Topological strings from lattice spins

sometimes with:

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Why study topological strings?

• They compute F-terms in superstring effective actions.
They organize (holomorphic and UV soft terms for) $4d, \mathcal{N} = 1$ string vacua.

• They count BPS states.

• They provide a toy model for defining physical strings.

• How is it that the Kodaira-Spencer theory exists?

• Z_{top} is a wavefunction.
in what Hilbert space, with what Hamiltonian?

Today

Using the "melting crystal", I'm going to try to formulate a UV completion of the Kodaira-Spencer theory

for some Calabi-Yau backgrounds:

A local two-dimensional lattice model

in a background field related to the choice of CY.

Facts from the topological string

The closed B-model is a theory of complex structure deformations ("Kodaira-Spencer theory"), i.e. of $\Omega \in H^{(3,0)}$.

On CY of the form A_1 singularity over Riemann surface

$$zy = F(u, v)$$

(This class of CYs is mirror to noncompact toric CY.)

the B-model can be described by a "locally-free" chiral boson at the free-fermion radius (i.e. or weyl fermion) (in a fixed conformal frame) on a non-commutative space – the embedding coordinates u, v of the RS are canonically conjugate $[X, P] = ig_s$. (Aganagic-Dijkgraaf-Klemm-Mariño-Vafa)

The boson determines the complex structure of the space on which it lives.

$$\partial\varphi = \int_{S^2} \Omega$$

The fermion operator at (u, v) inserts a D-brane wrapping $z = 0$ or $x = 0$.

The KS theory has many gauge symmetries $A \to A + \bar{\partial}\xi$, and confusing issues about normalizability of modes.

melting crystal

topological A model on $\mathbb{C}^3$

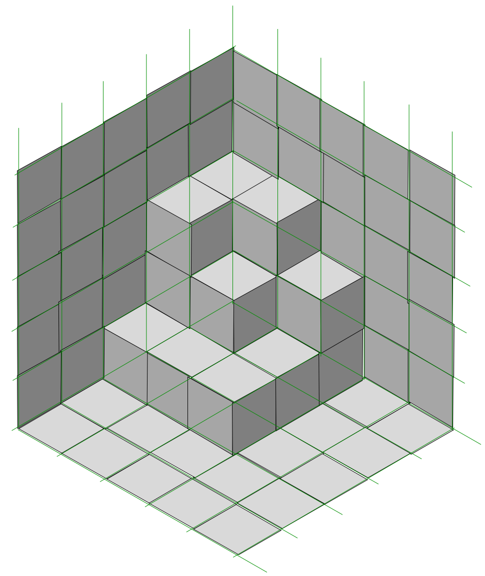
$$Z_{\text{closed}}(q = e^{-g_s}) = e^{\sum_{g=0} g_s^{2g-2} F_g} \simeq M(q)$$

(BCOV, Faber-Pandharipande, Gopakumar-Vafa, Mariño-Moore) with

$M(q) = \prod_{n=1}^{\infty}(1 - q^n)^{-n}$

$$M(q) = \sum_{\pi} q^{|\pi|}.$$

(Okounkov-Reshetikhin-Vafa)



• This picture can be overlaid on the toric diagram for $\mathbb{C}^3$.

the three directions are $|z_i|^2$.

The sum has an interpretation as a sum over blow-ups

(Maulik-Nekrasov-Okounkov-Pandharipande, Iqbal-Nekrasov-Okounkov-Vafa).

• The choice of origin isn't special (like constant B-field).

• The topological A-model on $\mathbb{C}^3$ requires some choice of boundary conditions at ∞.

Including (lagrangian) branes with $U(\infty)$ representations attached $\longrightarrow$ topological vertex (Aganagic-Klemm-Mariño-Vafa)

$$C_{R_1 R_2 R_3} = Z^{-1} \sum_{\pi \in S(R_1 R_2 R_3)} q^{|\pi| - |\pi_{R_1 R_2 R_3}|}$$

where $S(R_1 R_2 R_3)$ is the set of tableaux which asymptote to R_i on the x_i axis, and $\pi_{R_1 R_2 R_3}$ is the smallest such tableau.

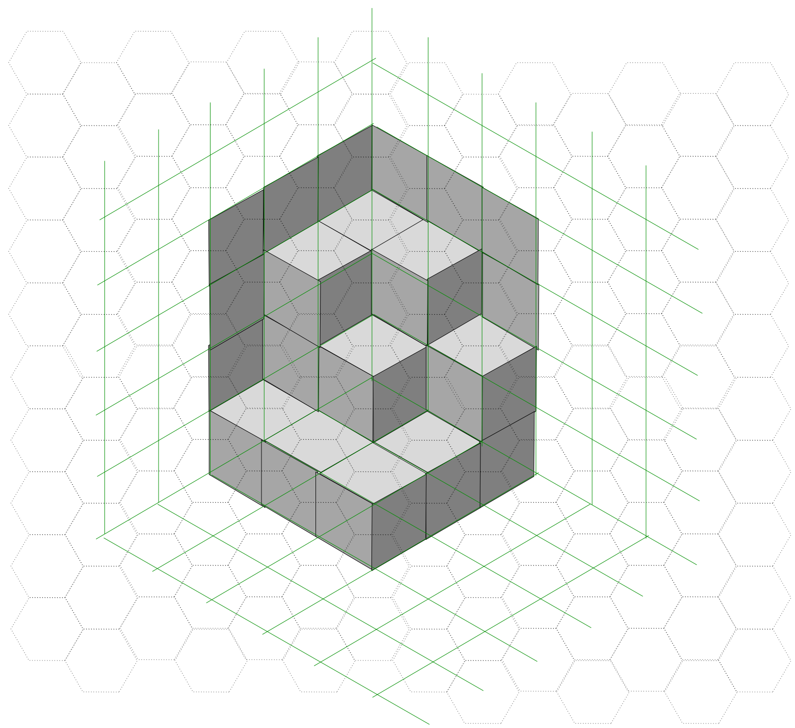
• There is a scaling limit shape (Kenyon, Okounkov):
If we assign size g_s to the boxes, at $g_s \to 0$ there is a limit shape which dominates the sum
which is the mirror B-model curve
$\{e^u + e^{-v} - 1 = 0\} \subset \{e^u + e^{-v} - 1 = xy\}$.

• Most importantly for me, there is a correspondence between

a. 3d tableaux π

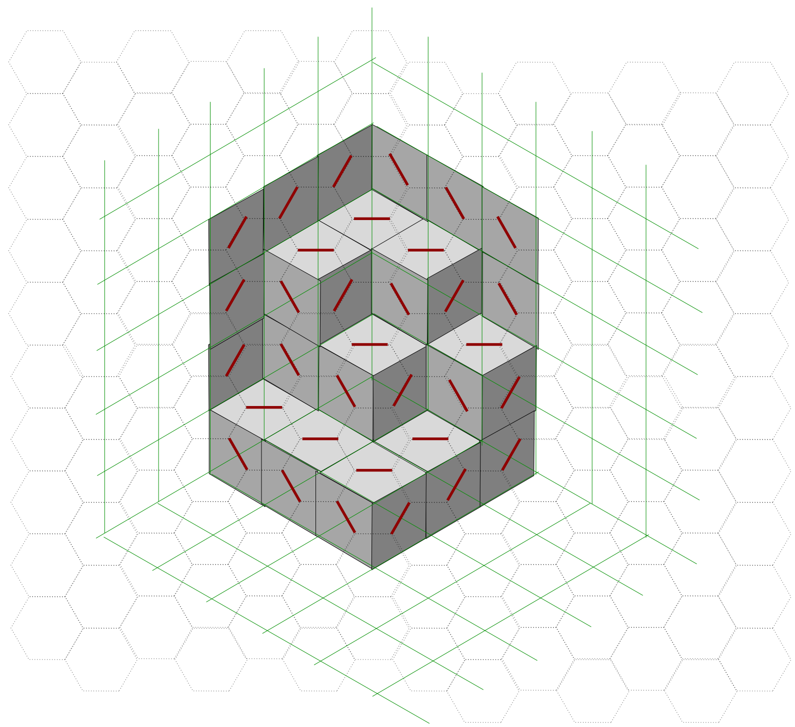
b. lozenge tilings of triangular lattice

(lozenge = two triangles sharing an edge)



c. dimer coverings of the dual hexagonal latttice

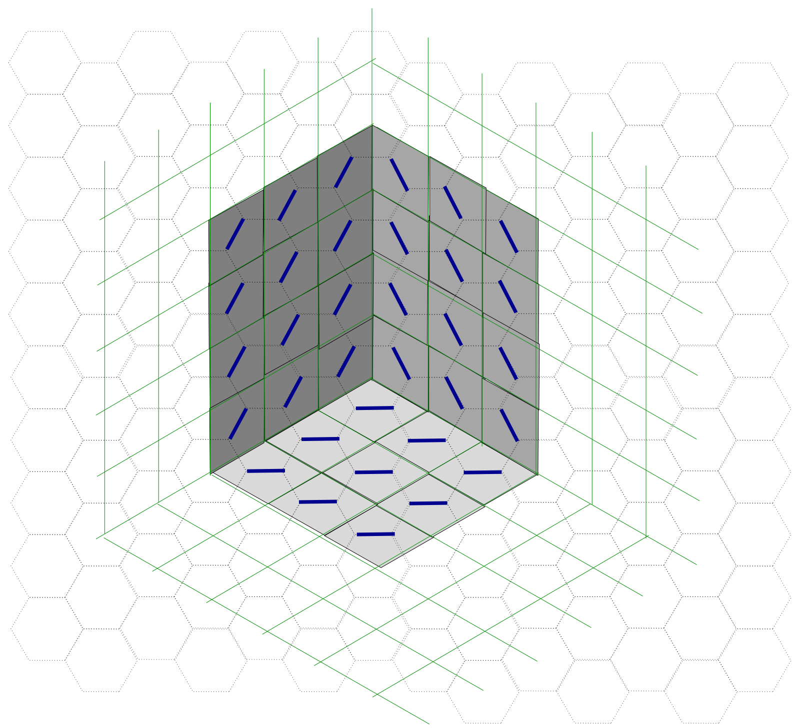
(a *dimer covering* is a pairing of adjacent vertices, so that every vertex is covered once.)



The b-c correspondence is: each lozenge contains a dimer. (b-c is bijective)

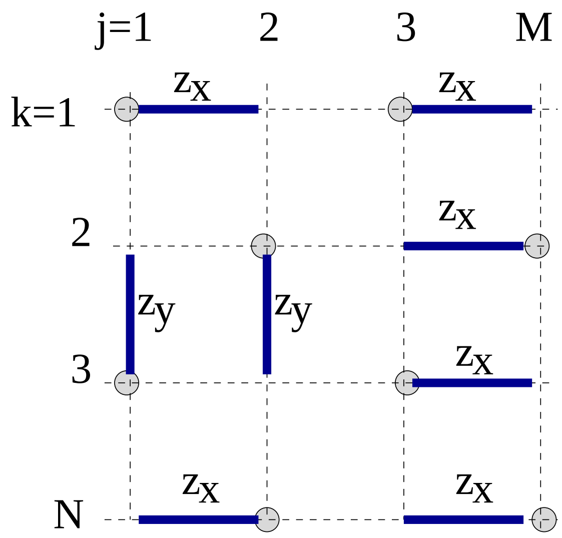
empty room

tableaux map to dimer and lozenge coverings which asymptote to the 'empty room':



Review of solution of uniform dimer model

(Temperley, Kasteleyn, Fisher, Stephenson, 1960)



$$Z(z_x, z_y) = \sum_C z_x^{n_x(C)} z_y^{n_y(C)}$$

($n_{x,y}$ are the number of horizontal/vertical dimers in the covering C)

Claim:

$$Z = \mathrm{Pf}\ \mathcal{D}(z_x, z_y)$$

$\mathcal{D}$ is $MN \times MN$.

$$\mathcal{D} = z_x \mathbb{1} \otimes J + i z_y J \otimes \mathbb{1}$$

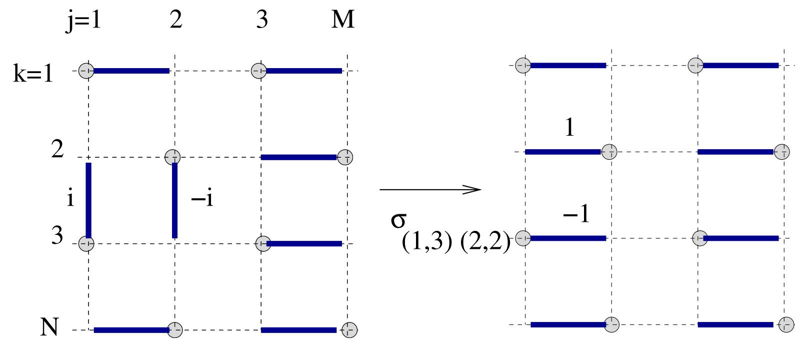
The first entry in the tensor product is the $M \times M$ row-space and the second is the $N \times N$ column space. J is the near-neighbor hopping matrix:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\ldots \\ -1 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 1 & \ldots \\ 0 & 0 & -1 & 0 & \ldots \end{pmatrix}$$

Basic idea: Nonzero entries in $\mathcal{D}$ correspond to links in the lattice. terms in the pfaffian correspond to dimer coverings.

$$\mathrm{Pf}\ \mathcal{D} = \frac{1}{2^n n!} \sum_{\sigma \in S_n} (-1)^\sigma \mathcal{D}_{\sigma_1 \sigma_2} \mathcal{D}_{\sigma_3 \sigma_4} \ldots \mathcal{D}_{\sigma_{n-1} \sigma_n}$$

the is are chosen so all coverings come with weight 1 (times $z_x \ldots$). Show that all come with same sign by comparing coverings related by allowed flips:



If the graph is bipartite (Made of two sublattices (white and black), vertices in each of which only touch vertices of the other.)

$$\mathcal{D} = \begin{pmatrix} 0 & D_{\circ\bullet} \\ D_{\bullet\circ} & 0 \end{pmatrix}.$$

More generally, the i multiplying the vertical dimers is replaced by the direction in the complex plane in which the edge points.

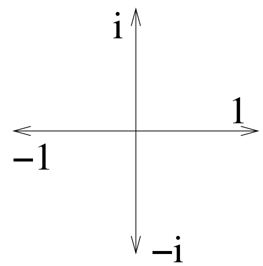
For the honeycomb lattice, $\omega^3 = 1$

$$D_{\bullet\circ} = q^{\partial_x} + \omega q^{\frac{1}{2}\left(\partial_x + \sqrt{3}\partial_y\right)} + \omega^2 q^{\frac{1}{2}\left(\partial_x - \sqrt{3}\partial_y\right)},$$

$$D_{\circ\bullet} = -q^{-\partial_x} - \omega^2 q^{-\frac{1}{2}\left(\partial_x + \sqrt{3}\partial_y\right)} - \omega q^{-\frac{1}{2}\left(\partial_x - \sqrt{3}\partial_y\right)}.$$

The important point

$\mathcal{D}$ is a latticization of a CHIRAL DIRAC OPERATOR!



$$\text{Pf } \mathcal{D} = \int \prod_{\text{sites}, v} d\psi_v \; e^{\sum_{v,v'} \psi \mathcal{D} \psi}$$

ψ are real Grassmans at every site of the lattice.
This is a latticization of a Euclidean path integral over a free 2d fermion field.

There is a Z_2 gauge symmetry in this system:

$$\psi_r \to \epsilon_r \psi_r, \quad \epsilon_r = \pm 1.$$

Solution

The eigenvalues of a matrix

$$A = X \otimes \mathbb{1} + \mathbb{1} \otimes Y$$

are of the form $\lambda_x + \lambda_y$, and the eigenvectors are just tensor products of the eigenvectors of X, Y:

$$A_{jj';kk'} v^x_{j'} v^y_{k'} = (\lambda_x + \lambda_y) v^x_j v^y_k.$$

$$Z^2 = \det A(q = 1) = \prod_{x,y} (z_x \lambda_x + i z_y \lambda_y)$$

$\lambda_{x,y}$ are the eigenvalues of the respective matrices

$X, Y \propto J$ have waves as eigenfunctions

$\lambda_{x,y}$ are Mth and Nth roots of unity.

dimer weights

So far, $q = 1$.

$$\text{3d tableaux} \leftrightarrow \quad \text{dimer coverings}$$

$$q^{|\pi|} \leftrightarrow \quad ?$$

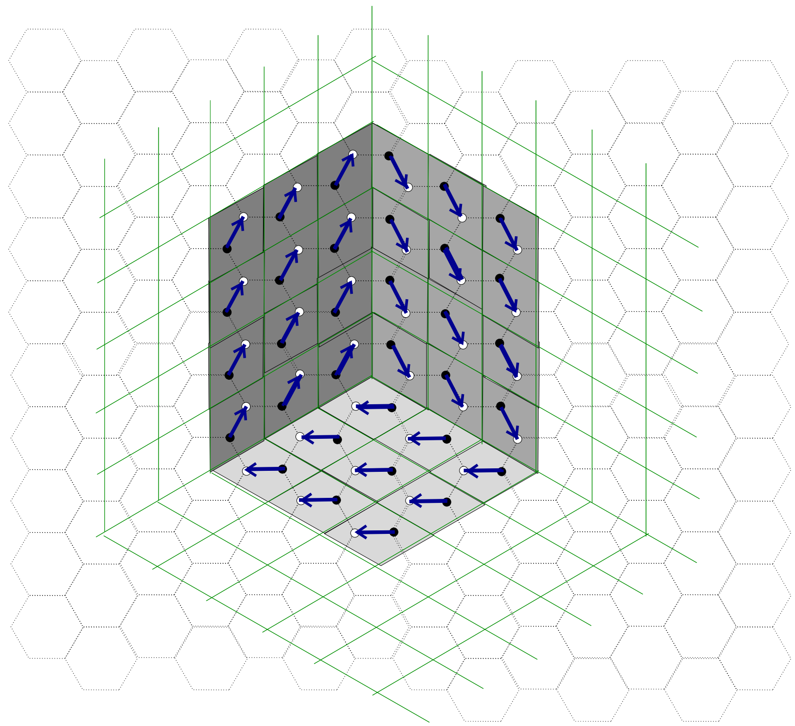
Honeycomb lattice is bipartite.$\Longrightarrow$ Dimers can be oriented.

Two dimer configurations on a bipartite lattice form a collection of *oriented closed loops*.

($\det A = (\mathrm{Pf}\ A)^2$, the determinant counts loops, the pfaffian counts dimers)

How to count boxes with dimers

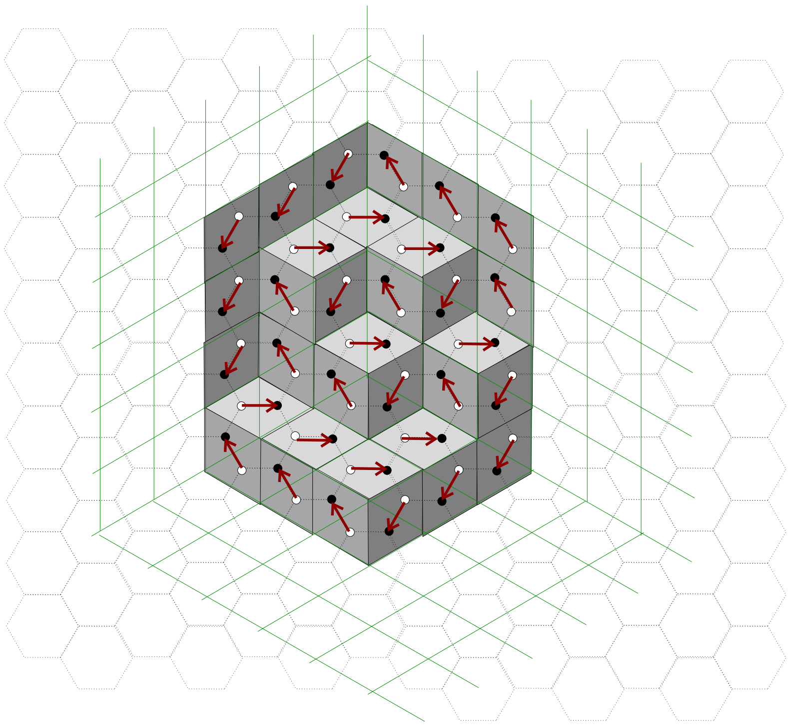
The number of boxes in a tableau is measured with respect to a reference configuration, the empty room.



The blue dimers are the empty-room tiling; on these the arrows go from black vertices to white.

How to count boxes with dimers

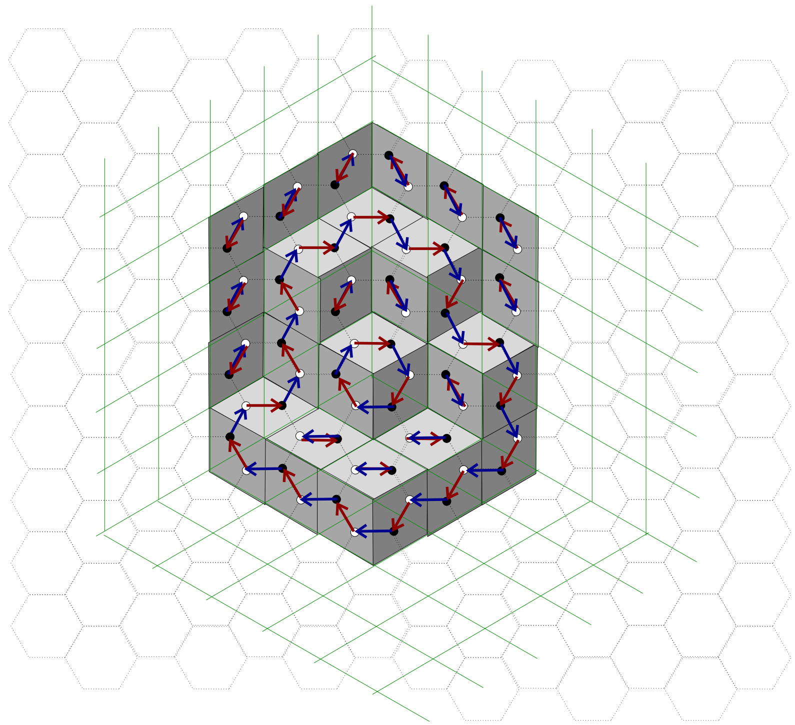
Def: A *difference* of two dimer coverings (π_1, π_2) is the set of oriented loops formed by superposing them with opposite arrow conventions.



The red dimers describe a particular 3d tableau X.

arrows go from white to black

How to count boxes with dimers



Number of boxes in X = total area inside the loops

(each plaquette counts for one unit of area).

Loops within other loops correspond to boxes which obscure one another.

This tableau contains $17 = 16 + 1$ boxes.

$q \neq 1$

So far, we've seen what $|\pi|$ is in terms of dimers. How do we modify the spin system to include this weight?

Answer: Deform the Dirac operator by the connection for a constant background magnetic field:

$$\mathrm{Area}(L) \times B = \int_{\mathrm{area}} F = \oint_L ds^\mu A_\mu$$

So those loops are Wilson loops.

The strength of the field is $e^{\int_\Sigma F} = q$ for each plaquette Σ.

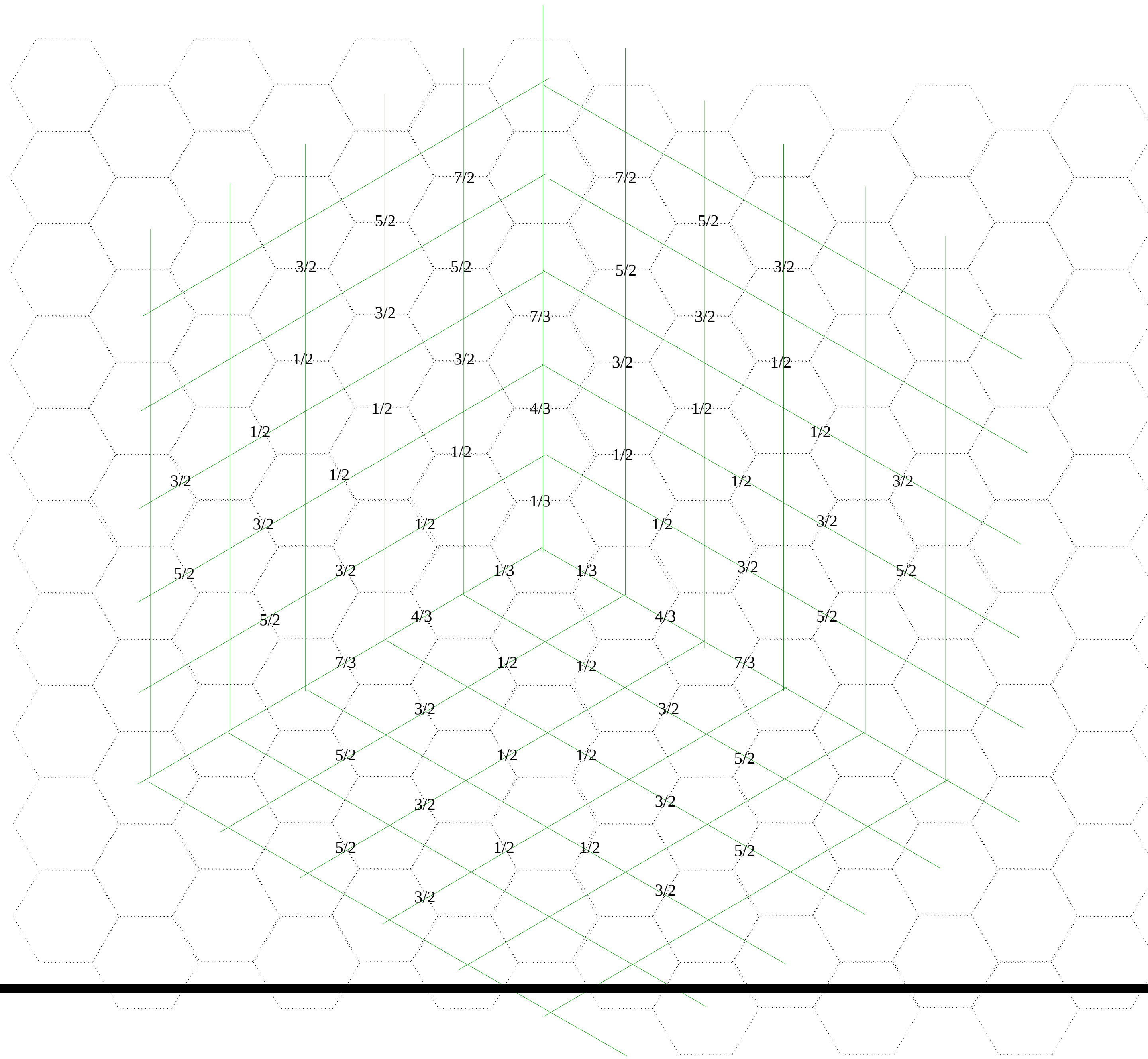
One gauge choice for this is (square lattice):

like $A_x = 0, A_y = Bx$ gauge

$$\mathcal{D}^{\bullet\circ}_{jk;j'k'}(q) = i\delta_{jj'}J_{kk'} + q^{k/2}J_{jj'}\delta_{kk'}$$

The background gauge field depends on the empty room.

z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Fermion doubling?

The Kasteleyn matrix is now of the form

$$\mathcal{D} = \begin{pmatrix} 0 & D_{\bullet\circ}(q) \\ D_{\circ\bullet}(q^{-1}) & 0 \end{pmatrix}$$

Not antisymmetric, no pfaffian.

$\psi D \psi$ depends only on the antisymmetric part of $\mathcal{D}$.

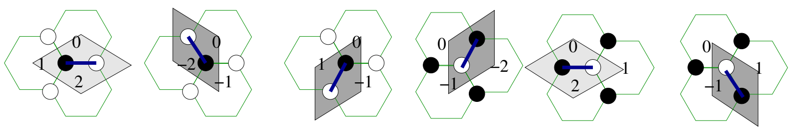
This would be fixed by complexifying the Grassmanns

$$\int d^2\psi_\bullet d^2\psi_\circ \; e^{\psi^\dagger \mathcal{D} \psi} = \det \mathcal{D} = |Z_{\mathrm{top}}|^2$$

But actually it is possible to study just the chiral part

$$\det D_{\bullet\circ} = \int d\psi_\bullet d\psi_\circ \; e^{\psi_\bullet D_{\bullet\circ}(q) \psi_\circ}$$

Bosonization



Local rules define the height function of a dimer covering:

Going around black verties counterclockwise the height variable increases when it doesn't cross a dimer; it jumps down by two when it crosses a dimer. The direction of increase is reversed around a white vertex.

This height variable is the height of the stack of boxes.

An uncovered site is a vortex for the height variable.

Observables

What are the observables of this system?

Monomer and dimer correlators

Simplest:

$$\langle \psi_r \psi_{r+e} \rangle = Z^{-1} \int d\psi \; e^{-S} \psi_r \psi_{r+e}$$

removes terms where ψ_r and ψ_{r+e} are pulled down from S.

This is the probability of having a dimer on edge **e**

The correlations are gaussian

$$\langle \psi_1 ... \psi_{2k} \rangle = \mathrm{Pf}_{rs} K_{rs}$$

with $K_{r,s} = \langle \psi_r \psi_s \rangle$. This is the inverse of the magnetic Kasteleyn matrix.

$$D_{r,s} K_{s,r'} = \delta_{r,r'}$$

A thing I think is true:

The kernel introduced by Okounkov and Reshetikhin

$$K_{r,s} = \int \frac{dz}{z} \int \frac{dw}{w} \frac{\sqrt{zw}}{z-w} z^{h_r} w^{-h_s} \Phi_{3D}(z, t_r) \Phi_{3D}^{-1}(w, t_s)$$

$$\Phi_{3D}(z,t) = \frac{\prod_{m=1}^{\infty}(1 - z^{-1}q^m)}{\prod_{m=t}^{\infty}(1 - zq^m)}, \quad (t \geq 0),$$

$$\Phi_{3D}(z,t) = \frac{\prod_{m=-t}^{\infty}(1 - z^{-1}q^m)}{\prod_{m=1}^{\infty}(1 - zq^m)}, \quad (t < 0),$$

$(h, t) = (z - (x + y)/2, x - y)$ satisfies this equation.

Classical limit

$q = e^{-g_s} \to 1$, with lattice spacing g_s.

Correlators are dominated by a saddle point (limit shape).

Local fluctuations of the height variable around this saddle point are gaussian. (Cohn-Kenyon-Propp, Kenyon, Kenyon-Okounkov)

The height variable determines the complex structure of the graph of the limit shape.

The height variable represents the same degree of freedom as the KS field.

What if you remove a spin?

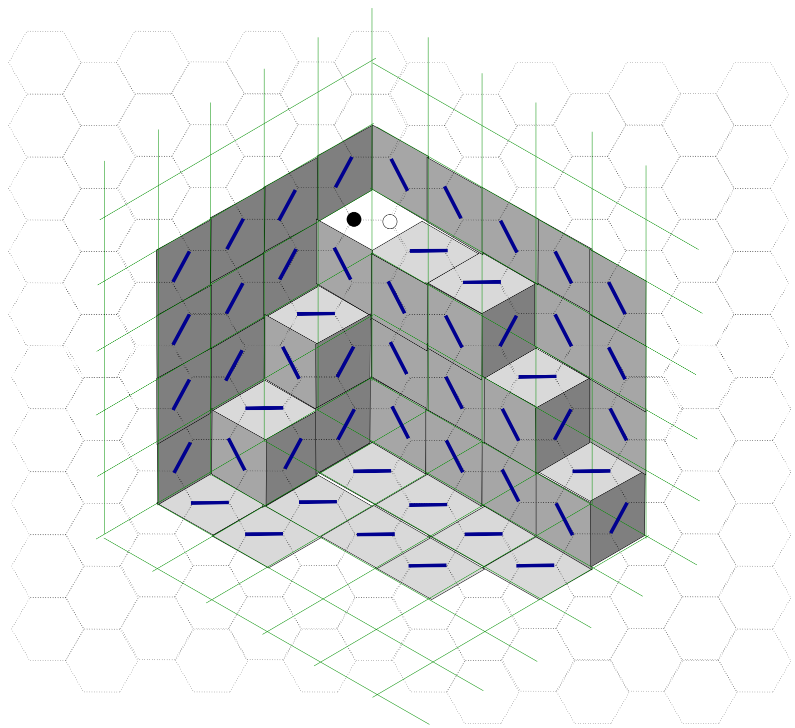
If you look at Fisher and Stephenson (1963) where they calculate the monomer 2-point function (in the uniform dimer model), you see them introduce a Z_2 wilson line in between the monomers
to keep the dimer coverings all contributing with the same sign.

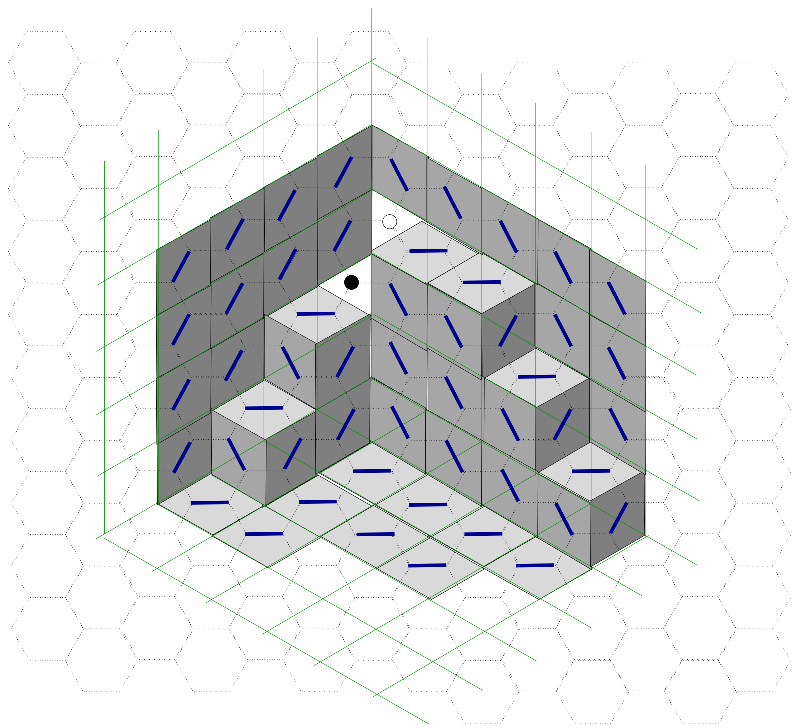
$$\psi^\circ(r)\prod_{i=0}^{k}\sigma_{r+e_i,r+e_{i+1}}\psi^\bullet_{r+e_k}$$

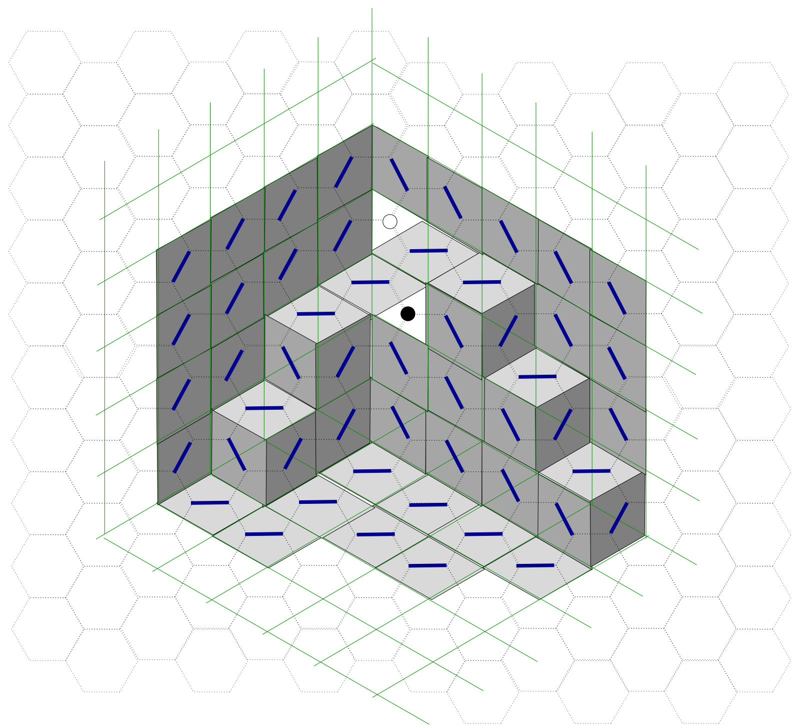
With the q-weight...?

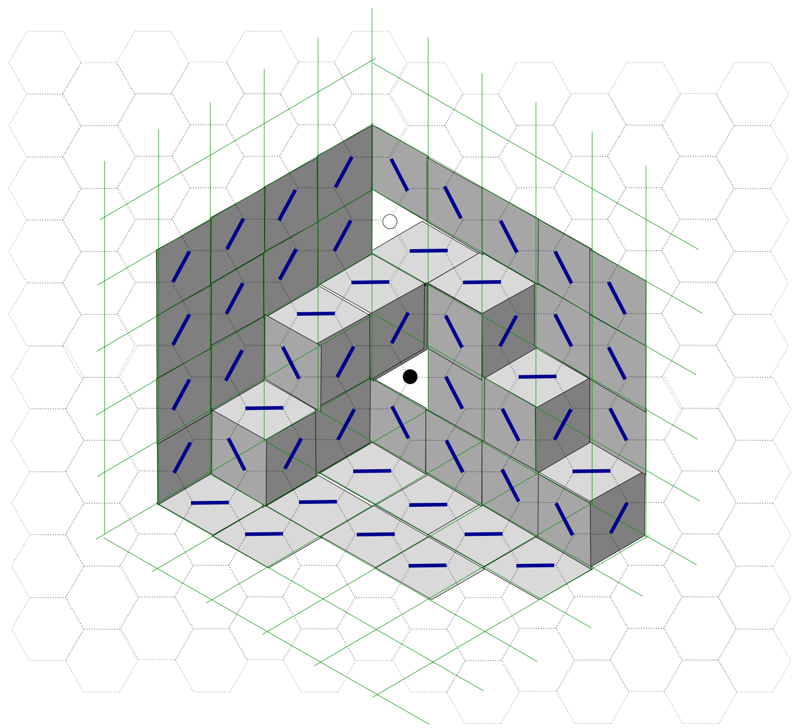
Different choices of route for this line differ by some closed loop C in the magnetized model, the ratio is $q^{A(C)}$.

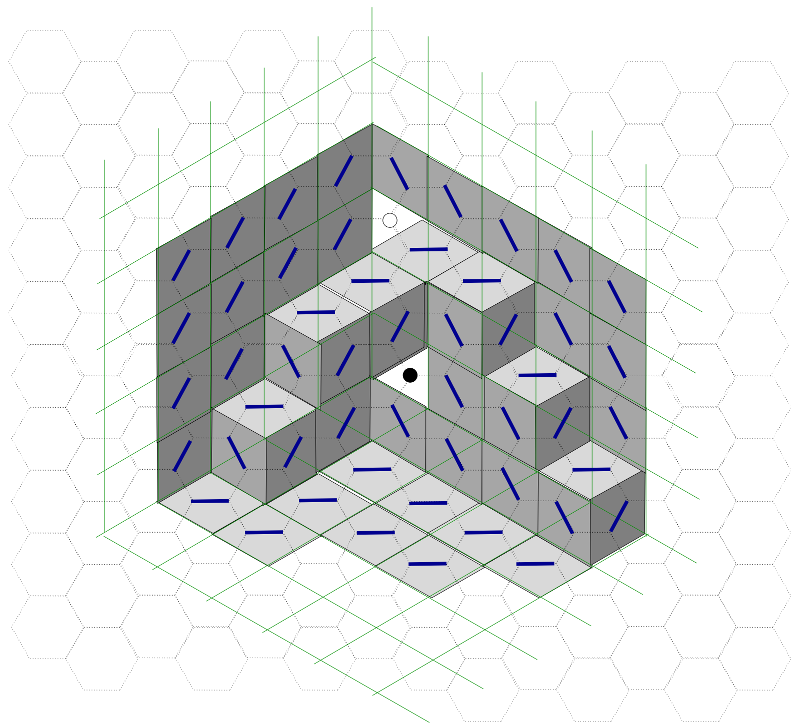
This is related to the fact that the height variable isn't well-defined near the monomer (it's a vortex).



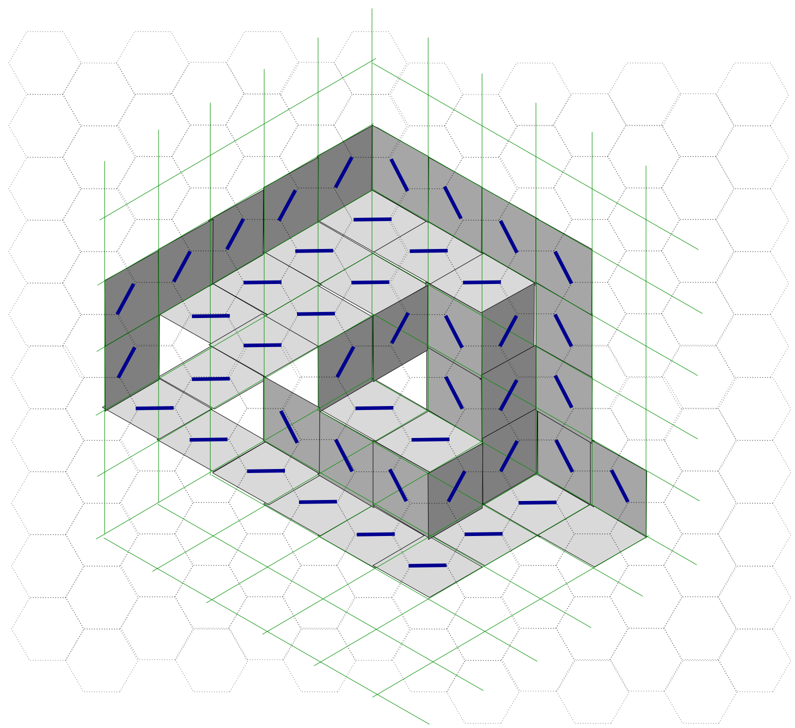




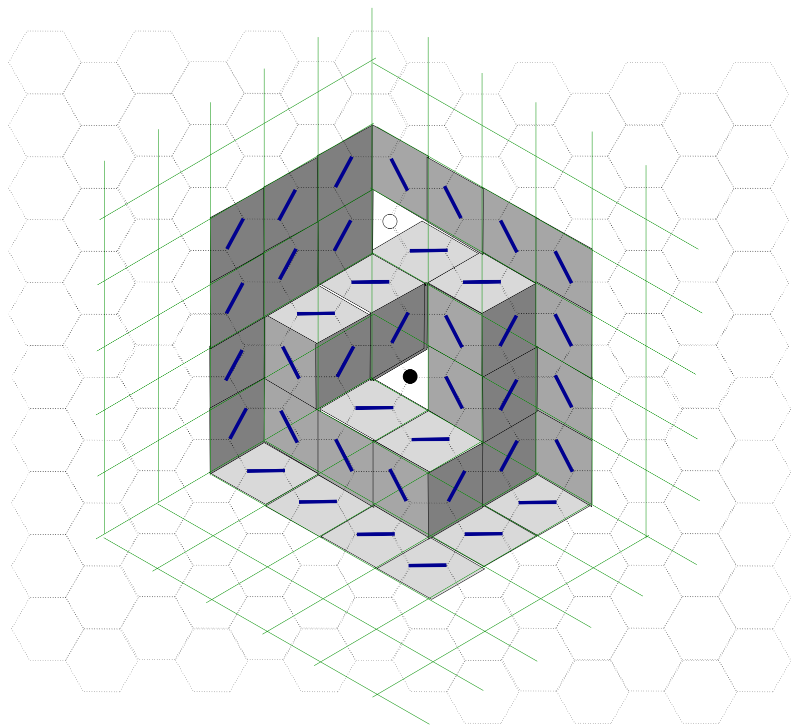




The separated monomers make an Escher staircase in the melting crystal.
By going up two steps and down three you can end up at the same height.



In this configuration you can go down without going up at all.



Intermediate steps in the calculation of dimer correlators involve these objects.

a word about D-branes

$$\sigma_{\mathbf{e}} \sim \mathbf{e}^{\psi_{r_-(\mathbf{e})}\psi_{r_+(\mathbf{e})}q^{f(\mathbf{e})}}$$

$$\Psi(r) \sim \left(\prod_{\mathbf{e}}^{r} \sigma_{\mathbf{e}}\right)\psi_r$$

$$Z(r) = Z^{-1}\langle\Psi_r\rangle$$

(Saulina-Vafa; Dijkgraaf-Sinkovics-Temurhan):

$$Z((0,a)) = \prod_{n=1}^{\infty}(1-q^{a+n})$$

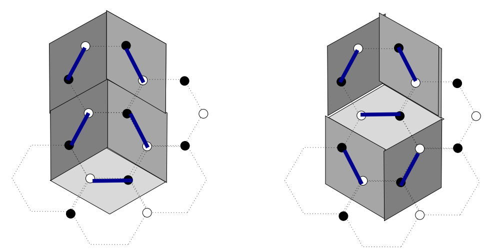
in the (h,t) coords above

Matches the string answer more or less.

Extension

The fermion description suggests a nonperturbative formulation of the topological string for other empty-room configurations.

For more general lattices, a nice characterization of 'empty room' configurations is that they have very few flippable plaquettes.



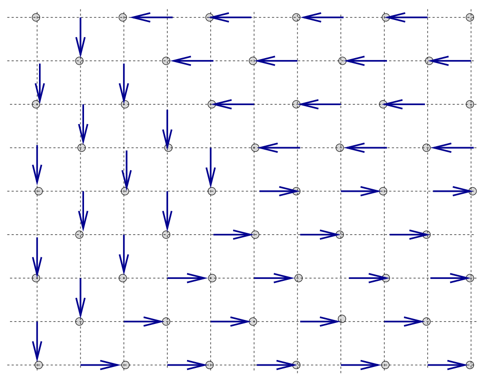
This lattice model can be formulated on any bipartite lattice. Even nonplanar ones (like for CYs with no global torus action) this requires picking holonomies of A around the one-cycles.

(and might be a little hard to solve.)

Universality

Clearly features of the scaling limit are universal.

What about before the asymptotic expansion in g_s?



e.g. 2d YM (Migdal), 3d lattice CS (Ooguri) are subdivision invariant.

Already at the continuum theory with finite lattice spacing.

Or: some dynamical mechanism for arrangement of lattice?

(chemistry)

Or: many nonperturbative definitions.

On the discretization of space

So possibly, the KS theory (a 6d gauge theory) exists in the UV because it's a lattice model.

It is tempting to extrapolate this short-distance completion to superstrings.

Stringy spacetime is made of little tetrahedrons, and this is why it's UV finite.

But:

topological string $= H^\star$(superstring)

BPS constraints on phase space $\longrightarrow$ noncommutative structure of space (Das-Jevicki-Mathur, Lin-Lunin-Maldacena, Mandal, Ooguri-Vafa-Verlinde)

BPS branes act like free fermions.

and BPS backgrounds can be described as a many-fermion ground state.

Replication

Chiral fermions on the lattice cry out for fermion doubling.

$$\longrightarrow |Z_{\rm top}|^2 ?$$

There is an interesting class of models with Ns at $\bullet$ sites, $\bar{N}$s at $\circ$ sites.
The connection can be made nonabelian.

$$\int d^N\psi^\bullet d^N\psi^\circ \; e^{\psi^\dagger D(A)\psi} = e^{S_{CS}(A)}$$

discrete Polyakov-Wiegmann formula.

Escape from two dimensions

Q: States of a 2d free fermion are labelled by young tableaux.
Whose states are labelled by 3d tableaux ?

A: The model obtained by adding an extra time direction.
$\int \prod d\psi \longrightarrow \int \prod [d\psi(t)]$
it's an antiferromagnetic spin system:
"quantum dimer model" (Rokhsar-Kivelson), a model of resonating valence bonds.
Its vacuum correlations are computed by the classical dimer model.

This is the same operation we perform to get "topological M-theory"
a mysterious 7d theory whose ground states are $CY \times S^1$,
excited states are more general G_2 manifolds

The end