Phases of matter
which are characterized by their edge states

lecture notes for SPT FRIDAYS at UCSD in May and June 2013

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[Main references: Senthil and Vishwanath, 1209.3058 [1]; Senthil and Xu, 1301.6172 [2] and some talks by those authors. A useful, brief review is the second part of [3]. See also this Journal Club for Condensed Matter Physics commentary by Matthew Fisher.]

0.1 Introductory remarks

These are notes for a series of lectures given at SPT FRIDAYs in May and June 2013.

The goal is to understand interacting phases of matter (in D spacetime dimensions) which can be characterized by their edge states (in $D-1$ spacetime dimensions). We can refine this understanding by specifying the symmetry group G of the Hamiltonian. Such phases are called symmetry-protected topological (or trivial) states, or SPT states, protected by the group G.

Notice that the possibility of characterizing a state this way implies that the $D-1$-dimensional models involved must not be realizable intrinsically in $D-1$ dimensions, consistent with the symmetries involved. If they could be so realized, preserving G, we could paint them on the surface of any bulk state, and they therefore could not be characteristic. This suggests a useful strategy for identifying obstructions to symmetry-preserving regulators of quantum field theory (QFT), which has been exploited so far in examples in [1, 4, 5, 6]. An attempt to systematize these obstructions is here [7].

Necessary features of a state which may be characterized by its edge include:

- A bulk energy gap. For example, without a gap, stuff we do at the boundary can have an influence on what happens in the interior, which falls off only like a power law, which we cannot control. In contrast, with an energy gap, the influence of modifications at the edge can be *parametrically* localized by increasing the gap.
- No topological order[1], and in particular no fractionalized bulk excitations. Such nontrivial bulk excitations would by definition not care about anything we do to the edge states, no matter how horrible.

These states have a group structure (explained in [3]). The group law is just composition: take the two summand states and stick them on top of each other, and let them interact in all ways consistent with the protecting symmetry G. The inverse of an SPT state is its mirror image; this allows the edge states to pair up.

[1] Let's provisionally say that a state has topological order if its groundstate degeneracy depends on the topology of space. The best definition of topological order in general dimension is, I think, an interesting open question. In 2+1 dimensions, nontrivial transformation under adiabatic modular transformations seems to capture everything [8]; topology-dependent groundstate degeneracy is a corollary.

A conjecture for the form of this group for a given G, D is given in [9]. There are exceptions to it [1, 4].

We will make our way towards studying bosons in 3+1 dimensions. These make an interesting point of focus because interesting states of bosons require interactions, lest they condense. 'Bosons' includes spin systems (via a map which can be found in cond-mat texts), in which context it is better to call these states (symmetry-protected) 'topological paramagnets' [1]. They are the basic ingredient in SPT states of fermions [10, 11, 12, 13, 14].

How can we characterize such states?

1. For symmetry groups which contain a $U(1)$ factor, we can couple them to external 'electromagnetic' gauge fields, and ask about the response. In 2+1 dimensions without time-reversal symmetry, this means a quantized Hall response; in 3+1 dimensions with time-reversal symmetry, this means a quantized magnetoelectric effect.

2. In the absence of a U(1) symmetry, we will have to do something else. One route to identifying a label on SPT states (which will allow us to distinguish them without trying to find every possible path between them in the space of Hamiltonians) is to ask what happens if we *gauge* (some subgroup of) the protecting symmetry [15, 16].

3. Another possibility is to focus on the weird stuff that goes on at the surface. Following [1], the point of view of defects in the would-be-bose-condensate – the dual vortices – turns out to be very fruitful. The completely trivial Mott insulator of the bosons is described in this langauge as a condensate of featureless vortices (in 2+1d; in 3+1d they are vortex loops). More interesting states are obtained if the vortex defects carry some decorations. A scheme for understanding all boson SPT states then follows by characterizing possible decorations of the vortices that can keep them from condensing without doing something else interesting (like breaking a symmetry or producing topological order).

1 Building blocks

The simplifying assumptions defining SPT states allow them to be built from lower-dimensional parts. This is called the 'coupled-layer' (or (I think misleadingly because of translation-invariance issues) 'network-model') construction. So it will be a good idea to understand lower-dimensional examples well. We begin with a brief tour through the examples which will appear the most. All of the ingredients (I think) involved in this tour will be used crucially in the later discussion!

1.1 $D = 0 + 1$ edge: A single spin $\frac{1}{2}$

A single spin-$\frac{1}{2}$ is arguably the simplest example of an unkillable state, and hence a possible edge state for an SPT state[2]. The general hamiltonian (up to a constant) is

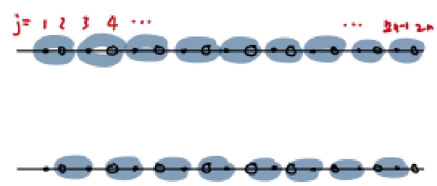
$$\mathbf{H} = \vec{h} \cdot \vec{\boldsymbol{\sigma}},$$

parametrized by the Zeeman field $\vec{h}$. If we insist on preserving the SU(2) spin rotation symmetry, the Hamiltonian has to be zero, and we are guaranteed a degeneracy. Alternatively, if we preserve the anti-unitary symmetry which acts by $\mathcal{T} : \vec{\boldsymbol{\sigma}} \mapsto -\vec{\boldsymbol{\sigma}}$ – the usual action of time-reversal on a spin – then we are also guaranteed this degeneracy of this *Kramers' doublet*.

So we may say that the degeneracy is protected by *either* SU(2) or $\mathbb{Z}_2^{\mathcal{T}}$.

Coherent state quantization of a spin-$\frac{1}{2}$ will play a useful role below. For an explanation of this, see §5.2.2 and §5.2.3 of my Physics 215C lectures or Dan Arovas' *Quantum Mechanic's Toolbox*.

1.2 $D = 1 + 1$ bulk: Haldane chains



Chains of integer spins provide the oldest known example of an SPT state. The important property of integer spins is that they don't faithfully represent all of SU(2) – only SO(3) actually acts on them. The chain of spin-1s is easiest to realize physically, but it will be easier to discuss a chain where each site has an additional singlet. So if you like, we are embedding the Hilbert space of each spin 1 into that of a pair of spin $\frac{1}{2}$s (as in the figure at right)[3]. The (Heisenberg) Hamiltonian

$$\mathbf{H}_{\text{Heisenberg}} = \sum_{j=1}^{2n-1} J_j \vec{\mathbf{S}}_j \cdot \vec{\mathbf{S}}_{j+1}$$

preserves spin SU(2) and $\mathbb{Z}_2^{\mathcal{T}}$. We'll let

$$J_j = \begin{cases} J_o, & j \text{ odd} \\ J_e, & j \text{ even} \end{cases}$$

The trivial phase is obtained by making spin singlets of spin $\frac{1}{2}$s associated to a single site (top figure); this is preferred when $J_o > J_e$. The non-trivial (Haldane) phase – preferred

[2] I say arguably because it is not in the thermodynamic limit

[3] So the model I am describing here is not exactly a spin-1 chain. For each pair of sites, we have $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. I find it easier not to project out the extra singlet.

when $J_o < J_e$ – is made by pairing spins from successive sites; this leaves behind an unpaired spin $\frac{1}{2}$ at each end, and realizes the edge theory mentioned in subsection §1.1.

Both of the previous states are gapped. The critical point between these two states occurs when $J_o = J_e$, and this is simply the spin $\frac{1}{2}$ (nearest-neighbor) Heisenberg model. This model has a gapless spin wave excitation, described by a certain 1+1d CFT, which (for this reason and many others) is the subject of 1.3.

These states have a useful continuum description in terms of a non-linear sigma model (NLSM) whose field variable is the would-be Neel vector $\hat{n} \in S^2$. This can be derived quite directly by coherent state quantization. The (euclidean) action is

$$S[\hat{n}] = \int \mathrm{d}x\mathrm{d}t \frac{1}{g^2} \partial_\mu \hat{n} \cdot \partial^\mu \hat{n} + \mathbf{i}\theta\mathcal{H} \ .$$

The *theta* term here is

$$\mathcal{H} = \int \mathrm{d}x\mathrm{d}t \frac{1}{\Omega_2} \hat{n}^a \partial_x \hat{n}^b \partial_t \hat{n}^c \epsilon_{abc}$$

where ϵ_{abc} is the completely antisymmetric tensor with nonzero entries equal to ± 1. This theta term (sometimes called Hopf term) is very topological. It is an integral of a total derivative. It therefore does not affect the equations of motion for $\hat{n}$, and it does not appear in Feynman rules. But it does matter for non-perturbative physics. It has many similarities to the θ term in gauge theory in $D = 3+1$, $S_{D=4}[A] \ni \mathbf{i}\theta \int_{M_4} \frac{\mathrm{tr} F \wedge F}{16\pi^2}$ (the coefficient of $\mathbf{i}\theta$ is also an integer if M_4 is closed.).

$\mathcal{H}$ counts the number of times the field configuration $\hat{n}$ maps the spacetime (which is a two-sphere for maps which approach a unique value at ∞) into the target space S^2. You can see this from the fact that the integrand in $\mathcal{H}$ is the area element of the image of the map. If we integrate over a closed spacetime manifold, $\mathcal{H}$ is an integer – it represents $\pi_{D=1+1}(S^2) = \mathbb{Z}$. Naively this means that physics only depends on θ modulo 2π. It is true of the bulk spectrum, which is determined by the partition function, obtainable from the partition function on a closed spacetime. But it is not true of all physics.

Classically, g is marginal (recall that a scalar field has engineering dimension zero in $D = 1+1$), but quantumly, this model is asymptotically free. More specifically, because the target space S^2 has positive curvature, its radius ($\propto g^{-1}$) goes toward zero in the IR. This is simple to see in perturbation theory (where the θ term has no effect) about small g (see *e.g.* Physics 215C lectures, §5.2).

What happens when $g \gg 1$? Then perturbation theory breaks down and θ can matter. For $\theta \in 2\pi\mathbb{Z}$, it is a nontrivial fact, analogous to the behavior of $D = 3+1$ Yang-Mills theory, that this produces, by dimensional transmutation, a mass gap,

$$\Lambda_{\text{Haldane}} \sim \Lambda_0 e^{-\frac{c_0}{4\pi g^2}}$$

where c_0 is a number and Λ_0 is the UV scale at which we specify g. I've called it Λ_{Haldane}, but the first evidence for it (at $\theta = 0$) came from Polyakov [17].

When $\theta = \pi$, the story is different.

For a review of this subject (especially from the QFT point of view which I am emphasizing), see Affleck, *Quantum spin chains and the Haldane gap*.

We can also allow NN interactions, terms in $\mathbf{H}$ of the form $J^{(2)}\vec{\mathbf{S}}_j \cdot \vec{\mathbf{S}}_{j+2}$. If we make $J^{(2)}$ big enough, we can close the bulk gap. In particular, near $J_2 \sim \frac{1}{4}J_1$ (like in the long-range Haldane-Shastry chain with $J^{(n)} \sim \frac{1}{2}^n$ which is critical) we have a ∞-order transition where the gap opens.

At a bit larger $J^{(2)}$, at the *Majumdar-Ghosh point* of the spin $\frac{1}{2}$ chain, the Hamiltonian is exactly solvable and has two groundstates related by a one-site translation. These states are valence bond states, just of the form of the spin-1 limit, but now the two states are degenerate, *spontaneously* breaking translation invariance.

We'll come back to the variation of the edge states within the bulk phase, and the important question of which edge phase transitions must be bulk phase transitions.

We can also add terms which go like powers of $\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j$, as in the AKLT models. For more on the space of spin-symmetric Hamiltonians on the spin $\frac{1}{2}$ chain see Dan Arovas' notes or the book by Auerbach.

1.3 $D = 1 + 1$ edge: $\mathrm{SU}(2)_1$ WZW conformal field theory

Just now we talked about SO(3) spin chains (*i.e.*̃spin chains where each site of the lattice is in a representation of SO(3), not a projective one, as per footnote 3), and the fact that they have a gapped phase and are described by a NLSM with $\theta = 2\pi$. More generally, the coefficient of the topological term is $\theta = 2s\pi$. I want to say more about the continuum QFT describing the gapless state that arises when $2s$ is odd. This model will play a very important role below.

This model has many different representations, related by various kinds of bosonization operations. For a longer review of 2d CFT in a related context, see Fradkin's 2d edition, chapter 7.

All these models have Virasoro central charge $c = 1$. For a discussion of the definition and importance of this quantity, see *e.g.* Ginsparg's lectures [18].

1. A single free boson 'at the SU(2) radius'. Recall that a free boson in 1+1d dimensions

enjoys a line of fixed points[4]; the coordinate along this line is called the 'Luttinger parameter' or the 'radius'; the latter interpretation being natural if we think of the scalar field as the coordinate in a sigma model on the circle. This parameter determines the spectrum of scaling dimensions of the vertex operators $e^{\mathbf{i}\alpha\cdot\phi}$.

A word about massless fields in two dimensions: Recall that the general solution to the wave equation in $D = 1 + 1$

$$0 = \Box\phi = \left(-\partial_t^2 + \partial_x^2\right)\phi \quad \text{is} \quad \underline{\phi}(x,t) = \phi_+(x+t) + \phi_-(x-t).$$

In euclidean time, $\phi_\pm$ depend (anti-)holomorphically on the complex coordinate $z \equiv x + \mathbf{i}\tau$ and the machinery of complex analysis becomes useful.

Consider a compact free boson $\phi \simeq \phi + 2\pi$ in $D = 1 + 1$ with action

$$S[\phi] = \frac{R^2}{8\pi}\int \mathrm{d}x\mathrm{d}t\partial_\mu\phi\partial^\mu\phi. \tag{1}$$

Notice that if we redefine $\tilde{\phi} \equiv R\phi$ then we absorb the coupling R from the action $S[\tilde{\phi}] = \frac{1}{8\pi}\int \mathrm{d}x\mathrm{d}t\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi}$ but now $\tilde{\phi} \simeq \tilde{\phi} + 2\pi R$ has a different period – hence 'radius'.

So: there is a special radius (naturally called the SU(2) radius) where new operators of dimension $(1,0)$ and $(0,1)$ appear, which are charged under the boson number current $\partial_\pm\phi$. Their dimensions tell us that they are (chiral) currents, and their charges indicate that they combine with the obvious currents $\partial_\pm\phi$ to form the (Kac-Moody-Bardakci-Halpern) algebra $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.

Exercise: $SU(2)_1$ current algebra from free bosons

Here you will verify that the model (1) does in fact host an $SU(2)_L \times SU(2)_R$ algebra involving *winding modes* – configurations of ϕ where the field winds around its target space circle as we go around the spatial circle. We'll focus on the holomorphic (L) part; the antiholomorphic part will be identical.

Define

$$J^\pm(z) \equiv :e^{\pm i\phi(z)}:, \qquad J^3 \equiv i\partial\phi(z).$$

The dots indicate a normal ordering prescription for defining the composite operator: no wick contractions between operators within a set of dots.

(a) Show that $J^3, J^\pm$ are single-valued.

(b) Compute the scaling dimensions of $J^3, J^\pm$. The scaling dimension Δ of a holomorphic operator in 2d CFT can be extracted from its two-point correlation function:

$$\langle\mathcal{O}^\dagger(z)\mathcal{O}(0)\rangle \sim \frac{1}{z^{2\Delta}}\ .$$

[4] actually the space of fixed points is a bit more complicated – it has a branched structure (see Figure 1 of [19]).

For free bosons, all correlation functions of composite operators may be computed using Wick's theorem and

$$\langle \phi(z)\phi(0)\rangle = \frac{1}{R^2}\log z.$$

(c) Defining $J^{\pm} \equiv \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)$ show that the operator product algebra of these currents is

$$J^a(z)J^b(0) \sim \frac{k\delta^{ab}}{z^2} + i\epsilon^{abc}\frac{J^c(0)}{z} + \ldots$$

with $k = 1$. This is the level-$k = 1$ SU(2)Kac-Moody-Bardakci-Halpern algebra.

(d) [Bonus tedium] Defining a mode expansion for a dimension 1 operator,

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

show that

$$[J^a_m, J^b_n] = i\epsilon^{abc}J^c_{m+n} + mk\delta^{ab}\delta_{m+n}$$

with $k = 1$, which is an algebra called Affine $SU(2)$ at level $k = 1$. Note that the $m = 0$ modes satisfy the ordinary $SU(2)$ lie algebra.

For more details (and the solutions and some applications in string theory) see problem 5 here.

2. By varying the radius of the boson above, we can reach 'the free-fermion radius', where the model is equivalent to a single Dirac fermion. Each value of the radius specifies a CFT with $c = 1$. Therefore, we see that a single free (complex) fermion is related to $\mathrm{SU}(2)_1$ by a marginal deformation.

   For more on abelian bosonization, see problem 2 here and problem 1 here.

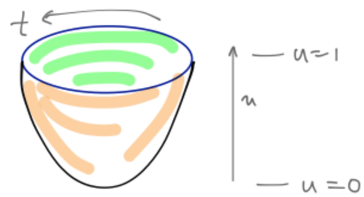
3. Non-linear sigma model on SU(2) with WZW term at level 1 [20]. In this model, the field variable $\mathfrak{g}$ takes values in the group SU(2). The SU(2) group manifold is S^3, as you can see by the mapping

$$\mathfrak{g} = \phi^0 \mathbb{1} + \mathbf{i}\vec{\boldsymbol{\sigma}}\cdot\vec{\phi}, \quad \phi_0^2 + \vec{\phi}^2 = 1.$$

   The non-abelian currents are [20]

$$J_+ = \frac{\mathbf{i}}{2\pi}\mathfrak{g}^{-1}\partial_+\mathfrak{g}, \quad J_- = -\frac{\mathbf{i}}{2\pi}\partial_-\mathfrak{g}\mathfrak{g}^{-1}.$$

   In order to write its action in a manifestly symmetric way, we have to extend the field into a (possibly fictitious) extra dimension whose coordinate is u. We do this in such a way that the real system lives at $u = 1$: $\vec{\mathfrak{g}}(t, x, u = 1) = \vec{\mathfrak{g}}(t, x)$, and $\mathfrak{g}(t, x, u = 0) = \mathbb{1} \in \mathrm{SU}(2)$ – it goes to the north pole at the other end of the extra dimension for all t. This means that the space is really a disk with the origin at $u = 0$, and the boundary at $u = 1$. Call this disk B, its boundary ∂B is the real spacetime.



We can write the WZW term in terms of the S^3-valued field $\phi^{0,1,2,3}$ as

$$\mathcal{W}[\phi] = \frac{2\pi}{\Omega_3} \int_{B_3} \phi^{\cdot} \wedge \mathrm{d}\phi^{\cdot} \wedge \mathrm{d}\phi^{\cdot} \wedge \mathrm{d}\phi^{\cdot} \epsilon_{....}.$$

The integrand here is the volume element of the image of a chunk of spacetime in the target S^3. If we integrate over the union of two balls with cancelling boundaries $B_3 \cup \bar{B}_3$, we get an integer multiple of 2π.

The coefficient k of $\mathcal{W}$ in the action must be an integer since B_3 and $\bar{B}_3$ give equally good definitions of $\mathcal{W}$, which differ by $2\pi k$, which will not affect the path integral if $k \in \mathbb{Z}$.

This WZW term is *less* topological than the theta term we discussed above, in the sense that it affects the equations of motion for $\mathfrak{g}(x,t)$.

The following table gives a comparison between theta terms and WZW terms for a field theory in D spacetime dimensions, on a spacetime M_D:

theta term	WZW term
$\mathcal{H} = \int_{M_D} h$ $h = \mathrm{d}q$	$\mathcal{W} = \int_{B_{D+1}} w, \quad \partial B_{D+1} = M_D$ $w = \mathrm{d}v$
Doesn't affect EOM	Affects EOM
Invisible in perturbation theory	Appears in perturbation theory, *e.g.* in beta functions
$\mathcal{H} \in \mathbb{Z}$ for M_D closed	Coefficient of $\mathcal{W} \in \mathbb{Z}$ in order for path integral to be well-defined.

4. Massless Schwinger model.

$$Z = \int [D\psi_\alpha D\bar{\psi}_\alpha DA_\mu] e^{-S}, \quad S = \int \mathrm{d}^2 x \left(\sum_{\alpha=1,2} \bar{\psi}_\alpha \gamma^\mu \left(\partial_\mu - \mathbf{i} A_\mu\right) \psi_\alpha - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \right)$$

This model has a manifest $U(N)$ symmetry, where N is the number of fermion species, acting by $\psi_\alpha \to U_{\alpha\beta}\psi_\beta$. Actually, the symmetry is bigger – we can act independently on left- and right-movers[5]

But the U(1) axial current

$$j_5^\mu \equiv \sum_\alpha \bar{\psi}_\alpha \gamma \gamma^\mu \psi_\alpha, \quad \gamma \equiv \gamma^t \gamma^x$$

is anomalous:

$$\partial_\mu j_5^\mu = -\mathbf{i} \frac{N}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}|_{N=2} \ .$$

$$A_\mu = \epsilon_{\mu\nu} \partial_\nu \theta^a + \partial_\mu \theta^b$$
$$\psi_\alpha \to e^{-\mathbf{i}\gamma\theta^a - \mathbf{i}\theta_b} \psi_\alpha \equiv \psi'$$

makes ψ' neutral [21].

The theta angle in the spin system *is* the theta angle in the gauge theory $\theta \epsilon_{\mu\nu} F^{\mu\nu}$.

One reason to care about this description (besides the fact that it comes from ancient high-energy literature [22]) is that it is a successful example of a *parton construction* – the introduction of a gauge theory as a description of a system of condensed matter. The fermions (in this context, 'spinons') are related to the original spins by

$$\vec{\mathbf{S}}_r = \frac{1}{2} \psi_r^\dagger \vec{\boldsymbol{\sigma}} \psi_r$$

where I am omitting the $\alpha = 1, 2$ index. In order for the model of ψs to describe the same Hilbert space, we must project onto states where there is one spin per site, and there are no charge ($\psi_r^\dagger \psi_r$) fluctuations ; this projection is implemented by A_t, the time component of the gauge field above.

A second reason to care about this description is that it illuminates what happens when we keep cranking up θ past 2π ([23], section 4). In 1+1d, an electrically-charged particle is like a capacitor plate, separating regions with different values of the electric field; the jump in the field is proportional to its electric charge. The theta term in 2d gauge theory $\theta \int \mathrm{d}x \mathrm{d}t \epsilon_{\mu\nu} F^{\mu\nu} = \theta \oint_{C_\infty} A$ is then a demand for an electric field of magnitude θ, since it's the source term associated with a particle of charge θ at infinity.

But an electric field can be screened by the (massless!) dynamical charges created by $\psi^\dagger$. If the field is bigger than $e/2$ then it is energetically favorable to pair create the particles and reduce theta. The particles then fly to the boundary.

This screening of theta has an implementation in the spin chain as well. A useful description of a higher-spin chain (associated with $\theta = 2s\pi$) can be obtained by decomposing the spin-s into a collection of spin $\frac{1}{2}$s. The screening process is described in terms of the formation of valence bonds between various constituent spin-$\frac{1}{2}$s.

[5]Recall that in $D = 1 + 1$ we may choose a basis for the Dirac gamma matrices (2×2) such that the Dirac equation takes the form

$$0 = \gamma^\mu D_\mu \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} D_+ \psi_- \\ D_- \psi_+ \end{pmatrix}$$

so that the spin chirality $\gamma \psi_\pm = \pm \psi_\pm$ ($\gamma = \gamma^t \gamma^x$) is correlated with the movingness $\psi_\pm = \psi_\pm(x_\pm)$. A nice choice is $\gamma^t = \mathbf{i}\boldsymbol{\sigma}^2, \gamma^x = \boldsymbol{\sigma}^1$.

Note that the fixed point of the sigma model with WZW term of level k has central charge

$$c = \frac{3k}{k+2}$$

which is $c = 1$ for $k = 1$ and $c = 3$ for $k = \infty$. This latter result makes sense because the fixed point for the radius of the target space is proportional to k. In the large k limit, the curvature therefore becomes small, and the model becomes a semiclassical model of weakly coupled coordinates on the target space, and must have $c =$ the number of dimensions of the target space.

Notice that the fixed point has a huge emergent symmetry group, $\mathrm{O}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. This is manifest in terms of the matrix representation, where the two SU(2)s are right and left action on the matrix. It is a weird fact that the lexicographic right and left actions on the matrix are also respectively represented by right- and left-moving chiral currents in spacetime.

This O(4) mixes the Neel order parameter $\vec{S}$ with VBS order:

$$g = \mathbb{1} \underbrace{\phi^0}_{\substack{\text{dimer}\\ \text{ordering}\\ \text{(spin singlet)}}} + \mathbf{i}\vec{\boldsymbol{\sigma}} \cdot \underbrace{\vec{n}}_{\substack{\text{Neel}\\ \text{ordering}\\ \text{(spin triplet)}}}$$

A physical consequence of this is that the spin-spin correlations should have the same power laws as those of the dimer ordering.

1.4 $D = 2 + 1$ bulk: Boson $\nu = 2$ IQH state

We can distinguish SPT states in 2+1 dimensions with a U(1) symmetry by their quantized Hall response. No fractionalization means that σ_{xy} must be an integer in the basic units of $\frac{e^2}{h}$. The argument for this actually has a stronger consequence [24] for bosons: σ_{xy} must be an *even* multiple of $\frac{e^2}{h}$. Thread 2π worth of localized magnetic flux through some region of the sample (as in the $\otimes$ at right). This means

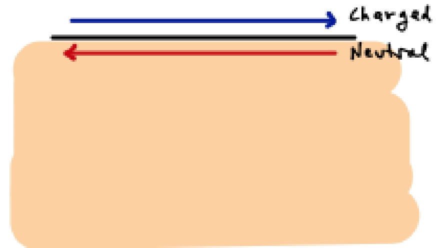
$$2\pi = \Delta\Phi = \int \mathrm{d}t \partial_t \left(\oint_{R|\partial R = C} \mathrm{d}\vec{a} \cdot \vec{B} \right) \overset{\text{Faraday}}{=} -\int \mathrm{d}t \oint_C \vec{E} \cdot \mathrm{d}\vec{\ell} \overset{j_r = \sigma_{xy} E_\varphi}{=} -\frac{1}{\sigma_{xy}} 2\pi \underbrace{\int \mathrm{d}t j_r}_{=\Delta Q}$$

which says that the inserted flux sucks in an amount of charge

$$\Delta Q = \sigma_{xy}.$$

This object is a localized excitation of the system, and hence must have bosonic statistics, by assumption of no fractionalization. But from the Bohm-Aharonov effect, it has statistics angle $\pi\sigma_{xy}$, which must therefore be in $2\pi\mathbb{Z}$.

A realization of the basic boson IQHE with $\nu \equiv \frac{\sigma_{xy}}{e^2/h} = 2$ is provided in [24]. In addition to electrical Hall conductivity, it has *no* thermal Hall conductivity. In general, this is proportional to $c_L - c_R$, the chiral central charge. These properties can be understood quite directly from the edge CFT, which is just the (non-chiral in matter content) $\mathrm{SU}(2)_1$ WZW model, with a funny assignment of symmetries. Specifically, although $c_L = c_R$, only the left mover ϕ_+ carries the U(1) charge. This means that preserving U(1), we can't backscatter, that is we can't add to the action terms like $\Delta S = g_\pm \cos(\phi_L \pm \phi_R + \alpha)$ which would lift the edge states.



2 Electromagnetic response for SPT states protected by $G \supset \mathbf{U}(1)$

First: recall free fermion topological insulators, protected by $\mathrm{U}(1) \ltimes \mathbb{Z}_2^{\mathcal{T}}$.

The effective field theory for any 3+1d insulator, below the energy gap, has the following form

$$S_{\text{eff}}[\vec{E}, \vec{B}] = \int \mathrm{d}^3x\mathrm{d}t \left(\epsilon \vec{E}^2 - \frac{1}{\mu} \vec{B}^2 + 2\alpha\theta \vec{E} \cdot \vec{B} \right) \tag{2}$$

where ϵ, μ are the dielectric constant and permittivity, and α is the fine structure constant. Flux quantization implies that

$$\frac{\alpha}{32\pi^2} \int_{M_4} \vec{E} \cdot \vec{B} \in \mathbb{Z}$$

is an integer for any closed 4-manifold. This means that the partition function is periodic

$$Z(\theta + 2\pi) = Z(\theta)$$

and hence the spectrum on a closed 3-manifold is periodic in θ. (As we will discuss, shifting θ by 2π is not so innocuous on a space with boundary or for the wavefunction.)

Time reversal acts by

$$\mathcal{T} : (\vec{E}, \vec{B}) \to (\vec{E}, -\vec{B})$$

which means $\theta \to -\theta$, which preserves the spectrum only for $\theta \in \pi\mathbb{Z}$. So time-reversal invariant insulators are labelled by a quantized 'magnetoelectric response' θ/π [25].

Now consider what happens on a space with boundary. The interface with vacuum is a domain wall in θ, between a region where $\theta = \pi$ (TI) and a region where $\theta = 0$ (vacuum).

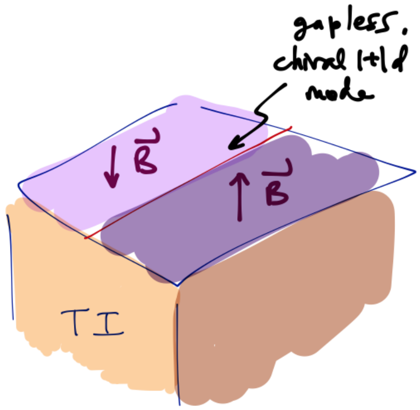
The electromagnetic current derived from (2) is[6]

$$j^{\mu}_{EM} = \frac{e^2}{2\pi h}\epsilon^{\mu\cdots}\partial.\theta\partial.A. + \cdots \tag{3}$$

where the $\cdots$ indicate contributions to the current coming from degrees of freedom at the surface which are not included in (2). If we may ignore the $\cdots$ (for example because the edge is gapped), then we find a surface Hall conductivity

$$\sigma_{xy} = \frac{e^2}{h}\frac{\Delta\theta}{2\pi} = \frac{e^2}{h}\left(\frac{1}{2} + n\right) \tag{4}$$

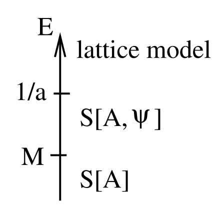
where $\Delta\theta$, the change in θ between the two sides of the interface, is a half-integer multiple of 2π. To be able to gap out the edge states, and thereby ignore the $\cdots$, it is sufficient to break $\mathcal{T}$ symmetry, for example by applying a magnetic field. There are two different ways of breaking $\mathcal{T}$; the 1+1d domain wall between these on the surface supports a *chiral* edge



mode.

The periodicity in $\theta \simeq \theta + 2\pi$ for the fermion TI can be understood from the ability to deposit an (intrinsically 2+1 dimensional) integer quantum Hall system on the surface. This changes the integer n in the surface Hall response (4). Following [24] we argued that a non-fractionalized system of bosons in 2+1d must have a Hall response which is an even integer; therefore a 3+1d boson TI has a θ parameter with period 4π.

The simplest short-distance completion of this model is a single massive Dirac fermion:



$$S[A,\psi] = \int \mathrm{d}^3x\mathrm{d}t\bar{\Psi}\left(\mathbf{i}\gamma^{\mu}D_{\mu} - m - \tilde{m}\gamma^5\right)\Psi.$$

It is convenient to denote $M \equiv m + \mathbf{i}\tilde{m}$.

$$\mathcal{T} : M \to M^{\star}$$

[6] A comment about notation: here and in many places below I refuse to assign names to dummy indices when they are not required. The $\cdot$s indicate the presence of indices which need to be contracted. If you must, imagine that I have given them names, but written them in a font which is too small to see.

so time reversal demands real M. Integrating out the massive Ψ produces an effective action for the background gauge field (and M) of the form above:

$$\log \int [D\Psi] e^{\mathbf{i} S_{3+1}[A,\psi]} = \frac{M}{|M|} \int \frac{\mathrm{d}^4 x}{32\pi^2} \epsilon^{abcd} F_{ab} F_{cd} + \cdots$$

The sign of M determines the theta angle. An interface between the TI and vacuum is a domain wall in M between a positive value and a negative value. Such a domain wall hosts a 2+1d massless Dirac fermion [26]. (The $\mathcal{T}$-breaking perturbation is just its mass, and the chiral edge mode in its mass domain wall has the same topology as the chiral fermion zeromode in the core of a vortex [27].)

A further short-distance completion of this massive Dirac fermion (as in the figure) comes from filling an integer number of bands with a nontrivial Chern-Simons invariant of the Berry curvature [28, 29, 25].

Otherwise – in a clean system – there is an odd number of surface Dirac cones. They can eat each other in pairs under allowed smooth deformations, so let's talk about the case with just one. What is this Dirac cone? The right way to contextualize it is: The Dirac cone is the critical theory arising at the transition between the two ways of breaking the protecting time-reversal symmetry (the two signs of the magnetic field, and hence the sign of the 2+1d mass, which would gap out the Dirac cone). This is a critical theory of a 2+1d free fermion integer quantum Hall plateau transition, where the Hall conductivity changes by e^2/h. And indeed that's what's happening as the surface Hall conductivity changes between $\pm\frac{1}{2}\frac{e^2}{h}$.

More generally, there are two important deformations of this statement.
(1) A surface chemical potential respects the symmetries, and replaces these cones with surface Fermi surfaces.

(2) Without translation invariance, there are other possibilities for the surface theory: namely other critical theories of the 2+1d IQH plateau transition. An important example is the critical point accessed by the Chalker-Coddington network model. This is definitely different from the Dirac cone, for example because it has correlation length critical exponent $\simeq 7/4$, whereas for the Dirac cone it is 1. CHECK .

If $\mathcal{T}$ is preserved at the surface it is still possible to gap the edge states, but only if there is surface topological order [10, 11, 12, 13, 14]. In these recent constructions of gapped edges of fermion SPTs with surface topological order, what is the surface Hall response? It seems like must be zero by $\mathcal{T}$. On the other hand, how can the gapped, $\mathcal{T}$-invariant edge cancel out the bulk contribution in (3)?! This is where the magic (magic here means non on-site symmetry realization on the surface) happens: $\mathcal{T}$ is represented in a really weird way on this gapped edge, which allows a nonzero surface contribution to the Hall current.

What exactly do these all these possible surface theories have in common? What would we have to change about one of them to force it to be separated by a *bulk* transition from the others?

3 Coupled-layer construction of SPT surface states

Let's begin by building a 2+1d gapped bulk from 1+1d edges. This will give us some insight about the transition between a nontrivial SPT and the trivial state. Then we'll do 3+1d bulk from gapped edges.

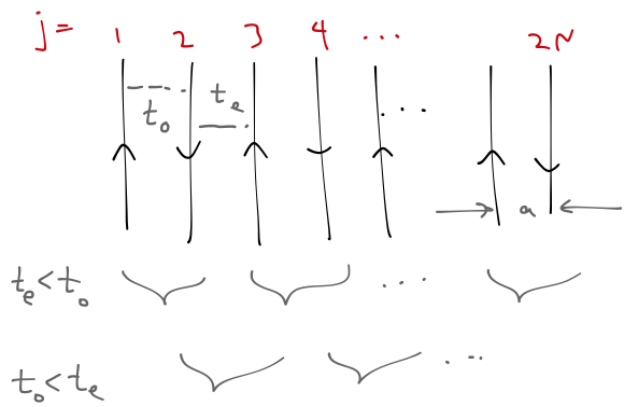
3.1 Coupled-layer construction for a fermion IQHE transition in $D = 2 + 1$

Following [1], let's make a model of *a* transition by which the Hall conductance of free fermions changes by $\frac{e^2}{h}$. (I say 'a' transition because there are others, for example in the absence of translation invariance.)

$$S = \int \mathrm{d}x \mathrm{d}\tau \sum_{j=1}^{2N} \mathcal{L}_j$$

$$\mathcal{L}_j = \underbrace{c_j^\dagger \left(\partial_\tau - \mathbf{i} s_j \partial_x\right) c_j}_{\text{IQHE edge}} - \underbrace{t_j \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}\right)}_{\text{interlayer hopping}}$$

$$t_j = \begin{cases} t_e, & j \text{ even} \\ t_o, & j \text{ odd} \end{cases} \quad , \quad s_j = -(-1)^j \ .$$



For each j we have one chiral $\nu = 1$ edge mode, that is a $D = 1 + 1$ complex chiral fermion. The t_j term is some back-scattering by which these layers hybridize and eat each other.

For $t_e < t_0$, all layers are paired and there's nothing left; $\sigma_{xy} = 0$.

For $t_e > t_0$, there are leftover chiral modes at the edges, $\sigma_{xy} = \frac{e^2}{h}$. The transition occurs at $t_e = t_o$, where we restore some extra translation symmetry. Taking a continuum limit $N \to \infty, Na$ fixed we find a 2+1d Dirac fermion.

3.2 Coupled-layer construction for a *boson* IQHE transition in $D = 2+1$

Still, following [1], now let's do the same for a boson integer Hall plateau transition. The $\nu = 2$ boson IQHE edge is described by the SU(2)$_1$ WZW model, with

$$S_{\nu=2\ \mathrm{edge}}[g] = \frac{1}{2\lambda}\int \mathrm{d}x\mathrm{d}\tau \mathrm{tr}\partial_\mu g^{-1}\partial^\mu g + \mathbf{i}\mathcal{W}_2[g]$$

where $\mathcal{W}_2$ is the 2d WZW term (which is written as an integral over a 3-ball). Notice that this model is actually not chiral; the only thing that's chiral about it, in its role as the boson $\nu = 2$ edge theory, is the way in which it realizes the boson number current. That is, the boson number current only has a right-moving part. This could be made explicit by coupling to an external U(1) gauge field; pretend we did this.

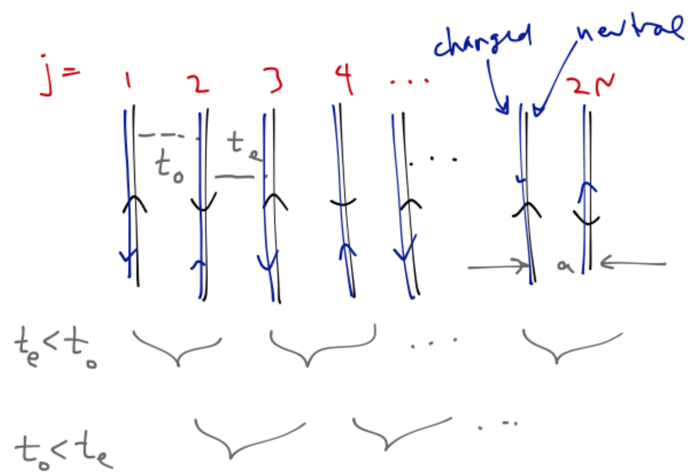
Now we make another lasagna, *i.e.* couple alternating layers of this stuff:

$$S = \sum_{j=1}^{2N}\left(S_{\nu=2\ \mathrm{edge}}[(-1)^j g_j] - t_j \mathrm{tr} g_j^\dagger g_{j+1} + h.c.\right) \tag{5}$$

The $(-1)^j$ affects only the WZW term, and is the analog of alternating the chirality in the fermion case. The hopping term here could be written in terms of the double-layer bosons as

$$\mathrm{tr} g_j^\dagger g_j \simeq \sum_{I=1,2} b_{j,I}^\dagger b_{j+1,I}$$

– notice that it preserves the independent boson number conservation laws. This term is happy when $g_j \sim g_{j+1}$.



Again for $t_e < t_0$, all layers are paired and there's nothing left; $\sigma_{xy} = 0$.
For $t_e > t_0$, there are leftover chiral modes at the edges, $\sigma_{xy} = 2\frac{e^2}{h}$. The continuum limit in these two cases is

$$S_{\nu=\frac{\theta}{\pi}}[g] = \frac{1}{2\kappa}\int \mathrm{d}^2x\mathrm{d}\tau \mathrm{tr}\partial_\mu g^\dagger \partial^\mu g + \mathbf{i}\theta\mathcal{H}_\theta[g] \tag{6}$$

with $\theta = 0, 2\pi$ respectively.

At $t_e = t_o$, the continuum limit gives [30]

$$S_{\Delta\nu\,=\,2}[g] = \frac{1}{2\kappa}\int \mathrm{d}^2x\mathrm{d}\tau \mathrm{tr}\partial_\mu g^\dagger \partial^\mu g + \mathbf{i}\pi\mathcal{H}_\theta[g]\ . \tag{7}$$

Here the object

$$\mathcal{H}_\theta[g] \equiv \frac{1}{6\Omega_3}\int \mathrm{d}^3x \mathrm{tr} g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g$$

represents $\pi_3(\mathrm{SU}(2) \sim S^3) = \mathbb{Z}$. Check my numerical prefactor; it should count the number of times the euclidean spacetime (we assume that $g(\infty) = \mathbb{1}$ so that it is topologically S^3) wraps the target space $\mathrm{SU}(2) \sim S^3$. Ω_n is the n-volume of the unit n-sphere.

Claim: this field theory, with $\theta = \pi$, has no trivial gapped state. It is either gapless or has a degenerate groundstate, as in the conclusion of the Lieb-Schulz-Mattis theorem. We'll discuss an instructive argument for this conclusion in section 4.

A useful historical question: Why were Matthew Fisher and Senthil [30] studying this construction in 2005? The answer is that they were looking for field theories with the features in the previous claim. More specifically, they were trying to find a representation of the deconfined quantum critical theory at the Neel-VBS transition (*i.e.* the continuous phase transition between a 2+1d Neel state and the valence bond solid (VBS) state of an antiferromagnet) which treated the Neel and VBS order on a more equal footing. Such a more equal footing is manifest in $D = 1+1$ via the symmetry enhancement of the the $\mathrm{SU}(2)_1$ WZW model.

Further claim [30]: at $\theta = \pi$, this model is dual to a non-compact $\mathbb{CP}^1$ model, famous from the study of the Neel-VBS deconfined quantum critical point.

Notice that so far we have preserved $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \supset \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ (and also the $\mathbb{Z}_2$ symmetry which interchanges the two layers of bosons $1 \leftrightarrow 2$). For purposes of making SPTs protected only by $\mathrm{U}(1)$ (and maybe $\mathbb{Z}_2^{\mathcal{T}}$) we should allow tunneling between the layers: $b_{j,1} \to b_{j+1,2}$.

3.3 Consolidation: iterative scheme

We can now extract an iterative scheme for constructing SPT states. Suppose we are given an SPT state in $D+1$ dimensions, protected by, say, a $\mathbb{Z}_2$ symmetry (say time reversal). Then we can destroy the state by breaking $\mathcal{T}$ in two different ways. For example, we can (and already did in §1.2) make spin chains by the same kind of coupled-layer construction, where now each layer is just a single spin-$\frac{1}{2}$. In this way the edge theory in D dimensions can be used to make (by coupling alternating layers) an SPT state in $D+1$ dimensions.

How to make the recursion step of the iterative scheme? We need to know the edge theory for the next higher dimension. The edge theory of a $D+2$ dimensional SPT can be made from the *critical* theory separating the $D+1$ dimensional SPT from the trivial state. The example with the spin-$\frac{1}{2}$s already shows this: we can break $\mathbb{Z}_2^{\mathcal{T}}$ and $\mathrm{SU}(2)$ by a Zeeman field $\mathbf{H}_{\mathrm{spin}} = \vec{h} \cdot \vec{\mathbf{S}}$; the critical theory is at $\vec{h} = 0$, a free spin-$\frac{1}{2}$!

Repeating this to get from a $D = 1+1$ edge to a $D = 2+1$ SPT, we couple layers of the $\mathrm{SU}(2)_1$ WZW model to get an O(3) NLSM with a topological term at $\theta = 2\pi$, describing the boson $\nu = 2$ IQH state. The critical theory separating this from the $\nu = 0$ trivial state is the O(3) NLSM at $\theta = \pi$.

A question you should ask at this point is: what's different about the $D+1$-dimensional critical theory (*e.g.* the $\mathrm{SU}(2)_1$ WZW model) when it arises as a critical point between $D+1$-dimensional SPTs (in the example above, the trivial and nontrivial SO(3) chains) as from when it arises on the surface of the $D+2$-dimensional SPT via a coupled-layer construction? In general, the answer is not clear to me. In the example above, the difference is merely the way the U(1) symmetry is realized. An intrinsically $1+1$-dimensional realization of the $\mathrm{SU}(2)_1$ WZW model cannot preserve the chiral U(1) symmetry at the lattice scale – it is an emergent symmetry.

In the case of the spin-$\frac{1}{2}$ at the surface of the SO(3) chain, the crucial thing is that the spin-$\frac{1}{2}$ is a *projective* representation of SO(3) (*i.e.* it acquires a minus sign under a 2π rotation).

Exercises: Majorana lasagna

1. Apply the coupled-layer construction beginning with a single Majorana mode as the $D = 0+1$ edge state to get a $D = 1+1$ SPT state. (The answer is here [31].)

2. [Warning: spoilers for the previous problem!] Do it again using the critical limit of the model above, that is, a massless 1+1d Majorana fermion. What 2+1d free-fermion SPT state do you make this way?[7]

---

4 Unkillable edge states

The key ingredient in the above discussion was the unkillability of the $D = 1+1$ $\mathrm{SU}(2)_1$ WZW model (while preserving the SU(2) symmetry). Here we review an argument showing that the $D = 2+1$ SO(4) sigma model with $\Theta = \pi$ shares this Bruce-Willis-like feature [32].

Practice run in $D = 1+1$: First we run the argument for the previous example: Consider the $D = 1+1$ O(3) NLSM in imaginary time. We already know that this system (for various θ) arises as in continuum limits of spin chains to which the Lieb-Schulz-Mattis theorem applies; here we discuss an argument for its conclusion which proceeds directly in the continuum.

[7] (Answer: a $p + \mathbf{i}p$ superconductor.)

1. At $\theta \in 2\pi\mathbb{Z}$, the model on a space without boundary has a gap, and a non-degenerate groundstate.

2. Put the system on a spatial interval, $I \equiv \{x \in [0, L]\}$. When $\theta = 0$, there is still nothing. When $\theta = 2\pi$, the θ term

$$\frac{\theta}{4\pi} \int \epsilon_{abc} n^a \mathrm{d}n^b \wedge \mathrm{d}n^c|_{\theta=2\pi} = \mathcal{W}_1$$

   check factors *is* a O(3) 0+1d WZW term (with level 1) for the boundary Neel vector. This is so simple it's tricky: in the usual construction of the WZW term, we introduce a fictitious extra dimension and write the thing as a total derivative (here $\epsilon_{abc} n^a \mathrm{d}n^b \wedge \mathrm{d}n^c$) integrated over that extra dimension. Here that extra dimension just happens to be the actual bulk space dimension! So this is a $D = 0 + 1$ particle on S^2 with a WZW term at level $k = 1$; that gives $k + 1 = 2$ states (for a reminder, see Physics 215C lectures, §5.2).

3. Now what happens to this system on the interval as we slowly tune θ from 2π to 0? There a symmetry that's restored at $\theta \in 2\pi\mathbb{Z}$ acting by $\vec{n} \to -\vec{n}$ (and also $\mathbb{Z}_2^{\mathcal{T}}$), but even for $\theta = 2\pi - \epsilon$, we can't kill the edge states because we haven't broken SU(2). The only way we can get rid of them – while preserving the SU(2) – is if we *add* degrees of freedom, someone for them to pair up with. This can only happen if there is a *bulk transition*. Another perspective on this unkillability is that the coefficient of the WZW term is quantized – you can't have a spin which isn't a half-integer. So for θ near 2π, we must flow back to the model at $\theta = 2\pi$.

4. There are two possibilities for the bulk transition, as usual: either it is continuous or it is not. The two possibilities are exemplified by the two figures in 1. Notice that by the action of time reversal

$$\text{Physics}(\theta) = \text{Physics}(2\pi - \theta)$$

   so if the phase transition is not at $\theta = \pi$, there must be two transitions. Ockham (following Kramers and Wannier [33]) suggests that the transition is at $\theta = \pi$.

   In the first scenario, the edge spin delocalizes from the edge at $\theta = \pi$, where it can join the continuum bulk critical theory at $\theta = \pi$. In the second scenario, the bulk has a two-fold-degenerate groundstate at $\theta = \pi$ associated with the level-crossing.

   So we arrive at the conclusion of the Lieb-Schulz-Mattis theorem about the model at $\theta = \pi$: either it is critical or it has a degenerate groundstate.

The phase diagrams associated to the two cases in $D = 1 + 1$ are:

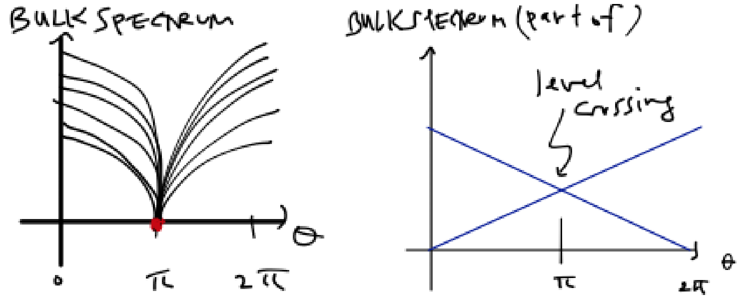
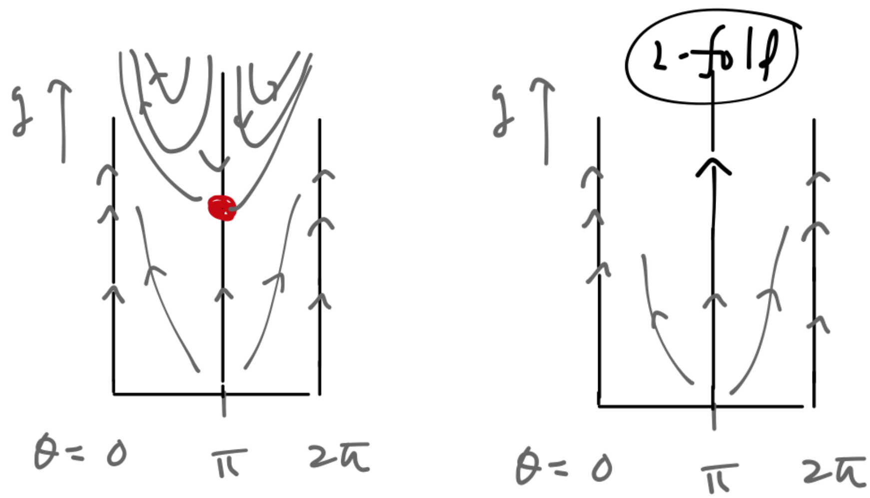


Figure 1: The two possibilities for the behavior of the bulk spectrum during the bulk transition between $\theta = 0$ and $\theta = 2\pi$.



In the figure at right, the end of the flow at $\theta = \pi$ is a gapped state with two-fold degenerate groundstate.

Notice that both kinds of transitions are possible between $\theta = 2\pi$ and $\theta = 0$, even in $D = 1 + 1$. Which is edge phase realized in a given model depends on :

- The bulk Hamiltonian within the SPT phase. In the spin chain case, we may add farther-neighbor terms in the spin hamiltonian. For example, by adding next-nearest-neighbor Heisenberg interactions with coefficient J_2, we may reach the Majumdar-Ghosh point at $J_2 = \frac{1}{2}J_1$. This model is exactly solvable and has two fully-dimerized groundstates related by translation invariance, which is thus spontaneously broken.
- Is there translation invariance? Alternating J_e/o terms (doubling the unit cell) change θ away from $\theta = \pi$ – recall the coupled layer argument. If there is *no* translation invariance at all (recall that for most of the states we discuss translations are not part of the protecting symmetry), the edge physics can be very different.

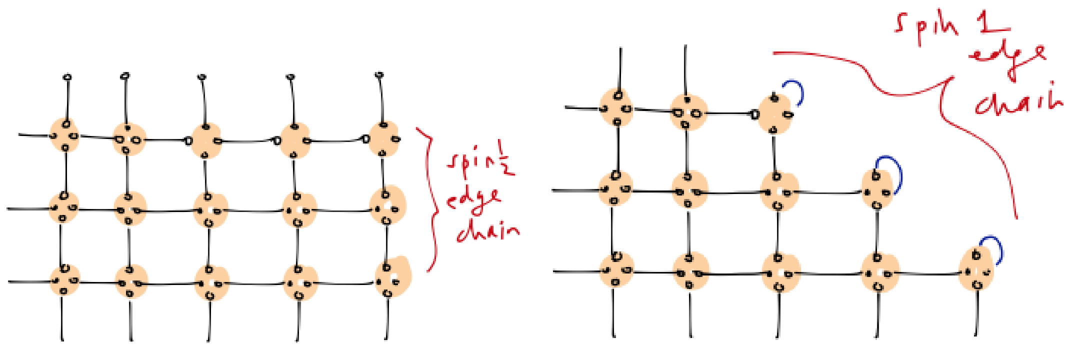
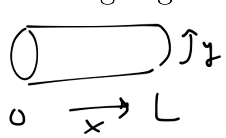


Figure 2: Each shaded site hosts four spin-$\frac{1}{2}$s. The AKLT state in bulk is the completely spin- and translation-symmetric state made by forming singlets along each bond.

- How do we terminate the lattice? Staircase? Armchair? Elephant? These are names for terminations of various lattices (OK I made up the last one). A dramatic example pointed out by Dan Arovas is in the square lattice AKLT model (that means four spin-$\frac{1}{2}$s at each site) it makes a big difference whether the normal to the edge is along a lattice vector or not:
- From the continuum point of view, the previous choice is related to the choice of boundary conditions in the sigma model, *i.e.* to what $\hat{n}$ does at the boundary.

The argument in $D = 2+1$: Now we follow the analogous steps for the $D = 2+1$ NLSM on S^3.

1. Again for $\theta \in 2\pi\mathbb{Z}$, the model has a gapped, non-degenerate groundstate on a space without boundary.
2. The analog of the interval is the cylinder, $S^1 \times I$. Let $x \in [0, L]$ and $y \simeq y + 2\pi R$. We are going to consider a regime where $L, R \gg \xi$, the bulk correlation length.

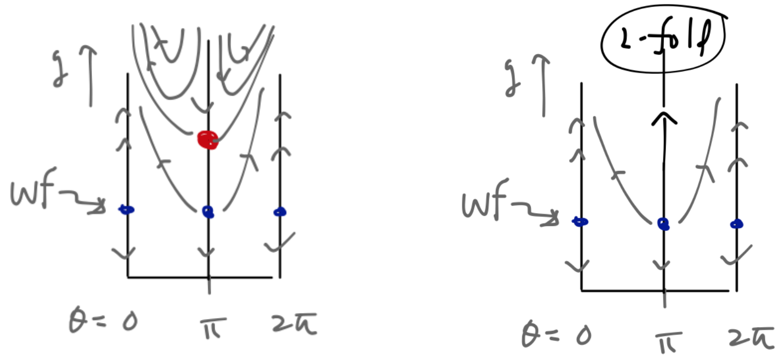
   

   At $x = 0, L$, we have a 1+1d O(4) NLSM with a WZW term $\mathcal{W}_2[\vec{n}]$ at $k = \frac{\theta}{2\pi}$. For $\theta = 0$, this flows to the strongly coupled gapped fixed point. For $\theta = 2\pi$ this flows to the SU(2)$_1$ WZW fixed point we've already heard so much about. For $\theta = 2\pi n, n > 1$ is there a screening phenomenon similar to the one for spin chains? It is possible, since the number of degrees of freedom of the fixed point theory grows with k.
3. Now tune θ from 2π to 0. The emergent symmetry $\mathrm{O}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ protects the edge CFT – there are no relevant perturbations of the SU(2)$_k$ CFT which preserve it.

To kill the edge CFT, we must couple left- and right-movers, and break the symmetry to the diagonal SU(2).

Since the bulk preserves the O(4) symmetry, killing the edge (which must happen somewhere as θ goes from 2π to 0) requires a bulk transition.

4. Again this transition may be continuous or not, producing the same two scenarios for the low-lying bulk spectrum as in Fig. 1. The corresponding phase diagrams are:



There is an alternative argument for this conclusion in [32], which proceeds by considering the strong-coupling limit of the sigma model on S^2 for various θ. This produces a quantum mechanical system similar to a particle on a ring (where the θ-term is just the winding number $\int d\tau \partial_\tau \phi$); when $\theta = \pi$ there is a degeneracy.

5 Ways of slicing the path integral

In the path integral on a closed spacetime manifold, the θ term multiplies an integer and therefore its value only matters mod 2π. The total value of θ does matter if the path integral is on a spacetime with boundary, either in space or time (a point which has been emphasized by Cenke Xu). A boundary of space means a real boundary; a boundary in time in the path integral means that the path integral is a functional of the boundary data – it computes a wavefunction, in 'position space'. For quantum mechanics of a single variable $q(t)$, this is manifested in the Feynman-Kac formula

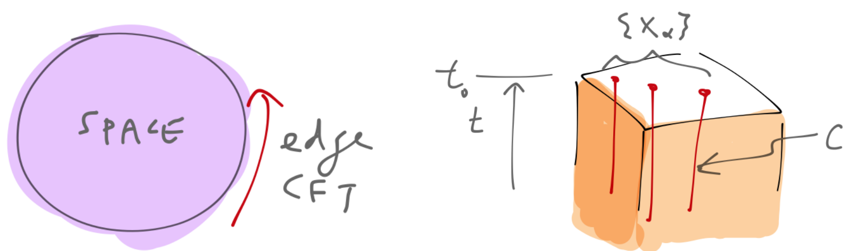
$$\psi(q) = \int_{q(t_0)=q} \prod_{t\in(-\infty,t_0)} dq(t) e^{-S_{\text{euclidean}}[q]} .$$

For a field theory, 'position-space wavefunction' means a wavefunctional $\Psi[\phi(x)]$, in

$$|\Psi\rangle = \int [D\phi(x)]\Psi[\phi(x)]|\phi(x)\rangle$$

where x labels *spatial* positions, and $|\phi(x)\rangle$ are coherent states for the field operator $\hat{\phi}(x)$. Which wavefunction? If the path integral is over a large euclidean time T before reaching the boundary, this is a *groundstate* wavefunction, since the euclidean time propagator $e^{-T\mathbf{H}}$ is a (un-normalized) projector onto lowest-energy states.

An important guiding concept in the study of interesting gapped states is that it is the *same stuff* living at a spatial boundary (edge modes) as at a temporal boundary (the wavefunction) [34, 35]. This perspective first arose (I think) in the context of quantum Hall states where famously [34] one can write groundstate and several-quasiparticle wavefunctions as correlation functions of certain operators in a 1+1d CFT, which is the same CFT that arises at a spatial edge. Why should this be true? It's because the bulk can be described by a path integral for a Chern-Simons gauge theory which has a certain WZW model living at its boundaries, wherever they are. For a spatial boundary, it produces a copy of that CFT at the boundary (roughly the group-valued CFT field g is related to the CS gauge field by $A = g^{-1}\mathrm{d}g$).



For a temporal boundary, the path integral expression for the wavefunctional (with some Wilson line insertions related to quasiparticles) takes the form

$$\Psi[g(x)] = \int_{A(t_0,x)=g^{-1}\mathrm{d}g} e^{\mathbf{i}S[A]} W[C] = \langle \prod_\alpha V_\alpha(x_\alpha) \rangle_{WZW}. \tag{8}$$

A too-brief explanation of this rich formula: the wilson line insertion is $W[C] = \mathrm{tr}_R \mathcal{P} e^{\mathbf{i}\oint_C A}$ where R is a representation of the gauge group G and $\mathcal{P}$ is path ordering. In (8), x_α are the locations where the curve C intersects the fixed-$t = t_0$ surface, and V_α are some operators in the CFT the appropriate representations R of G.

For spin chains, this point of view is used in [36] to construct spin chains whose continuum limit is the $\mathrm{SU}(2)_k$ WZW model with $k > 1$.

For 3+1d boson SPT states, the analogous bulk EFT is, instead of CS gauge theory, some weird BF theory or strongly-coupled sigma model, both of which we'll discuss below. At a spatial edge, we have some vortex excitations in $D = 2 + 1$. Correspondingly, the bulk wavefunctions will turn out to have a nice representation in a basis of states labelled by vortex loop configurations in $D = 3 + 1$.

6 (Neel) sigma model description of SPT wavefunctions

6.1 $D = 1+1$ Haldane chain

Take $D = 1+1$, $G = \mathrm{SO}(3)$, and let's study a field variable which is a 3-component unit vector $\hat{n} \in S^2$. The fact that $\pi_2(S^2) = \mathbb{Z}$ will play an important role. Think of this $\vec{n}$ as arising from coherent-state quantization of a spin chain. So take the (imaginary-time) action to be

$$S = \int d\tau dx \left(\frac{1}{g^2} \partial_\mu \hat{n} \cdot \partial^\mu \hat{n} + \mathbf{i} \frac{\theta}{4\pi} \epsilon_{abc} n^a \partial_\tau n^b \partial_x n^c \right).$$

We will focus on $\theta \in 2\pi\mathbb{Z}$. Recall from 1.2 that in this case the model has a gap. We would like to understand what is different between $\theta = 0$ and $\theta = 2\pi$.

Recall the role of the θ term: on a closed spacetime manifold M_D

$$Z_\theta(M_D) \equiv \int [Dn] e^{-S} = \sum_{n \in \pi_2(S^2)} e^{\mathbf{i}\theta n} Z_n$$

and $Z_\theta(M_D) = Z_{\theta+2\pi}(M_D)$. In particular, we can take $M_D = S^1 \times N_{D-1}$ to compute the partition function on any spatial manifold N_{D-1}. This means the bulk spectrum is periodic in θ with period 2π.

In contrast to the case of a closed manifold, if we compute the path integral on an (infinite) cylinder (*i.e.* with two boundaries, at $\tau = \pm\infty$),

$$\int_{\substack{\hat{n}(x,\tau=\infty) = \hat{n}(x) \\ \hat{n}(x,\tau=-\infty) = \hat{n}'(x)}} [D\hat{n}(x,\tau)] e^{-S[n(x,\tau)]} = \langle \hat{n}(x) | 0 \rangle \langle 0 | \hat{n}'(x) \rangle \tag{9}$$

then θ does matter, not just mod 2π. Notice that in expressions for functionals like $S[n(x,\tau)]$ I am writing the arguments of the function n to emphasize whether it is a function at fixed euclidean time or not. The fact that the theta term is a total derivative $\mathbf{i}\theta\mathcal{H} = \mathbf{i}\theta \int_{\partial M_D} w$ means that the euclidean action here is

$$S[n(x,\tau)] = \int_{M_D} d\tau dx \frac{1}{g^2} \partial_\mu \hat{n} \cdot \partial^\mu \hat{n} + \mathbf{i}\theta \int dx \left(w(n(x)) - w(n'(x)) \right).$$

The θ term only depends on the boundary values, and comes out of the integral in (16).

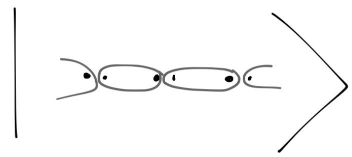
If we also take $g \to \infty$ there is nothing left in the integral and we can factorize the expression (16) to determine:

$$|0\rangle_{\theta=0} \propto \int [D\hat{n}(x)] |\hat{n}(x)\rangle = \prod_x |\ell = 0\rangle_x$$

is a product state; on the far RHS here we have a product of local singlets: $\int d\hat{n}|\hat{n}\rangle = |\ell = 0\rangle$ at each site. Here we used $\langle \hat{n}(x)|\hat{n}'(x)\rangle = \delta[n - n']$.

$$|0\rangle_{\theta=2\pi} \propto \int [D\hat{n}(x)] e^{\mathbf{i}\frac{\theta}{2\pi}\int \mathrm{d}x w(\hat{n}(x))} |\hat{n}(x)\rangle$$

Here $\int \mathrm{d}x w(\hat{n}(x)) = \mathcal{W}_1[\hat{n}]$ is the $D = 1+1$ WZW term; the role of the usually-fictitious extra dimension in writing this WZW term is now being played by the *real* euclidean time.

But the groundstate of the spin-one chain is the AKLT wavefunction, which we can denote graphically by: $|\text{AKLT}\rangle =$  This leads us to the conclusion that we can write the AKLT wavefunction (or at least a wavefunction in the same phase) in the coherent state basis as

$$|\text{AKLT}\rangle = \int [D\vec{n}(x)] e^{\mathbf{i}\mathcal{W}[\vec{n}]} |\vec{n}(x)\rangle \quad . \tag{10}$$

6.2 $D = 2+1$ $G = \mathbb{Z}_2$ boson SPT

The same logic can be used to produce a wavefunction for the new Ising paramagnet constructed in [15]. That paper provides an exactly solvable lattice model built on the Hilbert space

$$\mathcal{H}_{\text{ising}} = \otimes_{\text{sites } s \text{ of 2d lattice}} \text{span}\{|\alpha_s = \pm 1\rangle\} = \text{span}\{|D\rangle, D \in \{\text{domain wall configurations}\}\}$$

with a $\mathbb{Z}_2$-symmetric hamiltonian. The $\mathbb{Z}_2$ acts by $\alpha_s \to -\alpha_s, \forall s$. The usual Ising paramagnet can be described in this basis as a uniform superposition of domain walls

$$|\Psi_0\rangle = \sum_D |D\rangle \ .$$

The Levin-Gu state is

$$|\Psi_1\rangle = \sum_D (-1)^{N_D} |D\rangle$$

where N_D is the number of domain walls in the configuration D. [15] show that if we *gauge* the protecting $\mathbb{Z}_2$ symmetry, we get distinct topologically ordered states in the two cases. From Ψ_1, we get a system where the visons (endpoints of the domain walls, which are the same as π flux insertions of the $\mathbb{Z}_2$ gauge field) are semions.

[2] provide an effective field theory perspective on this state starting from the SO(4) NLSM in $D = 2+1$.

1. Take the action

$$S[\vec{\phi} \in S^3] = \int \mathrm{d}\tau \mathrm{d}^2 x \left(\frac{1}{g^2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} + \mathbf{i}\theta \mathcal{H}[\phi] \right)$$

Again $\int_{M_3,\text{ closed}} \mathcal{H}[\phi] \in \mathbb{Z}$ for smooth configurations $\phi(x,\tau)$. More precisely

$$\mathcal{H} \equiv \frac{1}{2\pi^2} \epsilon_{abcd} \phi^a \partial_\tau \phi^b \partial_x \phi^c \partial_y \phi^d = \frac{1}{12\pi^2} \epsilon_{abcd} \phi^a \mathrm{d}\phi^b \mathrm{d}\phi^c \mathrm{d}\phi^d$$

represents $\pi_3(S^3)$, and is a total derivative $\mathcal{H} = \mathrm{d}\mathcal{W}_2$.

2. Consider $\theta = 2\pi$. What about other values of θ ? $\mathcal{T}$ is not part of the protecting group, and $\vec{\phi} \to -\vec{\phi}$ is a symmetry for all θ. But $\theta \in 2\pi\mathbb{Z}$ will be preferred below. We can still ask about $\theta = 2\pi k$, with $k > 1$.

   By the same logic as in the previous subsection, the groundstate wavefunctional in the strong-coupling limit is

$$|\Psi\rangle_\theta \overset{g\to\infty}{\approx} \int [D\vec{\phi}(x)] |\vec{\phi}(x)\rangle e^{\mathbf{i}\frac{\theta}{2\pi}\mathcal{W}_2[\vec{\phi}(x)]}. \tag{11}$$

   The WZW term here has the identical form to the theta term in the action, but now it is interpreted as a functional of the *boundary* field configuration. Again the euclidean time direction is playing the role of the fictitious extra dimension.

3. Break the artifical $\mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathbb{Z}_2$ by adding some potential terms in the NLSM action. More specifically, decompose the coordinates on S^3 as $\phi^a = (\phi^0 \cos\alpha, \hat{n} \sin\alpha)$ with $\phi^0 = \pm 1, \hat{n} \cdot \hat{n} = 1$, and choose the potential to be minimized when $\phi^0 = 0$. Recall that for a single harmonic oscillator with $S = \int \mathrm{d}t \left(\frac{1}{2}\dot{q}^2 - V(q(t)) \right)$, the potential term $V(q) = \frac{1}{2}\omega^2 q^2$ changes the groundstate wavefunction by $\psi(q) \sim e^{-\omega q^2}$ (beware my factors of two). Adding this potential $V(\phi_0) \sim m^2 \phi_0^2(x,t)$ then weights the configurations with nonzero ϕ_0 with a factor of $e^{-m\int_{\text{space}} \mathrm{d}x \phi(x)}$ which suppresses their contribution; we can then take $m \to \infty$ and keep only configurations with $\phi_0 = 0$.

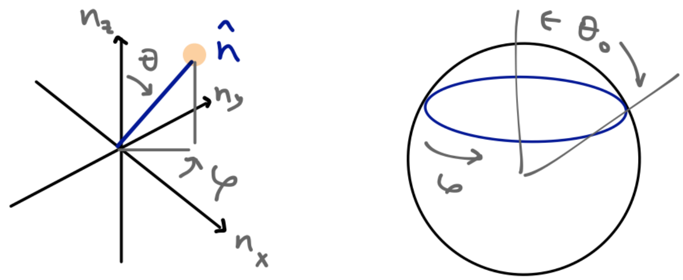
It is a general fact that restricting a WZW term for maps to a sphere to field configurations on the equator gives a theta term:

$$\mathcal{W}_d[\hat{\phi}|_{\text{equator}} = (0, \hat{n})] = \int \mathcal{H}_d[\hat{n}] \ .$$

An explicit example is:

$$\mathcal{W}_1[\vec{n} \in S^2] = \int_0^1 \mathrm{d}u \int_0^\beta \mathrm{d}t \epsilon^{\mu\nu} \epsilon_{abc} n^a \partial_\mu n^b \partial_\nu n^c = \int_0^\beta \mathrm{d}t \frac{1}{2} \left(1 - \cos\theta(t)\right) \partial_t \varphi(t) \ .$$

Here $\vec{n}(u = 0, t) = (0, 0, 1), \vec{n}(u = 1, t) = \vec{n}(t)$ and $\vec{n}(t + \beta) = \vec{n}(t)$. The polar coordinates are defined (as in the figure) by $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.



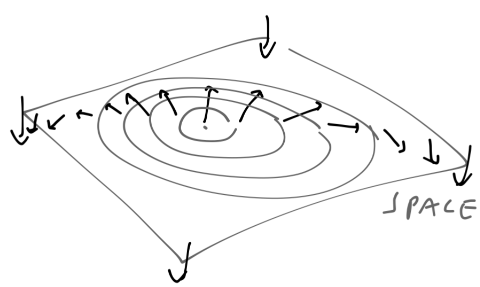
If we fix to some particular latitude $\theta(t) = \theta_0$ (such as the equator at $\theta_0 = \pi/2$) we get the $D = 0+1$ theta term:

$$\mathcal{W}_1[\vec{n}|_{\theta=\theta_0}] = \frac{1-\cos\theta_0}{2} \underbrace{\int_0^\beta \mathrm{d}t \partial_t \phi}_{\in 2\pi\mathbb{Z} = \text{winding \# around latitude}} = \frac{1-\cos\theta_0}{2} \int \mathrm{d}t \mathcal{H}_1[\varphi] \ .$$

More explicitly for the case at hand,

$$\mathcal{W}_2[\hat{\phi} = (0, \hat{n})] = \frac{1}{16\pi} \int \mathrm{d}^2 x \epsilon_{abc} \epsilon^{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c = \pi \int \mathcal{H}_2[\hat{n}] = \pi N_s.$$

check factors Notice that now $\hat{n} \to -\hat{n}$ symmetry (the symmetry of the actual spin system we want to keep) requires $\theta \in \pi\mathbb{Z}$. In the previous expression N_s is the *skyrmion number* of the configuration $\vec{n}$. A skyrmion is a configuration of an S^n-valued field on $\mathbb{R}^n$ which looks like this for $n = 2$:



It ties each point in space (labelled by its image under stereographic projection) to the corresponding point in the target sphere.

Claim: if we localize $\phi^0 = 0$, the wavefunction (11) becomes

$$|\Psi\rangle_\theta = \int [D\vec{n}(x)] |\vec{n}(x)\rangle \underbrace{e^{\mathbf{i}\frac{\theta}{2\pi} \int_{\text{space}} \mathcal{H}_2[\vec{n}(x)]}}_{=(-1)^{N_s}}$$

4. Now break the remaining $\mathrm{SO}(3) \times \mathbb{Z}_2 \to \mathrm{SO}(2) \times \mathbb{Z}_2$ by a potential which adds a mass for n^x, n^y, favoring $n^z = \pm 1$. As you can see from the figure, a skyrmion configuration is smooshed into a domain wall of n^z surrounding the location of the skyrmion. The wavefunction becomes

$$|\Psi_{\theta=2\pi}\rangle = \sum_{\{n^z\}} (-1)^{N_D} |\{n^z\}\rangle$$

where N_D is the number of domain walls in the configuration of spins $\{n^z\}$.

5. Finally, breaking the remaining SO(2) is innocuous and doesn't change the wavefunction for the n^z configurations.

A label on this state: In the case of SPTs where the protecting group contains a U(1) factor, we can often label distinct states by a *discrete* label associated with its EM response, its quantized Hall response in $D = 2 + 1$ (if $\mathcal{T}$ is not preserved) or its quantized magnetoelectric response $\frac{\theta}{2\pi}$ in $D + 3 + 1$. This gives a proof that a state is not adiabatically connected to the trivial one without having to think about every possible deformation, including all possible interactions. For this state protected by just $\mathbb{Z}_2$ we can't do this. Instead, following [15], we can find such a label by asking what happens if we gauge the $\mathbb{Z}_2$.

It is possible to see directly from the sigma model field theory that if we gauge the $\mathbb{Z}_2$ the visons will be semions [2]. The method to study discrete gauge theory in the continuum is to gauge a larger continuous group and Higgs it down to $\mathbb{Z}_2$.

To choose this larger continuous group, reorganize the field variables into an SU(2) matrix:

$$\mathfrak{g} = \phi^0 1\!\!1 + \vec{\phi} \cdot \vec{\boldsymbol{\sigma}} \mathbf{i} \in \mathrm{SU}(2)$$

(it's in SU(2) since $\det \mathfrak{g} = \det \begin{pmatrix} \phi^0 + \phi^3 & \phi^1 - \mathbf{i}\phi^2 \\ \phi^1 + \mathbf{i}\phi^2 & \phi^0 - \phi^3 \end{pmatrix} = \phi_0^2 + \vec{\phi}^2 = 1$.) in terms of which the action is

$$S[g] = \int \mathrm{d}^2x\mathrm{d}t \left(\frac{1}{g^2} \mathrm{tr}\mathfrak{g}^{-1}\partial_\mu \mathfrak{g}\mathfrak{g}^{-1}\partial^\mu \mathfrak{g} + \frac{\theta}{12\pi^2} \epsilon^{\cdots} \mathrm{tr}\mathfrak{g}^{-1}\partial_. \mathfrak{g}\mathfrak{g}^{-1}\partial_. \mathfrak{g}\mathfrak{g}^{-1}\partial_. \mathfrak{g} \right)$$

i.e. it is a 'principal chiral model for SU(2)'. check sphere factor The $\mathrm{O}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry acts on $\mathfrak{g}$ by

$$\mathfrak{g} \mapsto L^\dagger \mathfrak{g} R, \quad L, R \in \mathrm{SU}(2) \ .$$

Let's gauge the $U(1)_L \times U(1)_R$ subgroup generated by $\boldsymbol{\sigma}^z \otimes 1\!\!1$ and $1\!\!1 \otimes \boldsymbol{\sigma}^z$; this is accomplished by replacing the derivatives in the action by

$$\partial_\mu \mathfrak{g} \rightsquigarrow \partial_\mu \mathfrak{g} + \mathbf{i} a_\mu \boldsymbol{\sigma}^z \mathfrak{g} - \mathbf{i} b_\mu \mathfrak{g} \boldsymbol{\sigma}^z \tag{12}$$

where a, b are two new gauge fields. Notice that the order of $\mathfrak{g}$ and $\boldsymbol{\sigma}^z$ matters. We'll give a, b a mass by adding Higgs fields.

Claim:

$$\log \int [D\mathfrak{g}] e^{\mathbf{i}S[\mathfrak{g},a,b]} = \mathbf{i} \int \mathrm{d}^2x\mathrm{d}t \frac{2}{4\pi} \left(\epsilon^{\cdots} a.\partial a. - \epsilon^{\cdots} b.\partial b. \right) + \ldots \tag{13}$$

where ... is terms with more derivatives or more gauge fields. We knew this from §3.2 and in particular (6) – the coupling of each of a and b to $\mathfrak{g}$ in (12) is exactly the way the external gauge field couples to the sigma model variable of the $\nu = \pm 2$ boson IQHE state respectively, and (13) is the precisely the statement that this sigma model has $\nu = \pm 2$ Hall response.

A level k CS term attaches flux $\frac{2\pi}{k}$ to a unit electric charge. Because of the resulting Bohm-Aharonov phase, this changes the statistics angle of that charge by π/k. This means that the π-flux of the a, b gauge fields have $\pm\pi/2$ statistics – they are semions. Breaking $\mathrm{U}(1) \times \mathrm{U}(1) \overset{\langle\phi\rangle \neq 0}{\to} \mathbb{Z}_2$ (where ϕ has charge $(2, -2)$) doesn't change this.

comment about edge states from the sigma model description

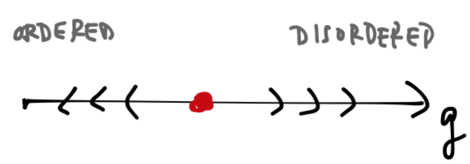
6.3 $D = 3 + 1$ Boson SPTs from Neel sigma models

In the previous construction, we (or rather [2]) added quite a bit of artificial symmetry to find a useful starting point. We (or rather [24]) did the same kind of thing in the discussion of the boson $\nu = 2$ IQHE SPT. Here we'll enlarge that boson symmetry even further, to $\mathrm{SO}(5)$ [2] and consider the $D = 3 + 1$ NLSM with field variable $\hat{n} \in S^4$, which has a fake $\mathrm{SO}(5) \supset \mathrm{U}(1) \times \mathrm{U}(1)$ symmetry. This is a device to make boson SPTs protected by $\mathrm{U}(1) \times \mathbb{Z}_2^\mathcal{T}$ or $\mathrm{U}(1) \ltimes \mathbb{Z}_2^\mathcal{T}$. We could have motivated by continuing the iterative coupled layer constructions one step further to $D = 3 + 1$, though I don't know how to get all of $\mathrm{SO}(5)$ this way. Perhaps SC Zhang does.

The action will be

$$S[\hat{n}] = \int \mathrm{d}^3x \left(\frac{1}{g^2} \partial_\mu \hat{n} \cdot \partial^\mu \hat{n} + \mathbf{i} \frac{\theta}{\Omega_4} \epsilon_{\cdot\cdot\cdot\cdot\cdot} n^{\cdot} \mathrm{d}n^{\cdot} \mathrm{d}n^{\cdot} \mathrm{d}n^{\cdot} \mathrm{d}n^{\cdot} \right) \tag{14}$$

where $\Omega_4 = \frac{8\pi^2}{3}$ is the volume of the 4-sphere. We will focus on $\theta = 2\pi$. The phase diagram is the same as for $\theta = 0$ (right).



The wavefunction at strong coupling is

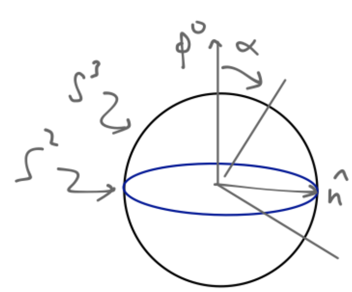
$$|\Psi\rangle_\theta \overset{g \to \infty}{\approx} \int [D\vec{n}(x)] e^{\mathbf{i}\theta \mathcal{W}_3[\hat{n}(x)]} |\hat{n}(x)\rangle,$$

where again $\mathcal{W}[n]$ looks the same as the theta term in (14) but interpreted as a functional of the boundary value, with τ as the fake extra dimension.

As before we reduce the symmetry in steps. First $\mathrm{SO}(5) \to \mathrm{SO}(4) \times \mathbb{Z}_2$ by

$$\vec{n} = (\sin\alpha\hat{\phi}, \cos\alpha\phi_0)$$

with $\hat{\phi} \in S^3$, $\hat{\phi}^2 = 1$, and $\phi_0 = \pm 1$. Add a potential to set $\alpha = \pi/2$, so $\phi_0 = 0$. Other decompositions may be interesting, too.



Next let's introduce the boson variables:

$$b_1 \equiv \phi_1 + \mathbf{i}\phi_2, \quad b_2 \equiv \phi_3 + \mathbf{i}\phi_4.$$

The U(1) that we wish to preserve acts by $b_I \to e^{\mathbf{i}\theta} b_I, I = 1, 2$. There are two possibilities for the $\mathbb{Z}_2^{\mathcal{T}}$ symmetry. Both act by $\phi_0 \to -\phi_0$ and $\sqrt{-1} \to -\sqrt{-1}$. They act differently on the bosons:

$$\mathbb{Z}_2^{(1)} : b_I \to b_I, \qquad \mathbb{Z}_2^{(2)} : b_I \to -b_I^{\star}.$$

The former commutes with the U(1) like XY-symmetric spin systems, while the former is the relevant realization in real boson systems, like ^{4}He.

The $\mathcal{W}$ term restricted to the equator $\phi_0 = 0$ becomes

$$\frac{\theta}{2\pi}\mathcal{W}_3[\hat{n} = (\hat{\phi}, 0)] = \frac{\theta}{24\pi^3}\epsilon_{....}\int_{\text{space}} \phi^{\cdot} \wedge \mathrm{d}\phi^{\cdot} \wedge \mathrm{d}\phi^{\cdot} \wedge \mathrm{d}\phi^{\cdot}$$

When $\phi = 2\pi$, this is

$$\mathcal{W}_3 = \pi N_s$$

where $N_s \in \pi_3(S^3)$ is the skymion number of the map $\vec{\phi}$: space $\to S^3$.

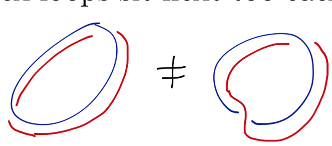
Claim: either choice for $\mathbb{Z}_2^{\mathcal{T}}$ enforces $\theta \in \pi\mathbb{Z}$, and hence reality of the wavefunction.

Penultimately, break SO(4) $\to$ U(1)$_1 \times$ U(1)$_2$, by adding a potential so that the field configurations are maps from space to the two-torus U(1)$_1 \times$ U(1)$_2$. We can label each such map by a configuration of two kinds of vortex loops. These vortex loops are just the loci where each boson vanishes:

$$V_I \equiv \{x \in \text{space} | b_I(x) = 0\}.$$

Since the boson is complex, $b_1(x) = 0$ is two real conditions on the three spatial coordinates x^i, so this locus is one-dimensional, i.e. a curve. And directly from the definition, these level-sets are closed loops.

And since we are already assuming that $\vec{\phi} \neq 0$ for energetic reasons, this means that $b_{1,2}$ don't simultaneously vanish. So the *linking number* of a configuration of the two kinds of loops is well-defined.

Finally, break U(1)$_1 \times$ U(1)$_2$ $\to$ U(1)$_{\text{diag}}$. The potential which does this makes the two kinds of vortex loops sit next too each other (but not too close). They can still wind around each other:  So we can think of the common vortex loop of the two kinds of bosons as a *ribbon*, stretching between the two kinds of loop. This assumes that the binding length for the two kinds of loops is smaller than the allowed separation of loops of the same species.

Claim: $N_s[b_1, b_2]$ is the total linking number $L_{12}(V_1, V_2)$ of the two species of vortex loops in the associated loop configuration. This means that

$$|\Psi\rangle_{\theta=2\pi} = \sum_{\{V_1, V_2\}} (-1)^{L_{12}(V_1,V_2)} |V_1, V_2\rangle \tag{15}$$

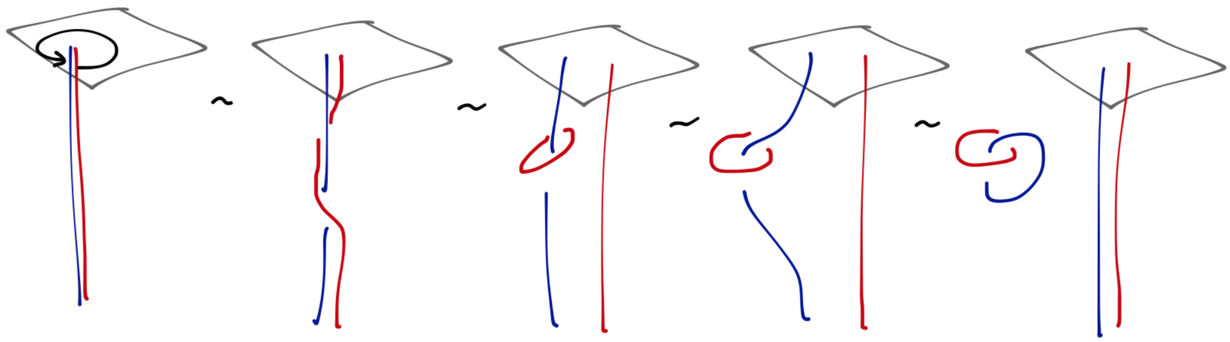


Figure 3: The end of the ribbon is a fermion, from [2]. In the first step, we rotate the red string around the blue one. The squiggles mean that the states associated with these configurations have the same amplitude in the groundstate, according to (17).

Exercise: Notice that the wavefunction in (17) is *not positive*. Explain how this is consistent with the theorem (due to Feynman?) that the groundstate wavefunction of a collection of bosons, with time-reversal symmetry, is positive definite.

As a consequence [2], in the presence of a boundary, an end of the vortex ribbon behave as a fermion. This statement is not precise because the surface may be gapless.

The *other* $D = 3 + 1$ **boson SPT:** Start from the same SO(5) NLSM, with $\theta = 2\pi$. But we will embed the real degrees of freedom differently into the $\vec{n}$ fields.

$$\vec{n} = (\mathrm{Re}\, b, \mathrm{Im}\, b, N_x, N_y, N_z)$$

$$\mathbb{Z}_2^{\mathcal{T}} : \sqrt{-1} \to -\sqrt{-1}, b \to -b^\star, \vec{N} \to -\vec{N}$$

In the core of a vortex line of b, i.e. a set of points where $b = 0$, we have $\vec{N} \cdot \vec{N} = \phi^2 = 1$ – inside the vortex line is an $\vec{N}$-field, a map from the vortex line into S^2. Each connected component of the vortex line is a circle, and living on it is a $D = 1 + 1$ O(3) NSLM; the theta term of the SO(5) model reduces to

$$2\pi \int_{\{b=0\}} \mathrm{d}t\mathrm{d}x \mathcal{H}[\phi = (0, 0, \vec{n})]$$

a $D = 1 + 1$ theta term, with $\theta = 2\pi$. This is the EFT of a Haldane chain living in the vortex line core.

Now what happens if one of these vortex lines terminates at a boundary of the system? It means that the Haldane chain has a boundary, which means there is a spin-$\frac{1}{2}$ living at the end: the surface vortex is a spin-$\frac{1}{2}$ two-state system. Before we break the fack SO(3) acting on $\vec{N}$ this is a representation of SO(3). But breaking the fake SO(3) symmetry is innocuous, since the Haldane chain is protected by $\mathbb{Z}_2^{\mathcal{T}}$, and the degeneracy of the edge state is, too – it is a Kramers' doublet.

6.4 Entanglement structure

The θ term affects the wavefunctional, and hence the manner in which the (nearby!) bulk degrees of freedom are (short-ranged-) entangled with each other. This means that fake boundaries can also detect a θ-dependence. What I mean by this is: the reduced density matrix in the groundstate associated with tracing out some region of the space A

$$\boldsymbol{\rho}_A \equiv \mathrm{tr}_{\mathcal{H}_{\bar{A}}} |\theta\rangle\langle\theta|$$

knows about θ. On general grounds, any density matrix can be written like a Boltzmann distribution

$$\boldsymbol{\rho}_A = \frac{1}{Z} e^{-\mathbf{K}}$$

with $\mathbf{K}$ Hermitian and Z determined by normalization $\mathrm{tr}\boldsymbol{\rho}_A = 1$. In this context $\mathbf{K}$ is called the *entanglement hamiltonian* (sometimes 'modular hamiltonian'). For gapped groundstates, very general arguments [37] (and also more specific arguments [38, 35]) relate the spectrum of $\mathbf{K}$ to the spectrum of actual edge states were the boundary of region A an actual boundary. (This connection was observed in quantum Hall states in [39].)

An interesting question raised by Zhoushen Huang is: can we use these strong-coupling sigma model wavefunctions to study the entanglement in the SPT groundstate in this limit? A given bulk phase hosts a large boundary phase diagram. Which edge phase will we get for the entanglement Hamiltonian? In the sigma model on a space with boundary we get to choose boundary conditions on the $\vec{n}$-field. This choice is absent for the entanglement calculation, both on the lattice and in the continuum. On the other hand, the choice of bulk Hamiltonian within the phase will affect both the edge spectrum and the entanglement spectrum. The strong-coupling limit of the sigma model is a particular choice of bulk Hamiltonian (actually, zero).

7 BF theory

The bulk EFT for $D = 3 + 1$ boson TIs has a description in terms of BF theory. This is a topological field theory where the dofs are a gauge field a and a 2-form potential B. For two species of bosons, there are two copies of each $I = 1, 2$ and

$$2\pi S[B, A] = \sum_I \epsilon^{\cdots\cdot} \frac{1}{2\pi} B^I_{\cdot\cdot} \partial_\cdot a^I_\cdot + \vartheta \sum_{IJ} \frac{K_{IJ}}{4\pi} \partial_\cdot a^I_\cdot \partial_\cdot a^J_\cdot \epsilon^{\cdots\cdot}.$$

Note that the theta angle ϑ here is not the same as the θ in the magnetoelectric resonse.

Whence these variables? Each copy is the 3+1d version of the charge-vortex duality, where for each boson current

$$j_\mu^{I=1,2} = \frac{1}{2\pi}\epsilon_{\mu\cdots}\partial_\cdot B_{\cdot\cdot}^I$$

which has $\partial^\cdot j_\cdot^I = 0$ as long as B is single-valued. B has to be a two form to soak up the indices in this equation. The magnetic field lines of a_μ are the vortex lines of b_I. If you like, the point in life of B is to say that a is flat; that is, its equation of motion is:

$$0 = \frac{\delta S}{\delta B} = \mathrm{d}a$$

Similarly, the EoM of a says B is flat.

Focus on the case $K = \boldsymbol{\sigma}^x$. One virtue of this effective action is that it reproduces the EM response. If we couple to an external U(1) gauge field $\mathcal{A}$ by

$$\Delta\mathcal{L} = \mathcal{A}_\mu\left(j_1^\mu + j_2^\mu\right)$$

then

$$\log\int[DaDB]e^{\mathbf{i}S[a,B,\mathcal{A}]} = \int\frac{2\vartheta}{16\pi^2}\mathrm{d}\mathcal{A}\wedge\mathrm{d}\mathcal{A} + \cdots$$

that is, the magneto-electric response is $\theta_{EM} = 2\vartheta$. So $\vartheta = \pi$ will be a nontrivial boson TI.

What is the groundstate wavefunction? As before we can cut open the path integral to find

$$\Psi_\vartheta[a(x), B(x)] = \underbrace{\Psi_{\vartheta=0}[a(x), B(x)]}_{\text{trivial Mott insulator}} \quad\times\quad \underbrace{\exp\mathbf{i}\frac{\vartheta}{8\pi^2}\int_{\text{space}} a_\cdot^I\partial_\cdot a_\cdot^J\epsilon^{\cdots}K^{IJ}}_{\overset{K=\boldsymbol{\sigma}^x}{=}\mathbf{i}\frac{\vartheta}{2}(\text{linking \# of }2\pi\text{ magnetic flux lines})} \quad .$$

I no longer believe the preceding discussion. Rather:

Choose $A_0 = 0$ gauge. Since A and B are conjugate variables, the analog of position space here is $|\vec{A}(x)\rangle$. For the same reason, we can only specify BCs on one or the other:

$$\int_{\substack{\vec{A}(x,\tau=\infty)=\vec{A}(x)\\ \vec{A}(x,\tau=-\infty)=\vec{A}'(x)}}[D\vec{A}(x,\tau)DB(x,\tau)]e^{-S[\vec{A}(x,\tau),B]} = \langle\vec{A}(x)|0\rangle\langle 0|\vec{A}'(x)\rangle \tag{16}$$

Notice that in expressions for functionals like $S[A(x,\tau)]$ I am writing the arguments of the function A to emphasize whether it is a function at fixed euclidean time or not. The fact that the theta term is a total derivative

$$\mathbf{i}\theta\mathcal{H} = \mathbf{i}\theta\int_{\partial M_D} w$$

(more precisely $\mathbf{i}\theta F \wedge F = c\mathbf{i}\theta \int_{\partial M_D} A \wedge F$)
means that the euclidean action here is

$$S[\vec{A}(x,\tau)] = \int_{M_D} \mathrm{d}\tau \mathrm{d}x \frac{1}{4\pi} B \wedge F + \mathbf{i}\theta \int \mathrm{d}x \left(w(n(x)) - w(n'(x)) \right).$$

The θ term only depends on the boundary values, and comes out of the integral in (16).

The integral over B^I is

$$\int [DB] e^{-\frac{1}{4\pi} \int B^I \wedge F^I} = \delta[F^I].$$

The delta functional on the RHS here sets to zero the flux of the gauge field for points in the interior of the cylinder.

After doing the integral over B, there is nothing left in the integral and we can factorize the expression (16) to determine:

$$\Psi[\vec{A}_I(x)] = \underbrace{\exp \mathbf{i} \frac{\vartheta}{8\pi^2} \int_{\text{space}} A^I_{\cdot} \partial_{\cdot} A^J_{\cdot} \epsilon^{\cdots} K^{IJ}}_{\overset{K=\boldsymbol{\sigma}^x}{=} \mathbf{i}\frac{\vartheta}{2} (\text{linking \# of } 2\pi \text{ magnetic flux lines})} \tag{17}$$

The conclusion is the same.

So again the figure above shows that the wavefunction of the end of the bound vortex lines (why are they bound?) acquires an extra minus sign under a 2π rotation.

The BF theory also knows about the edge states: the CS term at level 1 changes the statistics of the charges at the boundary.

An early confusing paper [40] claims that a similar BF theory is an effective topological field theory for *electronic* topological insulators. This interesting claim requires further development, especially with regard to the edge states.

8 Dual vortices

8.1 Charge-vortex duality

This is something that condensed matter theorists are supposed to know which I've had a hard time learning from the literature. Some references are: [41, 42, 43, 44, 45, 46, 47].

Consider a quantum system in $D = 2+1$ with a U(1) symmetry (a real symmetry, not a gauge redundancy). For example, focus on a complex bose field b with action

$$S[b] = \int \mathrm{d}t\mathrm{d}^2x \left(b^\dagger \left(\mathbf{i}\partial_t - \vec{\nabla}^2 - \mu \right) b - U(b^\dagger b)^2 \right) . \tag{18}$$

By Noether's theorem, the symmetry $b \to e^{\mathbf{i}\theta} b$ implies that the current

$$j_\mu = (j_t, \vec{j})_\mu = (b^\dagger b, \mathbf{i} b^\dagger \vec{\nabla} b + h.c.)_\mu$$

satisfies the continuity equation $\partial^\mu j_\mu = 0$.

This system has two phases of interest here. In the ordered/broken/superfluid phase, where the groundstate expectation value $\langle b \rangle = \sqrt{\rho_0}$ spontaneously breaks the U(1) symmetry, the goldstone boson θ in $b \equiv \sqrt{\rho_0} e^{\mathbf{i}\theta}$ is massless

$$S_{\text{eff}}[\theta] = \frac{\rho_0}{2} \int \left(\dot{\theta}^2 - \left(\vec{\nabla}\theta \right)^2 \right) \mathrm{d}^2x\mathrm{d}t, \quad j_\mu = \rho_0 \partial_\theta \ .$$

In the disordered/unbroken/Mott insulator phase, $\langle b \rangle = 0$, and there is a mass gap. A dimensionless parameter which interpolates between these phases is $g = \mu/U$; large g encourages condensation of b.

We can 'solve' the continuity equation by writing

$$j^\mu = \epsilon^{\mu\cdot\cdot} \partial . a. \tag{19}$$

where $a.$ is a gauge potential. The time component of this equation says that the boson density is represented by the magnetic flux of a. The spatial components relate the boson charge current to the electric flux of a. The continuity equation for j is automatic – it is the Bianchi identity for a – as long as a is single-valued. That is: as long as there is no magnetic charge present. A term for this condition which is commonly used in the cond-mat literature is: "a is non-compact."

The relation (19) is the basic ingredient of the duality, but it is not a complete description: in particular, how do we describe the boson itself in the dual variables? In the disordered phase, adding a boson is a well-defined thing which costs a definite energy. The boson is described by a localized clump of magnetic flux of a. Such a configuration is energetically favored if a participates in a superconductor – i.e. if a is coupled to a condensate of a charged field. The Meissner effect will then ensure that its magnetic flux is bunched together. So this suggests that we should introduce into the dual description a scalar field, call it Φ, minimally coupled to the gauge field a:

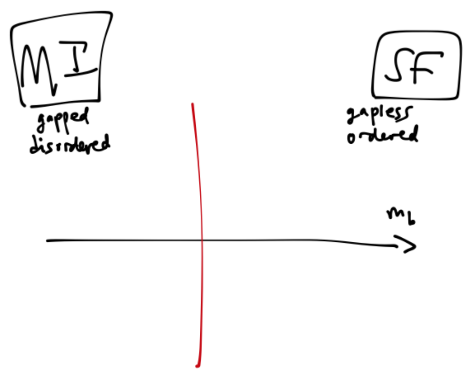
$$S[b] \leftrightsquigarrow S_{\text{dual}}[a, \Phi] \ .$$

And the disordered phase should be dual to a phase where $\langle \Phi \rangle \neq 0$, which gives a mass to the gauge field by the Anderson-Higgs mechanism.

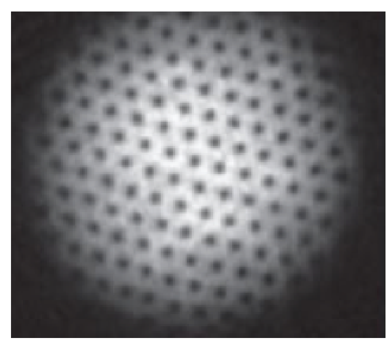
Who is Φ? More precisely, what is the identity in terms of the original bosons of the particles it creates? When Φ is not condensed and its excitations are massive, the gauge field is massless. This the Coulomb phase of the Abelian Higgs model $S[a, \Phi]$; at low energies, it is just free electromagnetism in $D = 2+1$. These are the properties of the ordered phase of b. (This aspect of the duality is explained in [48], §6.3.) The photon has one polarization state in $D = 2+1$ and is dual to the goldstone boson. This is the content of (19) in the ordered phase: $\epsilon^{\mu\cdot\cdot}\partial_\cdot a_\cdot = \rho_0 \partial_\mu \theta$ or $\star \mathrm{d}a = \rho_0 \mathrm{d}\theta$.

Condensing Φ gives a mass to the Goldstone boson whose masslessness is guaranteed by the broken U(1) symmetry. Therefore Φ is a disorder operator: its excitations are vortices in the bose condensate, which are gapped in the superfluid phase. The transition to the insulating phase can be described as a condensation of these vortices.

I should mention that these statements can also be made more precise in a lattice model whose continuum description is (18).



The vortices have relativistic kinetic terms, *i.e.* particle-hole symmetry. This is the statement that in the ordered phase of the time-reversal invariant bose system, a vortex and an antivortex have the same energy. An argument for this claim is the following. We may create vortices by rotating the sample, as was done in the figure at right (from Martin Zwierlein). With time-reversal symmetry, rotating the sample one way will cost the same energy as rotating it the other way.



This means that the mass of the vortices $m_V^2 \Phi^\dagger \Phi$ is distinct from the vortex chemical potential $\mu_V \rho_V = \mu_V \mathbf{i} \Phi^\dagger \partial_t \Phi + h.c.$. The vortex mass^2 maps under the duality to the boson chemical potential. Taking it from positive to negative causes the vortices to condense and disorder (restore) the U(1) symmetry.

To what does the vortex chemical potential map? It is a term which breaks time-reversal, and which encourages the presence of vortices in the superfluid order. It's an external magnetic field for the bosons. (This also the same as putting the bosons into a rotating frame.)

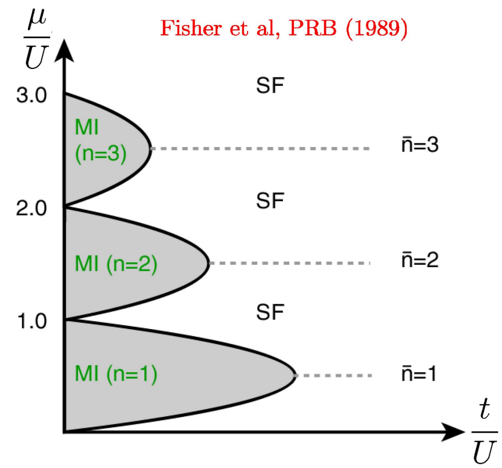
To summarize, a useful dual description is the Abelian Higgs model

$$S[a,\Phi] = \int \mathrm{d}^2x\mathrm{d}t \left(\Phi^\dagger \left((\mathbf{i}\partial_t - \mathbf{i}A_t - \mu)^2 + \left(\vec{\nabla} + \vec{A}\right)^2 \right) \Phi - \frac{1}{e^2} f_{\mu\nu} f^{\mu\nu} - V(\Phi^\dagger\Phi) \right).$$

We can parametrize V as

$$V = \lambda \left(\Phi^\dagger\Phi - v\right)^2$$

– when $v < 0$, $\langle\Phi\rangle = 0$, Φ is massive and we are in the Coulomb phase. When $v > 0$ Φ condenses and we are in the Anderson-Higgs phase.



I should mention that a fuller phase diagram for the bosons also allows us to vary the kinetic terms (the parameter t in the figure at right). This gives (even in mean field theory) the famous picture at right, with lobes of different Mott insulator states with different (integer!) numbers of bosons per site (which I borrowed from Roman Lutchyn). The description above is valid near the boundary of one of the MI phases. At the tips of the lobes are special points where the bosons b themselves have particle-hole symmetry (i.e. relativistic kinetic terms). For more on this diagram, see *e.g.* the original reference [42] or chapter 9 of [49].

In the previous discussion I have been assuming that the vortices of b have unit charge under a and are featureless bosons, *i.e.* do not carry any non-trivial quantum numbers under any other symmetry. If *e.g.* the vortices have more-than-minimal charge under a, say charge q, then condensing them leaves behind a $\mathbb{Z}_q$ gauge theory and produces a state with topological order. If the vortices carry some charge under some other symmetry (like lattice translations or rotations) then condensing them breaks that symmetry. If the vortices are minimal-charge fermions, then they can only condense in pairs, again leaving behind an unbroken $\mathbb{Z}_2$ gauge theory.

8.2 Lattice bosons in $D = 3 + 1$

Now let's talk about a description of bosons in $D = 3+1$ which includes the lattice structure. More specifically, let's think about a rotor description:

$$\mathbf{H} = -t \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) + U\hat{n}_i^2 .$$

Here $\hat{n}_i$ is the deviation from the average number of bosons. You should think of $b_i \sim e^{\mathbf{i}\theta_i}$ and $[\hat{n}_i, e^{\mathbf{i}\theta_i}] = n_i + 1$. Here $[\hat{n}_i, \theta_j] = \mathbf{i}\delta_{ij}$. The rotor description is an approximation which works if we don't have occasion to notice that nothing stops n_i from going negative.

The disordered, Mott insulating phase $\langle\theta\rangle = 0$ occurs at $t/U \ll 1$. A wavefunction for it is

$$|MI\rangle_{t=\infty} = \prod_i |\underbrace{n_i = 0}_{\text{no extra bosons}}\rangle \propto \prod_i \int_0^{2\pi} \mathrm{d}\theta_i |\{\theta_i\}\rangle$$

This is a trivial product state.

Perturbation theory in t/U gives

$$|MI\rangle_{K\equiv U/t\ll 1} \simeq \prod_i \int_0^{2\pi} \mathrm{d}\theta_i e^{-K\sum_{ij}\cos(\theta_i-\theta_j)} \prod_i |\{\theta_i\}\rangle$$

Here we are using a basis of states for the $D = 3+1$ quantum XY model which are lattice configurations of the $D = 3+1$ classical lattice XY model, $\{\theta_i\}$. The coefficient of a given basis vector is the associated Boltzmann weight.

Instead, we could use a basis labelled by the dual variables [50, 51, 52, 53, 41] – configurations of vortex loops $\{\vec{J}, \vec{A}\}$. This is the duality described in the previous subsection but in $D = 3+0$ instead of $D = 2+1$, and on the lattice. Here $\vec{J}$ is an integer living on sites of the dual lattice, labelling the number of vortex loops of θ entering from each direction; the fact that those loops are closed means $\vec{\nabla}\cdot\vec{J} = 0$. The role of the gauge field $\vec{A}$ is again that its magnetic field lines are the The $D = 3+0$ lattice duality of partition functions is

$$\int \prod_i \mathrm{d}\theta_i e^{-K\cos\theta_{ij}} \simeq \int \prod_{\text{links},\ell} \mathrm{d}\vec{A}_\ell \sum_{\{\vec{J}\}} e^{-\int \mathrm{d}^3x\left(\frac{1}{2K}\left(\vec{\nabla}\times\vec{A}\right)^2 + 2\pi\mathbf{i}\vec{A}\cdot\vec{J}\right)}$$

We can just stick the basis states in here to make a true statement about wavefunctions of the $D = 3+1$ quantum XY model:

$$\int \prod_i \mathrm{d}\theta_i e^{-K\cos\theta_{ij}} |\{\theta_i\}\rangle \simeq \int \prod_{\text{links},\ell} \mathrm{d}\vec{A}_\ell \sum_{\{\vec{J}\}} e^{-\int \mathrm{d}^3x\left(\frac{1}{2K}\left(\vec{\nabla}\times\vec{A}\right)^2 + 2\pi\mathbf{i}\vec{A}\cdot\vec{J}\right)} |\{\vec{A}, \vec{J}\}\rangle$$

The $K \to \infty$ ($t/U \to 0$) limit is a uniform superposition of vortex loops. It has positive weights. Usually the vortices don't carry any internal structure or quantum numbers. Their condensation then leads to a featureless Mott insulating state, like the product above, in which case the usual $|\{\theta_i\}\rangle$ basis is more useful.

The vortex perspective turns out to be quite useful when the vortices carry other quantum numbers.

9 Surface phase structure

At the surface of a $D = 3+1$ boson SPT the vortices are more interesting. Let's focus on the boson SPT protected by $U(1) \times \mathbb{Z}_2^{\mathcal{T}}$ where the vortices are fermions (created by a grassmann field operator c) and sketch features of the surface phase diagram (all within the same bulk phase), following closely [1].

- Break U(1), preserve $\mathbb{Z}_2^{\mathcal{T}}$: This surface superfluid is a vortex Mott insulator. It can be described by letting the fermionic vortices fill bands with zero net Chern number (required by $\mathcal{T}$). There is a surface goldstone mode, described by the dual photon a.
- Break U(1) and $\mathbb{Z}_2^{\mathcal{T}}$: Now we can have a surface superfluid which is a vortex Chern insulator: the fermionic vortices can fill bands with a net Chern number, and hence a quantized Hall response. The surface effective action, including the coupling to the external gauge field $\mathcal{A}$ is

  $$L[c, a, \mathcal{A}] = L[c, a] + \frac{1}{2\pi}\epsilon \partial a \mathcal{A} \ .$$

  This is the case with $\theta_{EM} = 2\pi$, and $\sigma_{xy} = \frac{e^2}{h}$.
- It is possible to preserve all symmetries and still gap out the edge states by condensing paired vortices. That is, let the vortices partially fill some bands and form a Fermi liquid with a cooper instability. This higgses the U(1) to $\mathbb{Z}_2$ and produces a weird version of the toric code, where the topological quasiparticles $e, m, \epsilon = e\bar{m}$ are respectively two charge-$\frac{1}{2}$ bosons and a neutral fermion.

A less sketchy description of these states can be obtained by implementing the duality transformation starting with two species of bosons. Let the field Φ_I create a vortex of boson b_I. An interesting dual surface action to consider is

$$L[\Phi_I, a_I, \beta] = \sum_{I=1,2} |\left(\partial_\mu - \mathbf{i}\left(a_\mu^I + \beta_\mu^I\right)\right)\Phi_I|^2 - V(|\Phi|^2) + \frac{\mathbf{i}}{\pi}\beta^1_{\cdot}\epsilon^{\cdot\cdot\cdot}\partial_{\cdot}\beta^2_{\cdot} + \mathbf{i}\mathcal{A}^I_{\cdot} j_I^{\cdot} \ .$$

Each pair of a_I, Φ_I is just as in the previous section, so the individual bose currents are represented as $j_\mu^I = \frac{1}{2\pi}\epsilon^{\mu\cdot\cdot}\partial_{\cdot} a^I_{\cdot}$. The role of the β gauge fields is to change the statistics of the vortices.

It is also interesting to consider the model obtained by dualizing only *one* of the species of bosons. These models are very similar to ones that arose in the study of deconfined critical points [54].

10 Locally open questions and omissions

- I didn't talk about exactly solvable lattice models of SPT phases [15, 55, 56, 14].
- What is microscopic description of the screening process in the 2d antiferromagnet on the cylinder which decreases θ by 2π (§4)? In the case of $D = 3 + 1$ boson SPT states, the process which jumps θ by its period (4π) is the 'adsorption' of a boson IQHE layer onto the surface.
- How to make a $B \wedge F$ representation of the 3+1d boson SPT where the vortex string has a Kramer's doublet inside?
- How to describe the state where the vortices are both fermions *and* Kramer's doublets in terms of the sigma model?
- What is the relation between the Neel sigma models we've discussed here and the spacetime lattice sigma models on the protecting group constructed by [9]?
- Can we give a more direct proof of the relation (10) between the AKLT wavefunction and the strong-coupling limit of the sigma model wavefunctional?
- What exactly is the difference between the strong-coupling limit of (a) nonlinear sigma models on S^n and that of (b) abelian gauge theory in $D = 3 + 1$ for the purposes of constructing wavefunctions? The analogous wavefunction in Maxwell theory is

  $$\Psi[\vec{A}(x)] = e^{\mathbf{i} \int_{\text{space}} A \wedge dA} .$$

  This wavefunction actually solves the Schrödinger equation for quantum Maxwell theory at *finite* coupling. There is even a non-Abelian version of it for which this is also true. There is even an analog for gravity called the 'Kodama state'. What's the catch? It's not normalizable as a wavefunction for photon fields [57]; attempting to quantize the model about this groundstate gives negative energy for one of the two circular polarization states. Why don't [2] run into this problem? They would have the same problem if they thought of their wavefunctions as wavefunctions for gapless magnons!

  The analog of the gapped, disordered phase of the NLSM is the confining phase of the Maxwell theory. So this logic suggests the existence of a topologically nontrivial confined state of Maxwell theory, with $\theta = 2\pi$. If all excitations are bosonic, this is distinct from $\theta = 0$. (If there are no charged particles, all $\theta \in 2\pi\mathbb{Z}$ are distinct?) What are its characteristics? Actually, we have already encountered this wavefunction in the BF section. It should also arise via a parton construction, as in [5, 58, 12].
- Speaking of parton constructions: for a long time it was believed (more defensible: I believed) that only deconfined phases of parton gauge theories were interesting for condensed matter – why would you want to introduce a gauge field if it is just going to be confined? Parton constructions for quantum Hall states are an easy and lucrative exception to this, in that the gauge field acquires a mass via a Chern-Simons term and so is massive but not confined. A more dramatic exception appears in [59, 60, 61, 62]

where confined states of parton gauge theories are used to describe superfluid phases in $D = 2+1$ in terms of variables that allow access to nearby fractionalized phases (and the degrees of freedom that become light at the intervening phase transitions).

More recently, confined states of parton gauge theories have been used to discuss $D = 3+1$ SPT states in [63, 5, 58, 12].

- Didn't [1] actually also classify superfluid phases with short-range entanglement, that is *topology-enriched symmetry-broken phases*?
- What is the crucial property shared by all possible edge states with the same bulk SPT state? Alternatively, what edge state properties can only be changed during a bulk transition? Some progress on this question is described in [64]. They say: "In the presence of a symmetry which remains unbroken in the thermodynamic limit, we prove that these edge modes, namely the additional ground states that arise whenever a boundary is present, carry equivalent representations of G within a phase."

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