2.$$

This quantity appears in the thermodynamics of the system. Recall that the *entropy* is the log of the number of states consistent with the information we have about the state of the system:

$$S(F) = \log(\#\text{ of states with macroscopic feature } F) \leq S_{\max} \sim V \log 2.$$

The biggest the entropy can be is the log of the total number of states of the system. The conclusion is that in a QFT, the maximum entropy goes like the volume of space.

The main point I want to convey in this final subsection is that quantum gravity is not like this. Here is a rough four-step argument for this crazy claim (due to 't Hooft and Susskind in the early 1990s). The key ingredient is:

Black Hole Thermodynamics.

1. A black hole is an object from which there is (classically) no escape, because the escape velocity is larger than the speed of light. The region from which there is no escape is bounded by the black hole's *event horizon*. Their existence is a direct consequence of (the universally attractive nature of) gravity.

2. If you throw together enough stuff in a small-enough region, you will make a black hole.

3. Consistent laws of thermodynamics in the presence of gravity require that we assign a black hole an entropy. The idea is: if we didn't assign an entropy to a black hole, we could violate the second law of thermodynamics and build perpetual-motion machines. We could do this just by throwing all our trash into a black hole. Since the high-entropy trash would just be gone, doing so would reduce the entropy of the world.

The entropy we must assign to a black hole in Einstein gravity is

$$S_{\rm BH} = \frac{\text{area of horizon}}{4l_P^2},$$

$l_P \equiv \sqrt{\frac{G_N\hbar^2}{c^3}}$ is the *Planck length*, the intrinsic length scale of the gravitational interactions.

With this realization (due to Bekenstein and Hawking), there is a consistent set of laws of thermodynamics including black holes. The second law is generalized to the statement that $S + S_{\rm BH}$ doesn't decrease (with overwhelming probability).

4. Now, let's ask: what is the maximum entropy in a gravitating system? I claim that

$$S_{\rm max}(\text{stuff in a region of space with surface area } A \text{ in a gravitating system}) = S_{\rm BH}(\text{biggest}) = \frac{A}{4l_P^2},$$

where $S_{\rm BH}$(biggest) is the entropy of the biggest black hole that fits in the region. The idea is: suppose this were not the case, *i.e.* we have a collection of stuff in the region in question with $E_{\rm stuff} < M_{\rm BH}$ and $S_{\rm stuff} > S_{\rm BH}$. Now throw some trash into the region. This increases the energy at fixed volume, so increases the density. If we do it enough, the system will gravitationally collapse, and form a black hole. But then the resulting entropy would be $S_{\rm final} \leq S_{\rm BH} \leq S_{\rm initial}$. In this way we would could violate the second law of thermodynamics, and build perpetual motion machines and save the world from the infestation of humans. So that means we probably can't do this.

But now recall that the maximum entropy is

$$S_{\rm max} \propto \#\text{ of dofs.}$$

We've just shown that for a system with black holes

$$S_{\rm max} \propto \#\text{ of dofs} \propto \frac{\text{area}}{l_P^2} \ll \frac{\text{volume}}{l_P^3}$$

where the last inequality is true for regions large enough compared to the Planck length.

This means that gravity has the same number of degrees of freedom as an ordinary system (field theory) living on the *boundary* of space. This is called the *Holographic Principle*.

This is a general principle about gravity, and makes us really want to know: *who* is this field theory living on the boundary of space? In general we don't know the answer.

There is one set of examples where we do know, which is for gravity with a negative cosmological constant. (In our world we have a very teeny positive cosmological constant.) In this case, solving Einstein's equations shows that space has a naturally-occurring boundary, which is frozen in a sense. Near this boundary, the geometry is anti-de Sitter space. Gravity in asymptotically-anti-de Sitter space (AdS) *is* a special kind of field theory (called a conformal field theory (CFT)) living on the one-lower-dimensional boundary. This AdS/CFT correspondence was discovered using string theory, and we know lots of examples and can check it in lots of ways. It is an explicit realization of the holographic principle.

I mentioned that black holes have an entropy. This means they must also have a temperature. But a hot body in outer space will radiate. This means that black holes evaporate, and it seems that all that is left is some thermal radiation labelled by the temperature. Where does the information about what made the black hole go when this happens? This is the black hole information problem.

There can be black holes in AdS, which (if they're small enough) behave just like black holes in flat space, and will evaporate. But that whole process is described by the dual CFT, which is an ordinary quantum system with unitary time evolution, from which information cannot disappear. So the AdS/CFT correspondence tells us the resolution of the black hole information problem (the information does not go away), but doesn't tell us exactly how it is resolved.

A word about *entanglement*: in an ordinary system (a field theory), the Hilbert space is a *tensor product* of the Hilbert spaces of the individual degrees of freedom. For example, the Hilbert space of one spin is $\mathcal{H}_2 = \text{span}\{|0\rangle, |1\rangle\}$, the vector space spanned by these two vectors, *i.e.*all linear combinations of the form $a_0 |0\rangle + a_1 |1\rangle$. If I have two spins, the Hilbert space is

$$\mathcal{H}_2 \otimes \mathcal{H}_2 = \text{span}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}.$$

The dimension of the Hilbert space for N spins is 2^N, and this is the number we were counting above.

A state of two spins of the form can be a product state like $|00\rangle$ or $|11\rangle$, or it can be some superposition like $|00\rangle + |11\rangle$. The latter state is said to be entangled. The *entanglement entropy* is a way to quantify this[51].

[51] If $\mathcal{H}_A = \text{span}\{|a\rangle, a = 1..N_A\}$ and $\mathcal{H}_B = \text{span}\{|b\rangle, b = 1..N_B\}$, then a general state of $\mathcal{H}_A \otimes \mathcal{H}_B$ is of the form

$$|w\rangle = \sum_{a=1}^{N_A} \sum_{b=1}^{N_B} w_{ab} |a\rangle \otimes |b\rangle .$$

So all the information is in the matrix w_{ab}. The entanglement entropy of A (or of B) is $S = -\sum \lambda^2 \log \lambda^2$ where $\{\lambda\}$ are the eigenvalues of the matrix w.

If we have degrees of freedom spread over space, we can ask about the entanglement of some region with its complementary region.

But notice that the very notion of entanglement between regions of space required the degrees of freedom to be extensive, as in a field theory. So we don't really know what we mean by the entanglement of a region in a theory of gravity.

The temperature of a black hole. A black hole in $D = 3 + 1$ looks like

$$ds^2_{\rm Sch} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ . \tag{7.1}$$

$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the unit 2-sphere. The event horizon is located at a zero of the emblackening factor $f(r)$. For a generic (non-extremal) black hole, $f(r) \overset{r\sim r_H}{\sim} 2\kappa(r - r_H)$ has a single zero at the horizon (this defines the *surface gravity* κ), and the metric near the horizon can be rewritten in the form:

$$\begin{aligned} ds^2_{\rm Sch} &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \\ &\approx \kappa^2 R^2 dt^2 + dR^2 + r(R)^2 d\Omega^2 \\ &= -R^2 d\eta^2 + dR^2 + \dots \\ &\approx -dT^2 + dZ^2 + dX^2 + dY^2 \ . \end{aligned}$$

For definiteness I used the form of the Schwarzschild black hole in $D = 4$: $f = 1 - r_H/r$, $r_H = 2GM$, $R(r) = \sqrt{r(r - 2GM)} + \dots$, $\kappa = 1/4GM$, $\eta = \kappa t$, $T = R\sinh\eta$ and $Z = R\cosh\eta$. Notice that after the transformation, we obtain the Minkowski space $\mathbb{R}^{3,1}$. Rindler space is Minkowski space in the coordinate frame of a uniformly accelerating observer, whose worldline sits at fixed R. η is called the Rindler time; you can see from the last step that it is the rapidity from the point of view of the final Minkowski space. What we get is shown in Fig. 12. Notice that the two lines defined by $\{r = 2GM, \eta = \infty\}$ and $\{r = 2GM, \eta = -\infty\}$ divide the space into four regions { I, II, III, IV }.

The simplest way to see that a black hole has a temperature is to think about the euclidean geometry. Let $\theta = -\mathbf{i}\eta$. Then the metric is

$$ds^2 = R^2 d\theta^2 + dR^2 + ...$$

which is just flat space in polar coordinates. But this space is only actually flat if $\theta \equiv \theta + 2\pi$. Otherwise $R = 0$ is the tip of a cone, at which there is localized curvature. Such curvature does not solve the vacuum Einstein equations. So the black hole requires euclidean time to be periodic with $\tau = -\mathbf{i}t \equiv \tau + 2\pi\kappa$. The periodicity of euclidean time is $\beta = 1/T$ (recall that the thermal partition function is $\mathrm{tr} e^{-\beta H}$, and $e^{-\beta H}$ generates a

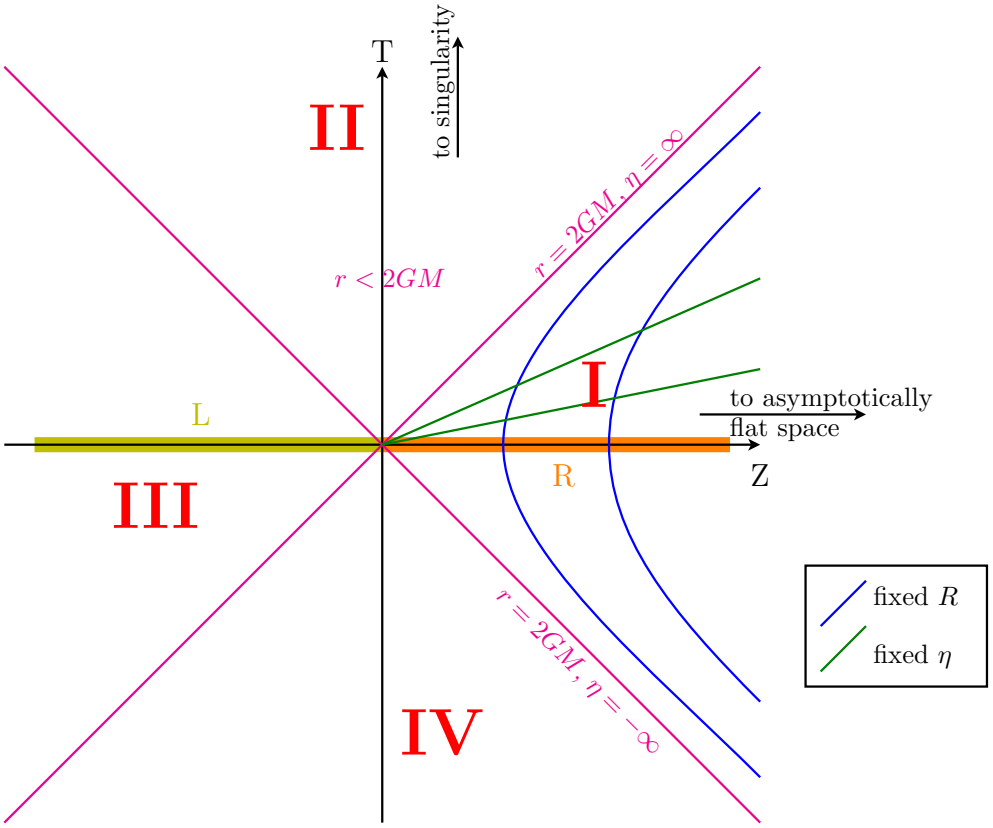


Figure 12: The geometry of the near-horizon region of a generic black hole is Rindler space. [Figure by Wing-Ko Ho]

translation in euclidean time by an amount β), so $T = \frac{1}{2\pi\kappa}$ is the temperature of the black hole.

We can understand this in more detail as follows. Region I is the region outside the horizon of the black hole. The key observation is that region I is self-contained. That is, region II and III can't communicate with it, while information from region IV passes through the line $\{r = 2GM, \eta = -\infty\}$ and hence corresponds to *initial data*. Therefore, at $T = 0$, the degrees of freedom at $Z < 0$ (marked by L on the figure) have no effect on region I. Thus we should trace over them when computing stuff in region I.

Now, Claim: Let $|\text{g.s.}\rangle$ be the ground state of any quantum field theory in Minkowski space, and let $T_{\mu\nu}$ be its stress-energy tensor. Its reduced density matrix on region I is

$$\rho_R = \text{tr}_L |\text{g.s.}\rangle\langle\text{g.s.}| = \frac{1}{\mathcal{Z}} e^{-2\pi H_R}$$

where H_R is the Rindler Hamiltonian:[52]

$$H_R = \int_{\text{constant } T} d^d x T_{\eta T}$$

and $\mathcal{Z} = \text{tr}_R\, e^{-2\pi H_R}$.

Notice that the density matrix we obtained is not a pure state but is entangled with the L d.o.f. (hence $S = -\text{tr}\rho_R \ln \rho_R \neq 0$). Much more specifically, it is a thermal density matrix with temperature $T_{\text{Rindler}} = 1/2\pi$.

Proof of Claim [Unruh, Bisognano-Wichmann 1975-6]. We will write the argument for a scalar field ϕ, but the argument is general. So consider a scalar field ϕ in Rindler space. A complete set of commuting d.o.f's at $T = 0$ is

$$\phi(x, y, z) = \begin{cases} \phi_R(x, y, z) & \text{for } z > 0 \\ \phi_L(x, y, z) & \text{for } z < 0 \end{cases}$$

ϕ_R and ϕ_L commute because they are at spacelike separated points. Any wavefunctional of the field ϕ can then be written as $\Psi = \Psi[\phi_L, \phi_R] = \langle \phi_L \phi_R | \Psi \rangle$.

The ground state is just the Minkowski space vacuum. The Feynman-Kac formula gives us a path integral representation of its wavefunctional:

$$\Psi_{\text{g.s.}}[\phi_L, \phi_R] = \frac{1}{\sqrt{\mathcal{Z}}} \int_{x^0>0, \phi(\vec{x}, x^0=0)=(\phi_L, \phi_R)} [d\phi] e^{-S_{\text{Eucl}}[\phi]}$$

This equation is written using constant-$x^0 = -iT$ time slices. We can gain some insight by instead slicing up the path integral by constant euclidean Rindler time $\theta (= -i\eta)$ (see Fig. 13(a) for illustration); euclidean Rindler time is just the angular coordinate in the x^0, Z plane. This yields:[53]

$$\Psi_{\text{g.s.}}[\phi_L, \phi_R] = \frac{1}{\sqrt{\mathcal{Z}}} \int_{\substack{0<\theta<\pi \\ \phi(\theta=0)=\phi_R \\ \phi(\theta=\pi)=\phi_L \\ \partial_R \phi(R=0)=0}} [d\phi] e^{-S_{\text{Eucl}}[\phi]} = \frac{1}{\sqrt{\mathcal{Z}}} \langle \phi_L | \, e^{-\pi H_R} \, | \phi_R \rangle \tag{7.2}$$

The last condition $\partial_R \phi(R = 0) = 0$ is simply a requirement that the function ϕ is regular at the origin. We identify the RHS of (7.2) $\langle \phi_L | \, e^{-\pi H_R} \, | \phi_R \rangle$ as the transition amplitude.

[52] Note that $[H_R] = 1$ since the Rindler time is dimensionless $[\eta] = 1$.

[53] We are transforming the variable that the function takes, rather than the function itself, and it is the function that's being integrated. Thus there is no Jacobian involved here.

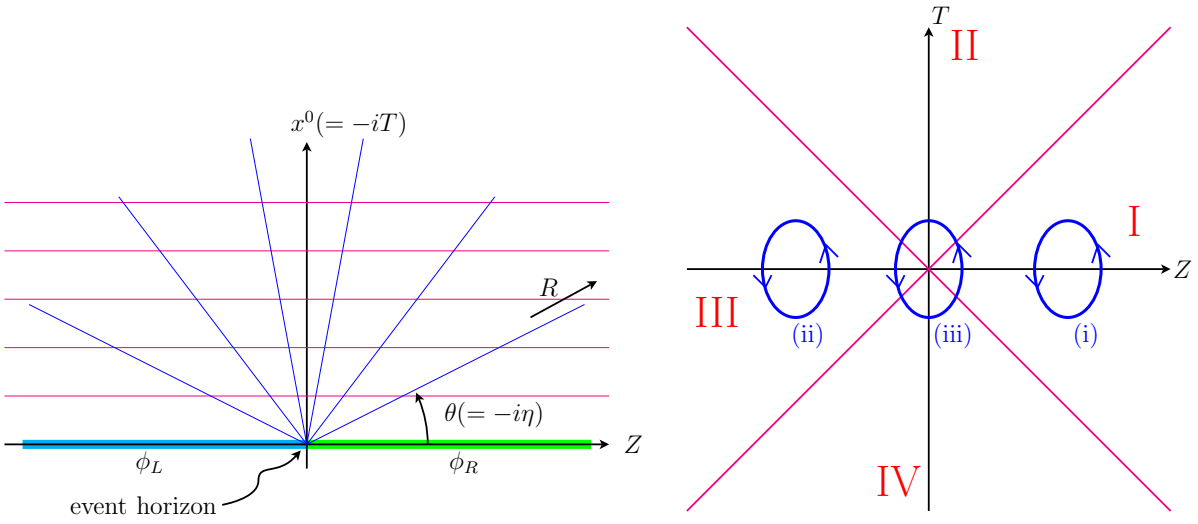


Figure 13: (a) Change in slicing sequence. The magenta lines are the old x^0 slices while the blue lines are the new θ slices. (b) Fluctuation in different regions of the Rindler space. [Figures by Wing-Ko Ho]

Hence the matrix elements of $\rho_R = \mathrm{tr}_L|\mathrm{gs}\rangle\langle\mathrm{gs}|$ are

$$\begin{aligned}\langle\phi_R|\,\rho_R\,|\phi'_R\rangle &= \int [d\phi_L]\Psi^*[\phi_L,\phi_R]\Psi[\phi_L,\phi'_R] \\ &= \left(\frac{1}{\sqrt{\mathcal{Z}}}\right)^2 \int [d\phi_L]\,\langle\phi_R|\,e^{-\pi H_R}\,|\phi_L\rangle\,\langle\phi_L|\,e^{-\pi H_R}\,|\phi'_R\rangle \\ &= \frac{1}{\mathcal{Z}}\,\langle\phi_R|\,e^{-2\pi H_R}\,|\phi'_R\rangle\end{aligned}$$

■

Some results and comments:

1. Forgetting for a moment about the motivation from black holes, this shows that a constantly accelerated observer[54] in $\mathbb{R}^{n,1}$ sees a *Unruh radiation* with temperature $T_{\text{proper}} = \dfrac{1}{2\pi R} = \dfrac{a}{2\pi}$. (The factor of R comes from the fact that the proper time is $\tau = R\eta$, so $\partial_\tau = \frac{1}{R}\partial_\eta = \frac{1}{R}H_R$.)

[54]Here a is the constant acceleration, defined by $a^2 \equiv g^{\mu\nu}a_\mu a_\nu$, where $a^\mu = \partial_\tau^2 X^\mu$ is the proper acceleration vector, and τ is the proper time along the worldline. A uniformly accelerating observer follows a trajectory $T = R\sinh\eta, Z = R\cosh\eta$ with fixed R, and the proper time satisfies $d\tau^2 \equiv -ds^2 = R^2 d\eta^2$ along the worldline. This equation shows that $a = \frac{1}{R}$. Thanks to Bowen Shi for pointing out an error here.

2. By the equivalence principle, the same happens outside a black hole. The corresponding Schwarzschild time is $t = \eta/\kappa$, which implies that $T_{\rm BH} = T_{\rm Rindler} = \dfrac{\kappa}{2\pi}$. This determines the temperature of the black hole in terms of the horizon radius, or equivalently in terms of its mass, and fixes the proportionality constant in the relation between entropy and area.

3. Where does the thermal radiation come from? Very heuristically, there are three types of fluctuations (see Fig. 13(b)):
(i) is just ordinary pair production fluctuation, with lifetime $\propto \hbar/E$
(ii) is something that we need not care about, if we live in region I
(iii) corresponds to stuff that enters at $\eta = -\infty, R = 0$ and falls back at $\eta = +\infty, R = 0$. In Schwarzschild time this particle stays forever. Hence this particle is real.

4. What we have calculated is the density matrix of a quantum field in a neighborhood outside $r \sim r_H$. Not all energy will get to ∞ because the black hole is like a potential well. The rate of particle emission includes what is called a *greybody factor* to account for this. The more complicated calculation done by Hawking includes this factor, but has the huge drawback of only applying to free fields.

5. *Information paradox.* A black hole can form from a dictionary or worse a pure quantum state, and then (apparently) evaporate into thermal radiation. The thermal density matrix is determined by one number only.
Q1. Is this a unitarity evolution?
Q2. Entropy results from coarse-graining. What are the microstate?
Q3. How is the information stored in the black hole?
Via AdS/CFT it is possible to answer Q1 and Q2.

Holographic duality. [This is a highly condensed version of these notes, which in turn are a condensed version of this course. The logic is from here.]
Bold Assertion:
(a) Some ordinary quantum field theories (QFTs) are secretly quantum theories of gravity.
(b) Sometimes the gravity theory is classical, and therefore we can use it to compute interesting observables of the QFT.

Here are three facts which make the Assertion seem less unreasonable.

1) First we must define what we mean by a quantum gravity (QG). As a working definition, let's say that a QG is a quantum theory with a dynamical metric. In enough dimensions, linearizing equations of motion (EoM) for a metric usually reveals

a propagating mode of the metric, some spin-2 massless particle which we can call a 'graviton'.

So at the least the assertion must mean that there is some spin-two graviton particle that is somehow a composite object made of QFT degrees of freedom. This statement seems to run afoul of the Weinberg-Witten no-go theorem, which says:
Theorem [Weinberg-Witten]**:** A QFT with a Poincaré covariant conserved stress tensor $T^{\mu\nu}$ forbids massless particles of spin $j > 1$ which carry momentum (*i.e.* with $P^\mu = \int d^D x T^{0\mu} \neq 0$).

You may worry that the assumption of Poincaré invariance plays an important role in the proof, but the set of QFTs to which the Bold Assertion applies includes relativistic theories.

Like any good no-go theorem, it is best considered a sign pointing away from wrong directions. The loophole in this case is blindingly obvious in retrospect: the graviton needn't live in the same spacetime as the QFT.

2) is the Holographic Principle described above. This suggests that the gravity theory should live in a spacetime with an extra dimension.

3) A beautiful hint as to the possible identity of the extra dimensions is this. Wilson taught us that a QFT is best thought of as being sliced up by length (or energy) scale, as a family of trajectories of the renormalization group (RG). A remarkable fact about this is that the RG equations for the behavior of the coupling constants as a function of RG scale u are *local* in scale:

$$u\partial_u g = \beta(g(u)) \ .$$

The beta function is determined by the coupling constant evaluated at the energy scale u, and we don't need to know its behavior in the deep UV or IR to figure out how it's changing. This fact is basically a consequence of locality in ordinary spacetime. This opens the possibility that we can associate the extra dimension suggested by the Holographic idea with energy scale. This notion of locality in the extra dimension actually turns out to be much *weaker* than what we will find in AdS/CFT, but it is a good hint.

To summarize, we have three hints for interpreting the Bold Assertion:

1. The Weinberg-Witten theorem suggests that the graviton lives on a different space than the QFT in question.

2. The holographic principle says that the theory of gravity should have a number of degrees of freedom that grows more slowly than the volume. This suggests that the quantum gravity should live in more dimensions than the QFT.

3. The structure of the Renormalization Group suggests that we can identify one of these extra dimensions as the RG-scale.

Clearly the field theory in question needs to be strongly coupled. Otherwise, we can compute and we can see that there is no large extra dimension sticking out.

Next we will make a simplifying assumption in an effort to find concrete examples. The simplest case of an RG flow is when $\beta = 0$ and the system is self-similar. In a Lorentz invariant theory (which we also assume for simplicity), this means that the following scale transformation $x^\mu \to \lambda x^\mu$ ($\mu = 0, 1, 2, ...d-1$) is a symmetry. If the extra dimension coordinate u is to be thought of as an energy scale, then dimensional analysis says that u will scale under the scale transformation as $u \to \frac{u}{\lambda}$. The most general $(d+1)$-dimensional metric (one extra dimension) with this symmetry and Poincaré invariance is of the following form:

$$ds^2 = \left(\frac{\tilde{u}}{\tilde{L}}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{d\tilde{u}^2}{\tilde{u}^2} L^2 \quad .$$

We can bring it into a more familiar form by a change of coordinates, $\tilde{u} = \frac{\tilde{L}}{L} u$:

$$ds^2 = \left(\frac{u}{L}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} L^2 \quad .$$

This is AdS_{d+1}[55]. It is a family of copies of Minkowski space, parametrized by u, whose size varies with u (see Fig. 14). The parameter L is called the 'AdS radius' and it has dimensions of length. Although this is a dimensionful parameter, a scale transformation $x^\mu \to \lambda x^\mu$ can be absorbed by rescaling the radial coordinate $u \to u/\lambda$ (by design); we will see below more explicitly how this is consistent with scale invariance of the dual theory. It is convenient to do one more change of coordinates, to $z \equiv \frac{L^2}{u}$, in which the metric takes the form

$$ds^2 = \left(\frac{L}{z}\right)^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right) \quad . \tag{7.3}$$

These coordinates are better because fewer symbols are required to write the metric. z will map to the length scale in the dual theory.

So it seems that a d-dimensional[56] conformal field theory (CFT) should be related to a theory of gravity on AdS_{D+1}. This metric (7.3) solves the equations of motion of

[55] It turns out that this metric also has conformal invariance. So scale and Poincaré symmetry implies conformal invariance, at least when there is a gravity dual. This is believed to be true more generally (see Polchinski's 1987 paper), but there is no proof for $D > 1+1$. Without Poincaré invariance, scale invariance definitely does *not* imply conformal invariance; indeed there are scale-invariant metrics without Poincaré symmetry, which do not have have special conformal symmetry.

[56] Note that I am forced to violate my convention for d and D in either the bulk or the boundary of this correspondence. Below I use D the number of spacetime dimensions of the boundary.

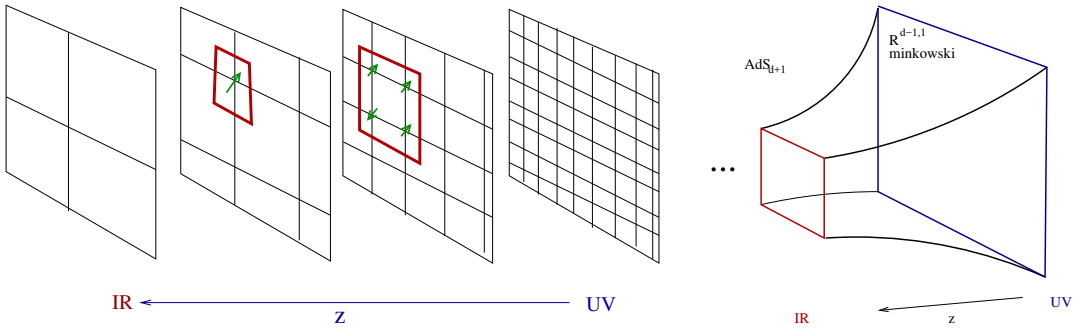


Figure 14: The extra ('radial') dimension of the bulk is the resolution scale of the field theory. The left figure indicates a series of block spin transformations labelled by a parameter z. The right figure is a cartoon of AdS space, which organizes the field theory information in the same way. In this sense, the bulk picture is a hologram: excitations with different wavelengths get put in different places in the bulk image.

the following action (and many others)[57]

$$S_{\text{bulk}}[g,\dots] = \frac{1}{16\pi G_N}\int d^{D+1}x\sqrt{g}\,(-2\Lambda + \mathcal{R} + \dots) \quad . \tag{7.4}$$

Here, $\sqrt{g} \equiv \sqrt{|\det g|}$ makes the integral coordinate-invariant, and $\mathcal{R}$ is the Ricci scalar curvature. The cosmological constant Λ is related by the equations of motion

$$0 = \frac{\delta S_{\text{bulk}}}{\delta g^{AB}} \quad \Longrightarrow \quad R_{AB} + \frac{D}{L^2}g_{AB} = 0 \tag{7.5}$$

to the value of the AdS radius: $-2\Lambda = \frac{D(D-1)}{L^2}$. This form of the action (7.4) is what we would guess using Wilsonian naturalness (*i.e.*, the 'Landau-Ginzburg-Wilson paradigm' or the EFT strategy): we include all the terms that respect the symmetries and redundancies (in this case, this is general coordinate invariance), organized by decreasing relevantness, *i.e.* by the number of derivatives. The Einstein-Hilbert term (the one with the Ricci scalar) is an irrelevant operator: $\mathcal{R} \sim \partial^2 g + (\partial g)^2$ has dimensions of length^{-2}, so G_N here is a length^{D-1}, the Planck length: $G_N \equiv \ell_{pl}^{D-1} \equiv M_{pl}^{1-D}$ (in units where $\hbar = c = 1$). The gravity theory is classical if $L \gg \ell_{pl}$. In this spirit, the $\dots$ on the RHS denote more irrelevant terms involving more powers of the curvature. Also hidden in the $\dots$ are other bulk fields that vanish in the dual of the CFT vacuum (*i.e.* in the AdS solution).

This form of the action (7.4) is indeed what comes from string theory at low energies and when the curvature (here, $\mathcal{R} \sim \frac{1}{L^2}$) is small (compared to the *string tension*, $\frac{1}{\alpha'} \equiv \frac{1}{\ell_s^2}$; this is the energy scale that determines the masses of excited vibrational

[57] For verifying statements like this, it can be helpful to use Mathematica or some such thing.

modes of the string), at least in cases where we are able to tell. The main role of string theory in this business (at the moment) is to provide consistent ways of filling in the dots.

But wait: in a theory of gravity, the space-time metric is a dynamical variable, and we only get to specify the boundary behavior. The AdS metric above has a boundary at $z = 0$. This is a bit subtle. Keeping x^μ fixed and moving in the z direction from a finite value of z to $z = 0$ is actually infinite distance. However, massless particles in AdS (such as the graviton discussed above) travel along null geodesics; these reach the boundary in finite time. This means that in order to specify the future evolution of the system from some initial data, we have also to specify boundary conditions at $z = 0$. These boundary conditions will play a crucial role in the discussion below.

So we should amend our statement to say that a D-dimensional conformal field theory is related to a theory of gravity on spaces which are *asymptotically* AdS_{D+1}. Note that this case of negative cosmological constant (CC) turns out to be much easier to understand holographically than the naively-simpler (asymptotically-flat) case of zero CC, which has a null boundary. Let's not even talk about the case of positive CC (asymptotically de Sitter), which has a boundary in the future.

Different CFTs will correspond to such theories of gravity with different field content and different bulk actions, *e.g.* different values of the coupling constants in $S_{\rm bulk}$. The example which is understood best (and was found first) is the case of the $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in four dimensions. This is dual to maximal supergravity in AdS_5 (which arises by dimensional reduction of ten-dimensional IIB supergravity on $AdS_5 \times S^5$). In that case, we know the precise values of many of the coefficients in the bulk action. This will not be very relevant for our discussion below. An important conceptual point is that the values of the bulk parameters which are realizable will in general be discrete[58]. This discreteness is hidden by the classical limit.

We will focus on the case of relativistic CFT, but let me emphasize here that the name 'AdS/CFT' is a very poor one: the correspondence is much more general. It can describe deformations of UV fixed points by relevant operators, and it has been extended to cases which are not even relativistic CFTs in the UV: examples include fixed points with dynamical critical exponent $z \neq 1$, Galilean-invariant theories and theories which do more exotic things in the UV like the 'duality cascade'.

Counting of degrees of freedom. [Susskind-Witten] We can already make a check of the conjecture that a gravity theory in AdS_{D+1} might be dual to a QFT in D dimensions. The holographic principle tells us that the area of the boundary in Planck

[58] An example of this is the relationship (7.9) between the Newton constant in the bulk and the *number* of species in the field theory, which we will find next.

units is the number of degrees of freedom (dof), *i.e.* the maximum entropy:

$$\frac{\text{Area of boundary}}{4\text{G}_\text{N}} \stackrel{?}{=} \text{number of dof of QFT} \equiv N_d \quad .$$

Is this true? Yes: both sides are equal to infinity. We need to regulate our counting.

Let's regulate the field theory first. There are both UV and IR divergences. We put the thing on a lattice, introducing a short-distance cut-off ϵ (*e.g.*, the lattice spacing) and we put it in a cubical box of linear size R. The total number of degrees of freedom is the number of cells $\left(\frac{R}{\epsilon}\right)^{d-1}$, times the number of degrees of freedom per lattice site, which we will call 'N^2'. The behavior suggested by the name we have given this number is found in well-understood examples. It is, however, clear (for example from the structure of known AdS vacua of string theory) that other behaviors N^b are possible, and that's why I made it a funny color and put it in quotes. So $N_d = \frac{R^d}{\epsilon^d} N^2$.

The picture we have of AdS_{D+1} is a collection of copies of d-dimensional Minkowski space of varying size; the boundary is the locus $z \to 0$ where they get really big. The area of the boundary is

$$A = \int_{\mathbb{R}^{d-1},\ z\to 0,\ \text{fixed } t} \sqrt{g} d^d x = \int_{\mathbb{R}^{D-1},\ z\to 0} d^d x \frac{L^{D-1}}{z^d} \quad . \tag{7.6}$$

As in the field theory counting, this is infinite for two reasons: from the integral over x and from the fact that z is going to zero. To regulate this integral, first put $x \simeq x + R$ in a box again, and second evaluate not at $z = 0$ but rather cut it off at $z = \epsilon$. This idea is that the boundary of AdS is associated with the UV behavior of the field theory, and that cutting off the geometry at $z = \epsilon$ is a UV cutoff (not identical to the lattice cutoff, but close enough for our present purposes). Given this,

$$A = \int_0^R d^d x \frac{L^d}{z^d}|_{z=\epsilon} = \left(\frac{RL}{\epsilon}\right)^d \quad . \tag{7.7}$$

The holographic principle then says that the maximum entropy in the bulk is

$$\frac{A}{4G_N} \sim \frac{L^d}{4G_N}\left(\frac{R}{\epsilon}\right)^d . \tag{7.8}$$

We see that the scaling with the system size agrees – the both-hand-side goes like R^d. So AdS/CFT is indeed an implementation of the holographic principle. We can learn more from this calcluation: In order for the prefactors of R^d to agree, we need to relate the AdS radius in Planck units $\frac{L^d}{G_N} \sim (LM_{pl})^d$ to the number of degrees of freedom per site of the field theory:

$$\boxed{\frac{L^d}{G_N} = N^2} \tag{7.9}$$

up to numerical prefactors.

An important conclusion from this calculation is that the gravity theory is classical $L \gg \ell_P$ precisely when the number of degrees of freedom at each point of space in the QFT, N^2, is large.

Preview of the AdS/CFT correspondence. Here's the ideology:

$$\begin{array}{ccc} \text{fields in AdS} & \longleftrightarrow & \text{local operators of CFT} \\ \text{spin} & & \text{spin} \\ \text{mass} & & \text{scaling dimension } \Delta \end{array}$$

In particular, for a scalar field in AdS, the formula relating the mass of the scalar field to the scaling dimension of the corresponding operator in the CFT is $m^2 L_{AdS}^2 = \Delta(\Delta - D)$.

One immediate lesson from this formula is that a simple bulk theory with a small number of light fields is dual to a CFT with a hierarchy in its spectrum of operator dimensions. In particular, there need to be a small number of operators with small (*e.g.* of order N^0) dimensions. (If you are aware of explicit examples of such theories, please let me know.) This is to be distinguished from the thus-far-intractable case where some whole tower of massive string modes in the bulk are needed.

Now let's consider some observables of a QFT (we'll assume Euclidean spacetime for now), namely vacuum correlation functions of local operators in the CFT:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle \quad .$$

We can write down a generating functional $Z[J]$ for these correlators by perturbing the action of the QFT:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \sum_A J_A(x)\mathcal{O}_A(x) \equiv \mathcal{L}(x) + \mathcal{L}_J(x)$$
$$Z[J] = \left\langle e^{-\int \mathcal{L}_J} \right\rangle_{CFT} \quad .$$

where $J_A(x)$ are arbitrary functions (sources) and $\{\mathcal{O}_A(x)\}$ is some basis of local operators. The n-point function is then given by:

$$\left\langle \prod_n \mathcal{O}_n(x_n) \right\rangle = \prod_n \frac{\delta}{\delta J_n(x_n)} \ln Z\Big|_{J=0} \quad .$$

Since $\mathcal{L}_J$ is a *UV* perturbation (because it is a perturbation of the *bare* Lagrangian by *local* operators), in AdS it corresponds to a perturbation near the boundary, $z \to 0$. (Recall from the counting of degrees of freedom above that QFT with UV cutoff $E <$

$1/\epsilon \longleftrightarrow$ AdS cutoff $z > \epsilon$.) The perturbation J of the CFT action is encoded in the boundary condition on bulk fields.

The idea (often referred to as GKPW) for computing $Z[J]$ is then, schematically:

$$Z[J] \equiv \left\langle e^{-\int \mathcal{L}_J} \right\rangle_{CFT} = \underbrace{Z_{\rm QG}[\text{b.c. depends on } J]}_{=???} \underset{N\gg 1}{\sim} e^{-S_{\rm grav}}\Big|_{\text{EOM, b.c. depend on } J} \ . \tag{7.10}$$

The middle object is the partition function of quantum gravity. We don't have a very useful idea of what this is, except in perturbation theory and via this very equality. In a limit where this gravity theory becomes classical, however, we know quite well what we're doing, and we can do the path integral by saddle point, as indicated on the RHS of (7.10).

An important point here is that even though we are claiming that the QFT path integral is dominated by a classical saddle point, this does not mean that the field theory degrees of freedom are free. How this works depends on what kind of large-N limit we take to make the gravity theory classical. In the kind of vector-like large-N limit that we've studied this quarter, the large-N expectation values can be computed by saddle point. In contrast, in a theory where the degrees of freedom are $N \times N$ matrices, such as a Yang-Mills theory with gauge group $\mathsf{SU}(N)$, there are infinitely-many leading-order diagrams.

One nice conclusion that we can state is that black hole thermodynamics, for black holes in AdS, is the ordinary thermodynamics of the dual CFT. In particular, the canonical ensemble partition function is obtained by periodic identification of euclidean time:

$$e^{-\beta F} = \mathrm{tr}_{\rm CFT} e^{-\beta H_{\rm CFT}} = Z_{\rm QG}[\mathrm{bdy} S^1_\beta \times \mathbb{R}^d] \simeq e^{-S_{\rm grav}}|_{\rm BH}. \tag{7.11}$$

The solution to the bulk eom with these boundary conditions is the euclidean black hole. Its action then determines the free energy, and the entropy is

$$S = -\partial_T F = \frac{\text{Area}}{4G_N} = S_{\rm BH}. \tag{7.12}$$

This calculation was first done by Gibbons and Hawking, without the knowledge of whose free energy was being computed[59].

[End of Lecture 22]

[59] If you want to actually do the calculation there is a wrinkle you must know about: there are some boundary terms in the action. First, is a term proportional to the extrinsic curvature of the boundary, required to make the equations of motion consistent with the boundary conditions we are imposing. Second, we must add some counterterms to get a finite answer. You shouldn't be surprised by this, since the boundary is associated with the UV of a QFT. Adding all possible local terms and fixing their coefficients to cancel the divergences arising in the on-shell action will give you the right answer.