New Anomalies from the Fifth Dimension

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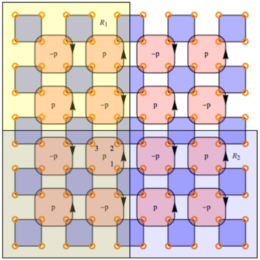
based on work with

S. M. Kravec and Brian Swingle



This talk is about 'emergeability'

[Senthil]

Crucial step in EFT: identify
- degrees of freedom
- realization of symmetry.

Extra *high-energy* requirement:

Given a microphysical $\mathcal{H}_{UV}$, (*e.g.* $\mathcal{H}_{UV} = \otimes_j \mathcal{H}_j$)

do these dofs act on (a subspace of) $\mathcal{H}_{UV}$?

Checks: all RG invariants must be the same.

Familiar hep-th avatar: 't Hooft anomaly matching.

A different kind of example: no gauge invariant fermion operators in an EFT for a bosonic system.

This might be considered a

High-Energy Problem for Low-Energy Physicists.

Many examples, *e.g.*:
- can a $p + ip$ superfluid be coupled to gravity without a spin structure?
- does a CFT always have a stress tensor?

High-energy physics point of view

Goal: Identify obstructions
to symmetry-preserving regulators of QFT,
by thinking about certain states of matter in one higher dimension
which have an energy gap
(*i.e.* $E_1 - E_{gs} > 0$ in thermodynamic limit).

These 'SPT (symmetry-protected topological) states'
[Wen et al; Reviews: Turner-Vishwanath, 1301.0330; Senthil, review of 'physics-based approach' 1405.4015]
are machines for producing such obstructions.

(Their study is also a useful step toward understanding more difficult states.)

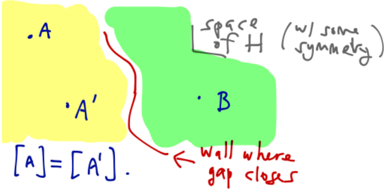
Plan

1. Introduction
2. Ideas about regularizing the Standard Model
   [Wen, J. Wang-Wen, You-BenTov-Xu]
3. Constraints on 3+1d QFT
   from symmetry-protected topological (SPT) states
   In particular, we'll identify constraints on manifest electric-magnetic duality symmetry.
   [Shauna Kravec, JM, 1306.3992, PRL
   work in progress with Brian Swingle]
4. A machine for *explicitly* realizing SPT states
   [Shauna Kravec, JM, Brian Swingle, in progress]

Realizations of symmetries in QFT and cond-mat

Basic Q: What are possible gapped phases of matter?

Def: Two gapped states are equivalent if they are adiabatically connected
(varying the parameters in the **H** whose ground state they are to get from one to the other, without closing the energy gap).



One important distinguishing feature: how are the symmetries realized?

Landau distinction: characterize by *broken* symmetries
e.g. ferromagnet vs paramagnet, insulator vs SC. ✓

Mod out by Landau: "What are possible (gapped) phases that don't break symmetries?" How do we distinguish them?

One (fancy) answer: symmetries can be *fractionalized*.
[Wen]: topological order.
This means emergent deconfined gauge theory, long-range entanglement.

Mod out by Wen, too

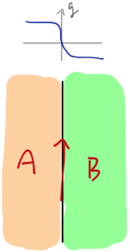
"What are possible (gapped) phases that don't break symmetries and don't have topological order?"

In the absence of topological order
('short-range entanglement' (SRE), hence simpler),
another answer: Put the model on the space with boundary.

A gapped state of matter in $d + 1$ dimensions
with short-range entanglement
can be (at least partially) characterized (within some symmetry class of hamiltonians) by (properties of) its edge states
(*i.e.* what happens at an interface with the vacuum,
or with another such state).

SRE states are characterized by their edge states

Rough idea: just like varying the Hamiltonian in time to another phase requires closing the gap $\mathbf{H} = \mathbf{H}_1 + g(t)\mathbf{H}_2$, so does varying the Hamiltonian in space $\mathbf{H} = \mathbf{H}_1 + g(x)\mathbf{H}_2$.



- Important role of SRE assumption: Here we are assuming that the bulk state has short-ranged correlations, so that changes we might make at the surface cannot have effects deep in the bulk.

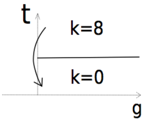
SPT states

Def: An *SPT state* (symmetry-protected topological state), protected by a symmetry group G is:
a SRE state, which is not adiabatically connected to a product state by local hamiltonians preserving G.

e.g.: free fermion topological insulators in 3+1d, protected by $U(1)$ and $\mathcal{T}$, have an odd number of Dirac cones on the surface.
One reason to care: if you gauge $\mathrm{H} \subset \mathrm{G}$, you get a state with topological order.

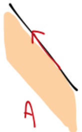
- Free fermion TIs classified [Kitaev: homotopy theory; Schneider et al: edge]

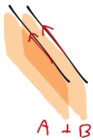
Interactions can affect the connectivity of the phase diagram in both directions:



- There are states which are adiabatically connected only via interacting Hamiltonians [Fidkowski-Kitaev, 0904.2197, Qi, Yao-Ryu, Wang-Senthil, You-BenTov-Xu].
- There are states whose existence requires interactions: *e.g.* Bosonic SPT states – w/o interactions, superfluid.

Group structure of SPT states

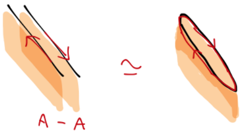




Simplifying feature:

SPT states (for given G) form a group:

-A : is the mirror image.



- With bulk topological order, bulk quasiparticles still nontrivial. Not a group.
- There can be many realizations of the edge of A, same 'SPT-ness'.
- The edge of A can be symmetric and gapped but topologically ordered.

Inverse of A cancels the SPT-ness of A's edge.

- [Chen-Gu-Wen, 1106.4772] conjecture: the group is $H^{D+1}(\mathrm{BG}, U(1))$.
- $\exists$ 'beyond-cohomology' states in $D \geq 3+1$ [Senthil-Vishwanath]
- The right group?: [Kitaev (unpublished), Kapustin, Thorngren].

Here: an implication of this group structure

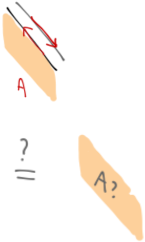
– which we can pursue by examples – is...

Surface-only models

Counterfactual:

Suppose the edge theory of an SPT state were realized *otherwise* – intrinsically in D dimensions, with a local hamiltonian.

Then we could paint that the conjugate local theory on the surface without changing anything about the bulk state. And then small deformations of the surface Hamiltonian, localized on the surface, consistent with symmetries, can pair up the edge states.



But this contradicts the claim that we could characterize the $D+1$-dimensional SPT state by its edge theory.

Conclusion: Edge theories of SPT_G states cannot be regularized intrinsically in D dims, *exactly* preserving on-site G –

"surface-only models" or "not edgeable".

[Wang-Senthil, 1302.6234 – general idea, concrete surprising examples of 2+1 surface-only states
Wen, 1303.1803 – attempt to characterize the underlying mathematical structure, classify *all* such obstructions
Metlitski-Kane-Fisher, 1302.6535; Burnell-Chen-Fidkowski-Vishwanath, 1302.7072]

Ideas about regularizing the Standard Model

Nielsen-Ninomiya result on fermion doubling

The most famous example of such an obstruction was articulated by Nielsen and Ninomiya:

It is not possible to regulate free fermions while preserving the chiral symmetry.

Recasting the NN result as a statement about SPT states

Consider free massive relativistic fermions in 4+1 dimensions (with conserved $U(1)$):

$$S = \int d^{4+1}x \bar{\Psi} \left(\partial\!\!\!/ + m \right) \Psi$$

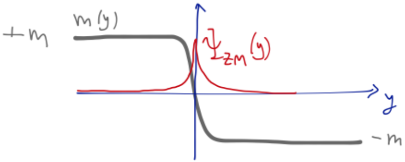
$\pm m$ label distinct Lorentz-invariant (P-broken) phases.

Domain wall between them hosts (exponentially-localized) 3+1 chiral fermions: [Jackiw-Rebbi, Callan-Harvey, Kaplan...]

One proof of this:

Couple to external gauge field $\Delta S = \int d^5 x A^\mu \bar{\Psi} \gamma_\mu \Psi$.

$$\log \int [D\Psi] e^{\mathbf{i} S_{4+1}[\Psi, A]} \propto \frac{m}{|m|} \int A \wedge F \wedge F$$



Galling fact: if we want the extra dimension to be finite, there's another domain wall with the antichiral fermions.

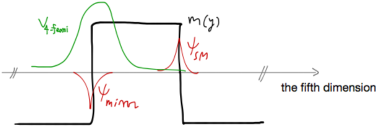
And if we put it too far away, the KK gauge bosons are too light...

Loophole in NN theorem

But the SM gauge group is *not* anomalous, shouldn't need extra dimensions.

Loophole: Interactions between fermions!

Old idea: add four-fermion interactions (or couplings to other fields) which gap the mirror fermions, but not the SM, and preserve the SM gauge group G.



These interactions should explicitly break all anomalous symmetries.

This requires a right-handed neutrino. [Preskill-Eichten 1986]: SU(5) and SO(10) lattice GUTs. [Many other papers ... recent work: Wen, J. Wang]

[Preskill-Eichten 1986]: Evidence for mirror-fermion mass generation without symmetry-breaking via eucl. strong coupling expansion.

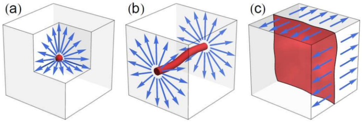
[Geidt-Chen-Poppitz]: numerical evidence for troubles of a related proposal in 1+1d.

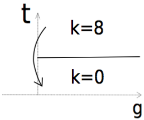
New evidence for a special role of $n_F = 16 \cdot n$

Collapse of free-fermion classification:
Dimensional recursion strategy [Wang-Senthil, Qi, Ryu-Yao, Wen, You-BenTov-Xu]:

1. Consider neighboring phase where G is spontaneously broken $\langle\phi\rangle \neq 0$.
2. Proliferate defects of ϕ to reach paramagnetic phase.
3. Must ϕ-defects carry quantum numbers which make the paramagnet nontrivial?

Initial step: [Fidkowski-Kitaev]
edge of $8\times$ majorana chain is symmetrically gappable.
same refermionization as shows equivalence of GS and RNS superstrings, SO(8) triality.





[You-BenTov-Xu]: In 4+1d, with many G, the collapse again happens at $k = 8 \simeq 0$ ($\to$ 16 Weyl fermions per domain wall.)

awkward: $G \supset \mathbb{Z}_2^{\mathcal{T}}$

Conclusion: This novel strategy for identifying obstructions to gapping the mirror fermions shows none when $n_F = 16n$.

Surface-only electrodynamics, by example

Strategy

Study a simple (unitary) gapped or topological field theory in 4+1 dimensions without topological order, with symmetry G.

Consider the model on the disk with G-inv't boundary conditions.

The resulting edge theory is
a "surface-only theory with respect to G"
– it cannot be regulated by a local $3+1$-dim'l model while preserving G.

This is the 4+1d analog of the "K-matrix approach" to 2+1d SPTs of [Lu-Vishwanath 12].

What does it mean to be a surface-only state?

These theories are perfectly consistent and unitary – they *can* be realized as the edge theory of some gapped bulk. They just can't be regularized in a local way consistent with the symmetries – without the bulk.

1. It (probably) means these QFTs will not be found as low-energy EFTs of solids or in cold atom lattice simulations.
2. Why 'probably'? This perspective does not rule out emergent ("accidental") symmetries, not explicitly preserved in the UV.
3. It also does not rule out symmetric UV completions that include gravity, or decoupling limits of gravity/string theory.

Some known examples of surface-only states

The first three examples are in $D = 2 + 1$, realized on the surface of $D = 3 + 1$ boson SPTs protected by time-reversal:

- Non-linear sigma model on S^2 with Hopf term at $\theta = \pi$

$$Z = \sum_{\text{instanton number, n}} (-1)^n Z_n$$

[model of high-T_c: Dzyaloshinskii-Polyakov-Wiegmann 88, surface only: Vishwanath-Senthil 12]

- "Algebraic vortex liquid": an insulating state of bosons (or a paramagnet) with massless fermionic vortices

[proposed by Fisher et al 06, surface only: Wang-Senthil 13]

- "All-fermion toric code": a version of $\mathbb{Z}_2$ gauge theory where $e, m, \epsilon \equiv em$ are all fermions.

[surface only: Burnell-Chen-Fidkowski-Vishwanath, Wang-Senthil]

- "All-fermion electrodynamics": a version of Maxwell theory where $e, m, \epsilon \equiv em$ are all fermions.
  In $D = 3 + 1$, with $G = \mathbb{1}$ [surface only: Wang-Potter-Senthil 13]

A simple topological field theory in 4+1 dimensions

Consider 2-forms B_{MN} in $4+1$ dimensions, with action

$$S[B] = \frac{K_{IJ}}{2\pi} \int_{\mathbb{R}\times\Sigma} B^I \wedge \mathrm{d}B^J$$

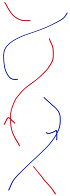
In $4\ell+1$ dims, K is a *skew*-symmetric integer $2N_B \times 2N_B$ matrix.

Note: $B \wedge \mathrm{d}B = \frac{1}{2}\mathrm{d}(B \wedge B)$.

Independent of choice of metric on $\mathbb{R} \times \Sigma_{2p}$.

Related models studied in: [Horowitz 1989, Blau et al 1989, Witten 1998, Maldacena-Moore-Seiberg 2001, Belov-Moore 2005, Hartnoll 2006]

[Horowitz-Srednicki]: coupling to string sources $\Delta S = \int_{\Gamma_I} B^I$ computes linking # of conjugate species of worldsheets Γ^I.



Simplest case ($N_B = 1$) is realized in IIB strings on $AdS_5 \times S^5$,

$B \equiv B_{NSNS}, C \equiv C_{RR}$:

$$S_{IIB} \ni \frac{1}{2\pi} \int_{AdS_5 \times S^5} F^{(5)}_{RR} \wedge B \wedge \mathrm{d}C = \frac{N}{2\pi} \int_{\mathbb{R}\times\Sigma} B \wedge \mathrm{d}C$$

Crucial hint: Type IIB S-duality acts by $B \leftrightarrow C$.

Strategy

1. Solve the model – when is it an EFT for an SPT state?
   Answer: when Pfaff$K = 1$.
2. Identify the edge states, and the symmetry G protecting them.
   (Whatever we get is surface-only with respect to G.)
   Answer: in the simplest realization, the edge theory is ordinary Maxwell theory, but with manifest electric-magnetic duality $(\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E})$.

Comments:

1. Breaking Lorentz symm. is not enough to allow this symmetry:
   the edge theory we find is exactly the manifestly-duality-invariant model of [Schwarz-Sen 94].
2. **Corollary:** it is not possible to gauge electric-magnetic duality symmetry.
   $\exists$ recent literature with continuum arguments for this impossibility:
   [Deser 1012.5109, Bunster 1101.3927, Saa 1101.6064]
3. A similar construction in 6+1 dimensions produces a sector of the infamous $(2, 0)$ SCFT on the edge.

All-fermion electrodynamics

So far, we've discussed 'pure' U(1) gauge theory (free).

A stronger obstruction can be found by adding matter

(Ends of strings are electric and magnetic charges)

Manifest duality symmetry
$\implies$ e, m and dyon $\epsilon \equiv em$
must have the same statistics.

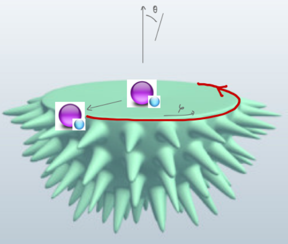
$$\begin{pmatrix} q_e \\ q_m \end{pmatrix} \to \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in \mathrm{SL}(2,\mathbb{Z})} \begin{pmatrix} q_e \\ q_m \end{pmatrix}$$

- If e, m are bosons, ϵ is a fermion!

'spin from isospin': [Jackiw-Rebbi]

$$\psi(x_1, x_2) = e^{\mathbf{i}\varphi}\psi(x_2, x_1)$$

$$\varphi = e\int_0^\pi d\varphi \underbrace{\mathcal{A}_\varphi(\theta = \frac{\pi}{2}, \varphi)}_{\text{Dirac monopole field}} \overset{\text{Dirac}}{=} \pi$$

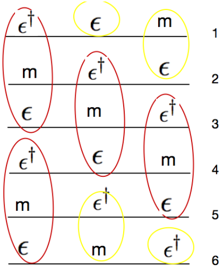


- All fermions is self-consistent......BUT

[Wang-Potter-Senthil] the all-fermion electrodynamics is not 'edgeable': on a space with boundary there is a unit-charge boson operator.

(Important assumption: no gauge invariant fermion operators on $\mathcal{H}_{UV}$.)

All-fermion electrodynamics



This is evidence for a 4+1d SPT
with no symmetry (analogous to Kitaev's E_8 state).

[work in progress:] Coupled-layer construction

following [Wang-Senthil] for all-fermion TC

produces trivial bosonic bulk, correct edge.

Idea: 'dyon string condensation' (like [Metlitski-Kane-Fisher])

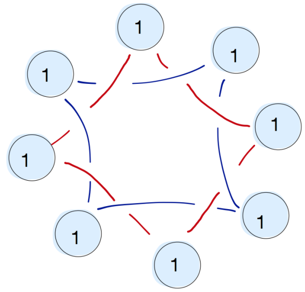
- Each layer is ordinary electrodynamics with bosonic charges.
- $b_i \equiv \epsilon_i^\dagger m_{i+1} \epsilon_{i+2}$ are mutually-local bosons.
- Condensing b_i (obliquely) *confines* $a_{i+1}, i+1 = 2...N-1$.
- At top layer: $m_1 \epsilon_2, \epsilon_1^\dagger m_1 \epsilon_2, \epsilon_1^\dagger$ survive, are fermions,

are electron, monopole & dyon of $U(1)_{odd}$.

$\exists$ a 4+1d local lattice model which realizes this construction.

In the bulk, continuum: this is the *BdC* theory with gapped string matter.

Like 'deconstruction':



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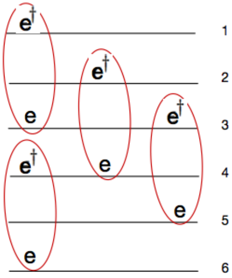

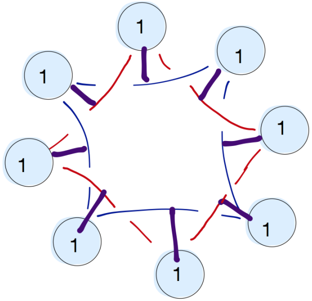

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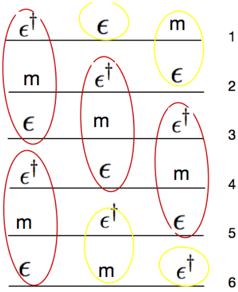


Dyon string condensation

If we did this: $\longrightarrow$

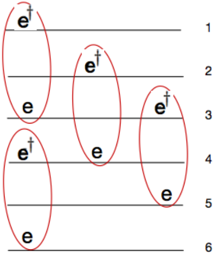
$\Delta H = V \sum_i (|\tilde{b}_i|^2 - v^2)^2$, $\tilde{b}_i \equiv e_i^\dagger e_{i+2} = v e^{\mathbf{i} a_{i,i+2}}$

would higgs $\prod_i \mathrm{U}(1)_i \to \mathrm{U}(1)_{\text{even}} \times \mathrm{U}(1)_{\text{odd}}$.

$4+1d$ Maxwell theory with $\mathrm{G} = \mathrm{U}(1)_{\text{even}} \times \mathrm{U}(1)_{\text{odd}}$, bulk photons.

Can dualize to 2-form potentials: $f^{o/e} = da^{o/e} = \star dC^{o/e}$

$$S = \sum_{\alpha=o,e} \int_{5d} \left(\frac{1}{g_\alpha^2} dC^\alpha \wedge \star dC^\alpha + C^\alpha \wedge \star j_m^\alpha \right).$$



Magnetic flux tubes of broken U(1)s collimate monopoles into *monopole strings* of 5d Maxwell.

If instead we do *this* : $\longrightarrow$

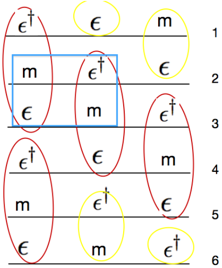
$b_i \equiv \epsilon_i^\dagger m_{i+1} \epsilon_{i+2}$ are mutually-local bosons.

$\Delta H = V \sum_i (|b_i|^2 - v^2)^2$. Condensing b_i (obliquely) *confines* $a_{i+1}, i+1 = 2...N-1$.

Binds monopole strings of $a^{e/o}$ to electric flux lines of $a^{o/e}$!

This is the effect of the additional term

$$\Delta S = \int \frac{1}{4\pi} C^e \wedge dC^o.$$



A solvable coupled-island construction of SPT states in $2 + 1$ dimensions

An SPT machine

A lot of effort has been put into classifying SPT states.

Fewer explicit constructions exist.

Useful *e.g.* for understanding the phase transitions between them, and the topologically-ordered states that result upon gauging (subgroups of) G.

Here: 2+1d $\mathbb{Z}_N$ paramagnets.

Virtues of our construction:

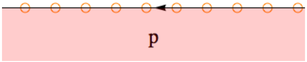
- Translation invariance not required. (Often translation invariance can protect an otherwise unprotected edge.)
- Uniform construction of domain wall operators.
  $\rightarrow$ [Levin-Gu] braiding statistics proof of nontriviality
- Illuminates connections between the few existing examples:
  [Levin-Gu ($\mathbb{Z}_2$), Chen-Liu-Wen ($\mathbb{Z}_2$), Chen-Gu-Liu-Wen (mysterious general formula?)]
- The 'duality' method of [Levin-Gu] was not available: gauging the bulk symmetry provides a (simpler?) construction of recent 'generalized string-net models' [Lin-Levin].

Non-onsite symmetry

An anomalous symmetry can be realized in a lattice model if it is *not on-site*: its action on one site depends on others.

This means you can't gauge it just by coupling to link variables (without coarse-graining first).

(Like chiral symmetry with staggered fermions.)



For example, at the edge of the Levin-Gu $\mathbb{Z}_2$ paramagnet,

$$\mathbf{S} = \prod_J \mathbf{X}_J \prod_J \mathbf{i}^{\frac{1}{2}(1-\mathbf{Z}_J\mathbf{Z}_{J+1})} = \prod_J \mathbf{X}_J \cdot \mathbf{i}^{\text{number of domain walls}}$$

A non-onsite symmetry $\mathbf{S}$ is nontrivial if $\mathbf{S} \neq \mathbf{U} \prod_j s_j \mathbf{U}^\dagger$ with $\mathbf{U}$ a *local symmetric unitary* (unitary evolution by a symmetric $\mathbf{H}$).

How to tell?? We will find a practical criterion below.

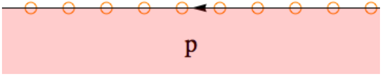
Focus on $\mathbb{Z}_N$ spins at each site:

$$\mathbf{XZ} = \omega \mathbf{ZX}, \quad \omega \equiv e^{\frac{2\pi \mathbf{i}}{N}} \quad \mathbf{Z}|n\rangle = \omega^n |n\rangle, .$$

$$\mathbf{P}_n(\mathbf{Z}) \equiv |n\rangle\langle n|, \quad \mathbf{X}|n\rangle = |n-1\rangle, \quad n = 0..N-1 \quad (\text{mod } N)$$

An SPT machine

Given desired action of
non-onsite symmetry on edge:



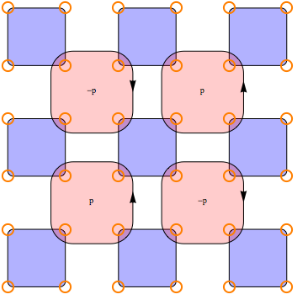
Couple together 'bags':

(Inspired by CZX model for $\mathbb{Z}_2$ [Chen-Liu-Wen, Swingle].)

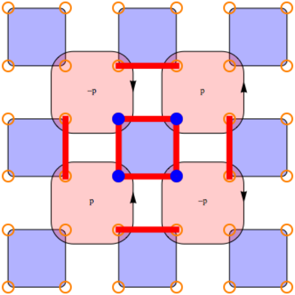
$$\mathbf{H}_{\text{CZX}} = -\sum_{\square} \mathbf{b}_{\square} + h.c.$$

$$\mathbf{b}_{\square} \equiv \mathbf{XXXX}\mathcal{P}. \qquad [\mathbf{b}_{\square}, \mathbf{b}_{\square'}] = 0$$

$$|\text{gs}\rangle = \prod_{\square} \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |nnnn\rangle$$



Think of each bag as a site.

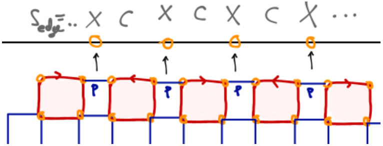


Symmetry-protected edge states

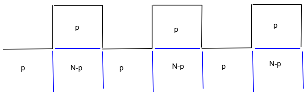
Note: $\mathbf{H}$ doesn't know about p; $\mathbf{S}$ does: ($[\mathbf{S}, \mathbf{H}_{\text{CZX}}] = 0$)

$$\mathbf{S} = \prod_j \mathbf{X}_j \left(\prod_{\text{bags}} \prod_j C_p \left(\mathbf{Z}_j, \mathbf{Z}_{j+1} \right) \right)$$

Not onsite on edge: $\longrightarrow$



Rough edge realizes the desired $\mathbf{S}$ on edge modes:



Claim of robustness: perturbing $\mathbf{H}_{\text{CZX}}$ by terms respecting $\mathbf{S}$, you cannot remove this edge stuff.

i.e. no local, symmetric unitary can make $|gs\rangle_{\text{edge}}$ a product state. (Gapless or symmetry-breaking degeneracy.)

Shortcoming (?): requires bipartite graph of connections between bags.

Note: I draw lattices for simplicity of drawing, but translation invariance is not at all necessary.

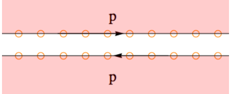
Bag link phases

$$\mathbf{S} = \prod_j \mathbf{X}_j \prod_{\text{bags}} \prod_j C_p(\mathbf{Z}_j, \mathbf{Z}_{j+1})$$

Wanted: A unitary operator $C(1,2) \equiv C(Z_1, Z_2)$ on two $\mathbb{Z}_N$-valued variables which satisfies the following three simple-looking conditions:

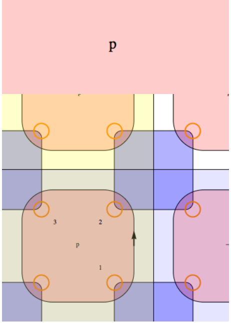
- group law : $\quad \mathbf{S}^N = \prod_j \left(\mathbf{Z}_j^\dagger \mathbf{Z}_{j+1}\right)^p$

only for closed loops: $\quad = 1$



- gappability : $\quad C_p(1,2)C_{-p}(1,2) = 1$

$=$

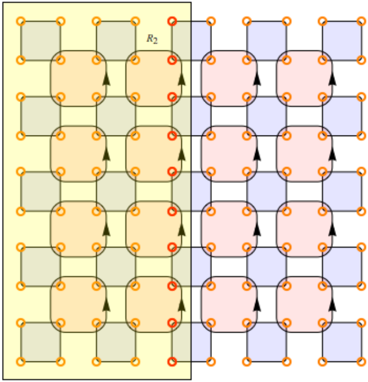


- non-triviality $\quad$ (flux braiding) :

$$\frac{X_1^{k_1} C_p^{k_2}(12) X_1^{-k_2}}{C_p^{k_2}(12)} \frac{X_3^{k_1} C_{-p}^{k_2}(23) X_3^{-k_1}}{C_{-p}^{k_2}(23)} \stackrel{?}{=} e^{2\pi \mathbf{i} \frac{2p}{N} k_1 k_2}.$$

A uniform construction of domain-wall operators

$$W(R) \equiv \prod_{j \in R \setminus \text{last col}} \mathbf{X}_j \prod_{\text{bags in } R} \mathcal{P} \cdot \prod_{j \in bag} C_p(\mathbf{Z}_j, \mathbf{Z}_{j+1})$$



Acts like **S** in the interior of R.

Threads $2\pi/N$-flux along its boundary.

Becomes the string operator in the topologically-ordered model with G gauged and deconfined.

Same C!

Solution of conditions 1-3:

$$C(\mathbf{Z}_1, \mathbf{Z}_2) = e^{\frac{2\pi \mathbf{i} p}{N^2} \sum_n n \mathbf{P}_n(\mathbf{Z}_1^\dagger \mathbf{Z}_2)} .$$

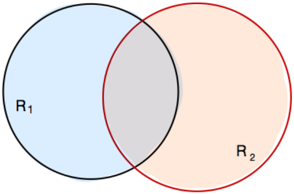
For $N = 2, 3$, this object appears in [Wang-Santos].

Braiding of flux insertions

[Levin-Gu]

$\mathbf{W}_R$ generate symmetries:

$$[\mathbf{H}, \mathbf{W}_R] = 0 \; .$$



If domain walls intersect *once*:

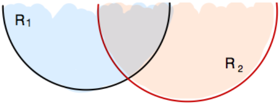
$$\mathbf{W}_{R_1}^{k_1} \mathbf{W}_{R_2}^{k_2} = e^{2\pi \mathbf{i} k_1 k_2 \frac{2p}{N^2}} \mathbf{W}_{R_2}^{k_2} \mathbf{W}_{R_1}^{k_1}$$

(annoying fine print:
this formula works for $k_1 k_2 \propto N$)

This must be represented on groundstates

(in fact, the whole spectrum)

$\implies$ nontrivial edge spectrum.



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(annoying fine print:
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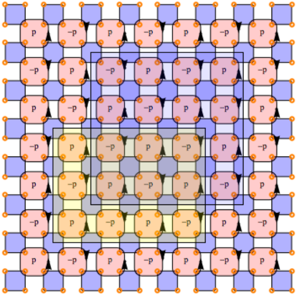
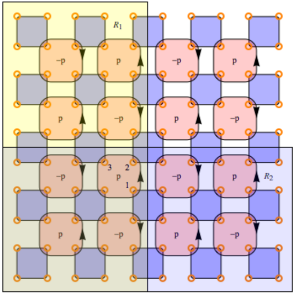
(in fact, the whole spectrum)

$\implies$ nontrivial edge spectrum.

This gives a very practical condition for nontriviality of C.





Coarse-graining transformation

step 1: change basis to on-site symmetry:

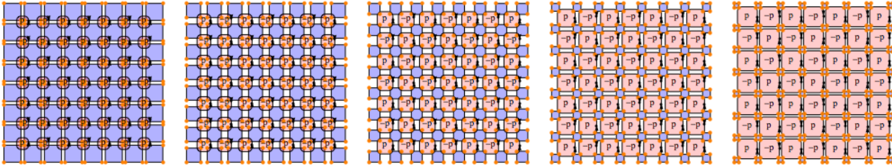
$$\mathbf{U}\mathbf{S}_1\mathbf{U}^\dagger = \prod_j \mathbf{X}_j$$

($\mathbf{U}$ known but, so far, ugly.)

Weirdness of $\mathbf{U}$: it's a local unitary, but not continuously connected to $\mathbb{1}$ by local symmetric unitaries.

In this basis, easy to gauge. Alternative[Swingle]: diagonalize action on bags.

step 2: project to low-energy hilbert space $= \mathbb{Z}_N$ spins on sites J of bag graph:



$$\mathbf{H} = -\sum_J \mathbf{X}_J u_J(\mathbf{Z}) + h.c.$$

$$|\mathrm{gs}\rangle = \mathbf{U} \otimes_J \left(\sum_n \frac{1}{\sqrt{N}} |n\rangle_J\right)$$

like [Levin-Gu] for $\mathbb{Z}_2$

should be interpreted in terms of fluctuating domain walls and junctions.

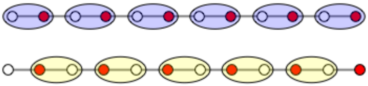
Non-onsite edge symmetry revisited

Chern-Simons description [Lu-Vishwanath]:

Edge is non-chiral boson, only the left-mover carries the $\mathbb{Z}_N$.

On the lattice, again the action is by *duality*:

$\mathbf{S}_1$ exchanges the Jordan-Wigner (para)fermions for the spins with the JW (para)fermions for the disorder operators.



Symmetric edge Hamiltonian for $\mathbb{Z}_N$:

$$\mathbf{H} = -\sum_J \left(\mathbf{X}_J + \mathbf{Z}^p_{J-1} \mathbf{X}_J \mathbf{Z}^p_{J+1} \right) + h.c.$$

JW solution for $\mathbb{Z}_2$: Always in ferromagnetic (topological) phase.

For $\mathbb{Z}_N$: Likely a simple lattice model for scalar with chiral $\mathbb{Z}_N$ symmetry.

Interesting generalizations

$G = \mathbb{Z}_N \times \mathbb{Z}_N$: $\exists$ a qualitatively different form of cocycle,

related to a CS term of the form $a \wedge db$ rather than $a \wedge da$.

Coupled island construction:

introduce two sets of $\mathbb{Z}_N$ variables at each site of a CZX lattice, $\mathbf{X}, \mathbf{Z}, \tilde{\mathbf{X}}, \tilde{\mathbf{Z}}$.

$$\mathbb{Z}_N \times \mathbb{Z}_N : \quad S = \prod_j \mathbf{X}_j \prod_{bags} C(\tilde{\mathbf{Z}}), \quad \tilde{S} = \prod_j \tilde{\mathbf{X}}_j \prod_{bags} C(\mathbf{Z}).$$

$$\boxed{\mathbf{W}_{R_1} \tilde{\mathbf{W}}_{R_2} = \omega^{\frac{p}{N}} \tilde{\mathbf{W}}_{R_2} \mathbf{W}_{R_1}} \quad \left(\mathbf{W}_{R_1} \mathbf{W}_{R_2} = \mathbf{W}_{R_2} \mathbf{W}_{R_1}, \tilde{\mathbf{W}}_{R_1} \tilde{\mathbf{W}}_{R_2} = \tilde{\mathbf{W}}_{R_2} \tilde{\mathbf{W}}_{R_1}\right.$$

U(1): our original goal was to make a solvable model of the boson integer quantum Hall state [Levin-Senthil, Senthil-Regnault, Barkeshli].

A rotor at each site:

$$[\mathbf{n}, e^{\mathbf{i}\theta}] = e^{\mathbf{i}\theta}, \quad \mathbf{X}(\varphi)_j \equiv e^{\mathbf{i}\varphi \mathbf{n}_j}, \mathbf{Z} = e^{\mathbf{i}\theta}, \quad \mathbf{X}(\varphi)\mathbf{Z}\mathbf{X}^\dagger(\varphi) = e^{\mathbf{i}\varphi}\mathbf{Z}.$$

$$\mathbf{b}_\square = \int_0^{2\pi} đ\theta \mathbf{X}(\theta)\mathbf{X}(\theta)\mathbf{X}(\theta)\mathbf{X}(\theta)\mathcal{P}$$

$$C_p(1,2) = e^{\mathbf{i}p \int_0^{2\pi} đ\theta \theta \mathbf{P}_\theta (\mathbf{Z}_1^\dagger \mathbf{Z}_2)}$$

Questions

- For $G = U(1)$, gauging the symmetry produces a model with $c_L \neq c_R$, but the model should still be solvable.
  Some tension with theorems of [Kitaev, honeycomb paper; Lin-Levin].
- We have not yet made precise the connection to group cohomology.
  The condition on the link phases that the DW commutator is a c-number should be the cocycle condition. Is it?
- Origin of bipartite restriction?!?
  In the continuum, there is no difference between $p \to -p$ and orientation reversal.
- Non-abelian G?
- 3d?
- What is anomalous about the all-fermion electrodynamics?

Concluding remark

Clearly the fruitful exchange of ideas between high energy theory and condensed matter theory continues.

The end

Thanks for listening.

Lattice model for the *BdC* theory

[Kravec, JM, Swingle, in progress]

• Put rotors $e^{\mathbf{i}b_p}$ on the *plaquettes* p of a 4d spatial lattice.

$e^{\mathbf{i}b_p}|n_p\rangle = |n_p + 1\rangle$.

$n_p \equiv \#$ of 'sheets' covering the plaquette.

• Put charge-k bosons $\Phi_\ell = \Phi^\dagger_{-\ell}$ on the *links* ℓ.

$[\Phi_\ell, \Phi^\dagger_\ell] = 1$

[Wegner, ..., Motrunich-Senthil,

Levin-Wen, Walker-Wang, Burnell et al]

$\Phi^\dagger_\ell$ creates a string segment.

$\Phi^\dagger_\ell \Phi_\ell \equiv \#$ of strings covering the link.

$$\mathbf{H} = -\underbrace{\sum_{\text{links},\ell\in\Delta_1} \Big(\sum_{p\in s(\ell)} n_p - k\Phi^\dagger_\ell\Phi_\ell\Big)^2}_{\mathbf{H}_1,\ \text{gauss law. happy when sheets close, or end on strings}} - \underbrace{\sum_{\text{volumes},\ v\in\Delta_3} \prod_{p\in\partial v} e^{\mathbf{i}b_p} + h.c.}_{\mathbf{H}_3\sim B^2,\ \text{makes sheets hop.}}$$

$$- \underbrace{\Gamma \sum_{p\in\Delta_2} n_p^2}_{\mathbf{H}_2\sim E^2.\ \text{discourages sheets.}} \underbrace{- t \sum_{p\in\Delta_2} e^{\mathbf{i}kb_p} \prod_{\ell\in\partial p} \Phi^\dagger_\ell + h.c.}_{\mathbf{H}_{\text{strings}},\ \text{hopping term for matter strings}} + V\left(|\Phi|^2\right)$$

When $\Gamma = 0, V = 0$, the model is solvable:

Soup of oriented closed 2d sheets, groups of k can end on strings.

Lattice boson model, cont'd

Condense $\Phi_\ell = ve^{\mathbf{i}a_\ell}$: $\mathbf{H}_{\text{strings}} = -\sum_p tv^4 \cos\left(kb_p - \sum_{\ell\in\partial p} a_\ell\right)$
$\implies \left(e^{\mathbf{i}b_p}\right)^k = \mathbb{1}$, $|n_p\rangle \simeq |n_p + k\rangle$.
Leaves behind k species of (unoriented) sheets.

Groundstate(s): equal-superposition sheet soup. k^{b_2} sectors for $\mathcal{I} = \mathbb{1}$.

Continuum limit.

$U(1) \overset{\text{Higgs}}{\to} \mathbb{Z}_k$ 2-form gauge theory: (subscripts indicate form degree)

$$L = \frac{tv^4}{2}\left(\mathrm{d}a_1 + kB_2\right) \wedge \star \left(\mathrm{d}a_1 + kB_2\right) + \frac{1}{g^2}\mathrm{d}B_2 \wedge \star \mathrm{d}B_2$$

$$\overset{\substack{\text{path integral} \\ \text{manipulations}}}{\simeq} \quad \frac{k}{2\pi} B \wedge \mathrm{d}C + \underbrace{\frac{1}{8\pi tv^4} dC \wedge \star \mathrm{d}C + \frac{1}{g^2}\mathrm{d}B \wedge \star \mathrm{d}B}_{\text{irrelevant perturbation, ignore when } E < tv^4, g}$$

with $\mathrm{d}C \simeq 2\pi tv^4 \star (\mathrm{d}\varphi + kB)$.

[Maldacena-Moore-Seiberg hep-th/0108152, Hansson-Oganesyan-Sondhi cond-mat/0404327]