The Matrix Harmonic Oscillator as a String Theory

with:

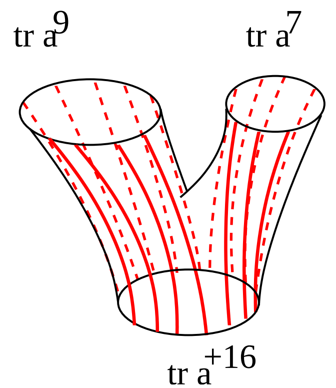
N. Itzhaki

hep-th/0408180

Introduction

't Hooft's demonstration of the planar structure of the feynman graphs of matrix models applies even to stupid theories, even free field theories. $\frac{1}{N} \propto g_s$

This is a feynman diagram in a free theory:



There is no guarantee that the worldsheet theory is tractable, or that it has a nice geometric interpretation as a sigma model.

In AdS/CFT, $\lambda_{\text{'tHooft}} = g^2_{YM} N = R^4_{AdS} \implies$ making the field theory free makes the geometry small.

But this doesn't mean CFT breaks down or is even hard – *e.g.* LG points.

To understand better how string theory emerges from gauge theory, let's start with as simple a gauge theory as possible:

$$S = \frac{1}{2} \int dt \ \mathrm{tr} \ \left((D_0 X)^2 - X^2 \right)$$

Today: A calculable worldsheet description of this string theory.

The role of N

There exist dualities between similar matrix models and $d \leq 2$ string theory interpretable as unstable open-string systems.

There is an important difference in the role of N.

random lattice models of 2d gravity: $N = \infty$

vs

summing over holes, triangulation of moduli space: $N = \frac{1}{g_s}$

Here it's exactly like AdS/CFT.

As it is for some zero-dimensional matrix models: Kontsevich, Penner, Dijkgraaf-Vafa.

some highlights

1. this is an explicit, calculable worldsheet description of a free gauge theory.

2. a (1/2-BPS) sector of $\mathcal{N} = 4$? ($\Longrightarrow$ part of $AdS_5 \times S^5$?)

3. string theory of the quantum Hall effect?

4. string perturbation theory truncates: at fixed total momentum only a finite number of genera contribute.

5. this string theory is *crazy*. The fact that sense can be made of it suggests that there are many related constructions which we would have discarded but shouldn't.

Plan

1. the matrix harmonic oscillator and its symmetries
2. a first look at the dual string theory
3. tree-level amplitudes
4. beyond tree level
5. conclusions

Some work with related motivations:

R. Gopakumar, `hep-th/0308184, 0402063`

Berenstein, `hep-th/0403110`

Aharony *et. al.*, `hep-th/0310285`

A. Karch *et. al.*, `hep-th/0212041, 0304107`

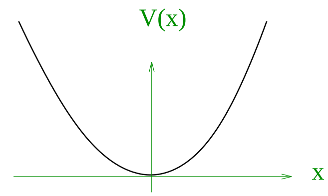
Dhar-Mandal-Wadia, `hep-th/0304062`

Fidkowski-Shenker, `hep-th/0406086`

H. Verlinde, `hep-th/....`

Matrix harmonic oscillator

Consider the gauged quantum mechanics of an $N \times N$ matrix harmonic oscillator,



$$S = \frac{1}{2} \int dt \text{ tr } \left((D_0 X)^2 - X^2\right)$$

$D_0 = \partial_0 + [A_0, \cdot]$ is covariant with respect to local $U(N)$ conjugations $X(t) \mapsto \Omega(t) X(t) \Omega^\dagger(t)$.

The gauge field acts as a Lagrange multiplier that projects onto singlet states.

Important distinction from $c = 1$ matrix model:

orbits in phase space are compact, the spectrum is discrete.

This model is solvable in at least two distinct ways:

in terms of free fermions (true for an arbitary potential).

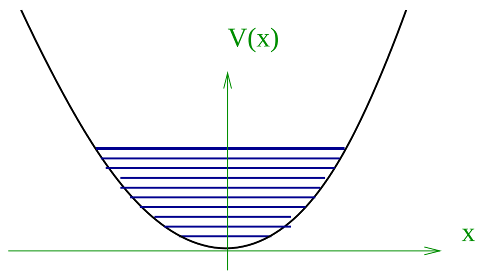
by matrix ladder operators (true for an arbitrary number of oscillators).

Fermions, briefly

$\Longrightarrow$ (Brezin-Itzykson-Parisi-Zuber) N free fermions in the harmonic oscillator potential.

$$H = \sum_{j=1}^{N} h_j$$

Energy eigenstates are Slater determinants:
states labelled by N integers k_n such that $0 \leq k_1 < k_2 < ... < k_N$, and the eigenvalues of the Hamiltonian are $E(k_i) = \sum_{i=1}^{N}(k_i + 1/2)$.



The vacuum energy is $E_0 = \frac{1}{2} + \frac{3}{2} + ... + \frac{(2N-1)}{2} = \frac{N^2}{2}$.

Bosons

Introduce matrix raising and lowering operators

$$a^i_j = \frac{1}{\sqrt{2}}\left(X^i_j + iP^i_j\right), \qquad a^{\dagger\ i}_j = \frac{1}{\sqrt{2}}\left(X^i_j - iP^i_j\right),$$

P is the momentum conjugate to the matrix X.

$$[a^i_j, a^{\dagger\ k}_l] = \delta^i_l \delta^k_j, \qquad [H, a^i_j] = -a^i_j, \qquad [H, a^{j\ \dagger}_i] = a^{j\ \dagger}_i.$$

The vacuum of H is defined by $a^i_j|0\rangle = 0$.

Excite $|0\rangle$ by $a^\dagger$ to make energy eigenstates.

Take traces to make singlets.

A useful basis of states: take $m_1 \geq m_2 \geq ... \geq m_r$

$$|\{m_n\}\rangle \equiv \prod_{n=1}^{r} \mathrm{tr}\,(a^{\dagger\ n})^{m_n}|0\rangle \quad \text{has} \quad E = \sum_{n=1}^{\infty} n m_n + E_0.$$

$m_i \leq N$ for linear independence: "stringy exclusion"

Finite N real, non-relativistic bosonization

At large N, excitations above the ground state are most easily described in terms of a chiral boson. Consider the partition function in the fermion description

$$Z(q,N) = \mathrm{tr}\; q^H = q^{\frac{N}{2}} \sum_{k_1=0}^{\infty} \sum_{k_2=k_1+1}^{\infty} \dots \sum_{k_N=k_{N-1}+1}^{\infty} q^{\sum_{n=1}^{N} k_n}.$$

Performing the sums sequentially (Boulatov-Kazakov)

$$Z(q,N) = q^{N^2/2} \prod_{n=1}^{N} \frac{1}{1-q^n}$$

the partition function of a 2d chiral boson, with $\alpha_0 = N$ whose excitations are truncated at level N.

$$H = \frac{\alpha_0^2}{2} + H_{\mathrm{osc}}, \qquad [H_{\mathrm{osc}}, \alpha_n] = n\alpha_n$$

Finite N real, non-relativistic bosonization

At large N, excitations above the ground state are most easily described in terms of a chiral boson. Consider the partition function in the fermion description

$$Z(q,N) = \mathrm{tr}\; q^{H} = q^{\frac{N}{2}} \sum_{k_1=0}^{\infty} \sum_{k_2=k_1+1}^{\infty} \cdots \sum_{k_N=k_{N-1}+1}^{\infty} q^{\sum_{n=1}^{N} k_n}.$$

Performing the sums sequentially (Boulatov-Kazakov)

$$Z(q,N) = q^{N^2/2} \prod_{n=1}^{N} \frac{1}{1-q^n} = q^{N^2/2} \prod_{n=1}^{N} \sum_{m_n=0}^{\infty} q^{n m_n}$$

the partition function of a 2d chiral boson, with $\alpha_0 = N$ whose excitations are truncated at level N.

$$H = \frac{\alpha_0^2}{2} + H_{\mathrm{osc}}, \qquad [H_{\mathrm{osc}}, \alpha_n] = n\alpha_n$$

$$\alpha_{-n} = \mathrm{tr}\ \left(a^\dagger\right)^n \qquad \alpha_n = \mathrm{tr}\ a^n, \qquad n > 0$$

are the modes of the bosons. This will be a tachyon of momentum m.
Later on: this chiral boson is the target-space field of the string description.
For finite N, there is a UV cutoff on the momentum modes of the chiral boson.

This is the same Hilbert space as that of two-dimensional Yang-Mills theory on a cylinder. But

$$(H_{\mathrm{SHO}} - E_0) = \sum_{i=1}^{r} m_i$$

rather than $C_2(R)$.

‘Observables’

The states made by products of traces aren’t orthogonal.

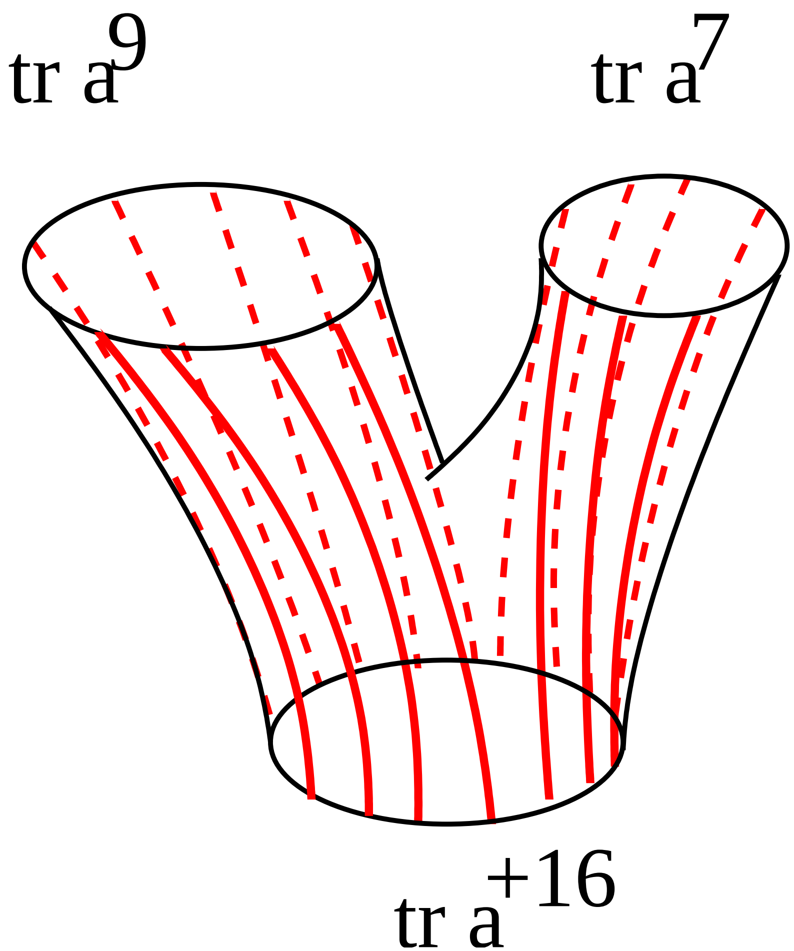
The observables are overlaps like

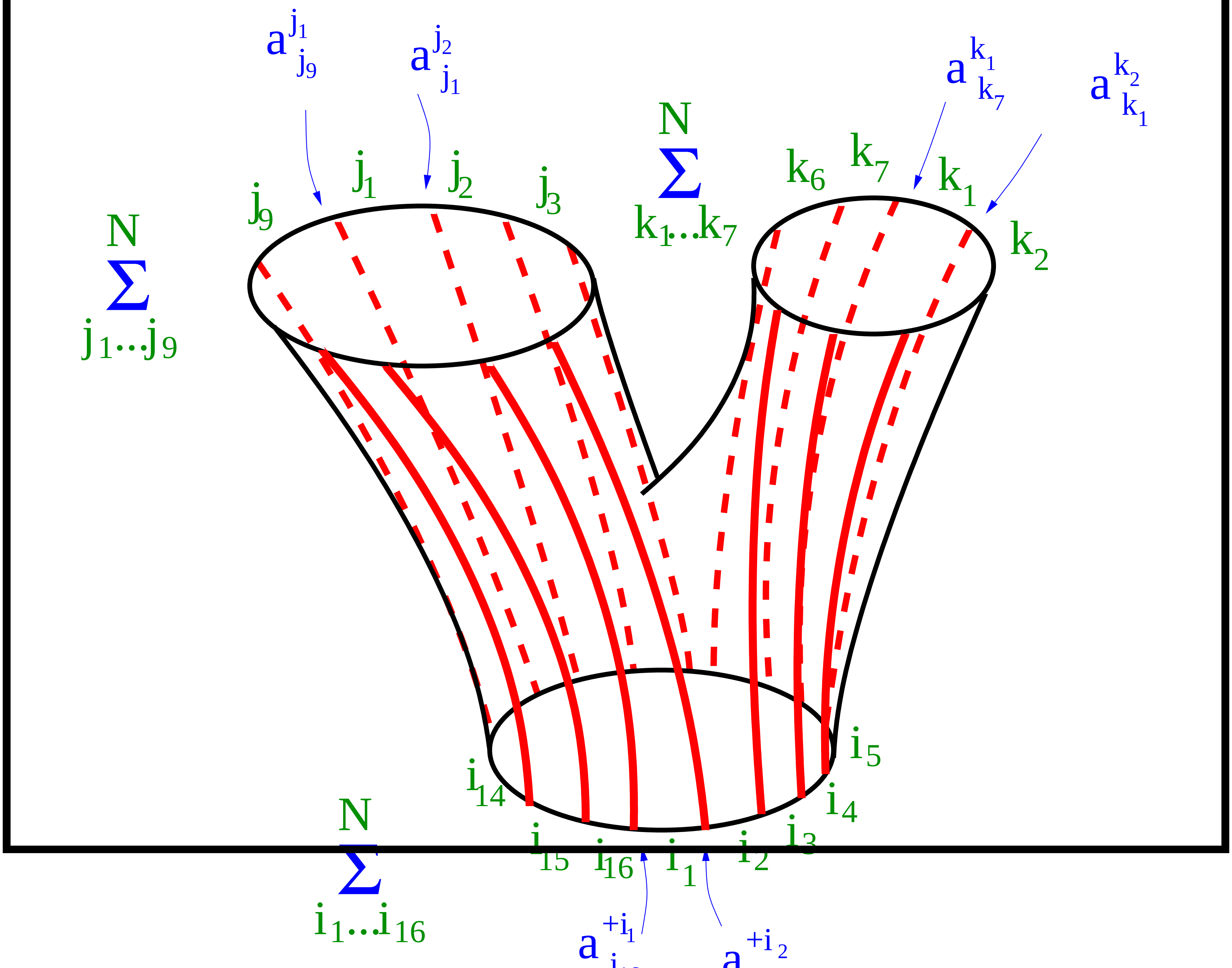
$$\mathcal{O}(k_i^+; k_j^-) \equiv \langle 0| \prod_i \mathrm{tr}\; a^{k_i^-} \prod_j \mathrm{tr}\; a^{\dagger\ k_j^+} |0\rangle$$

I’ve stripped off the time-dependence

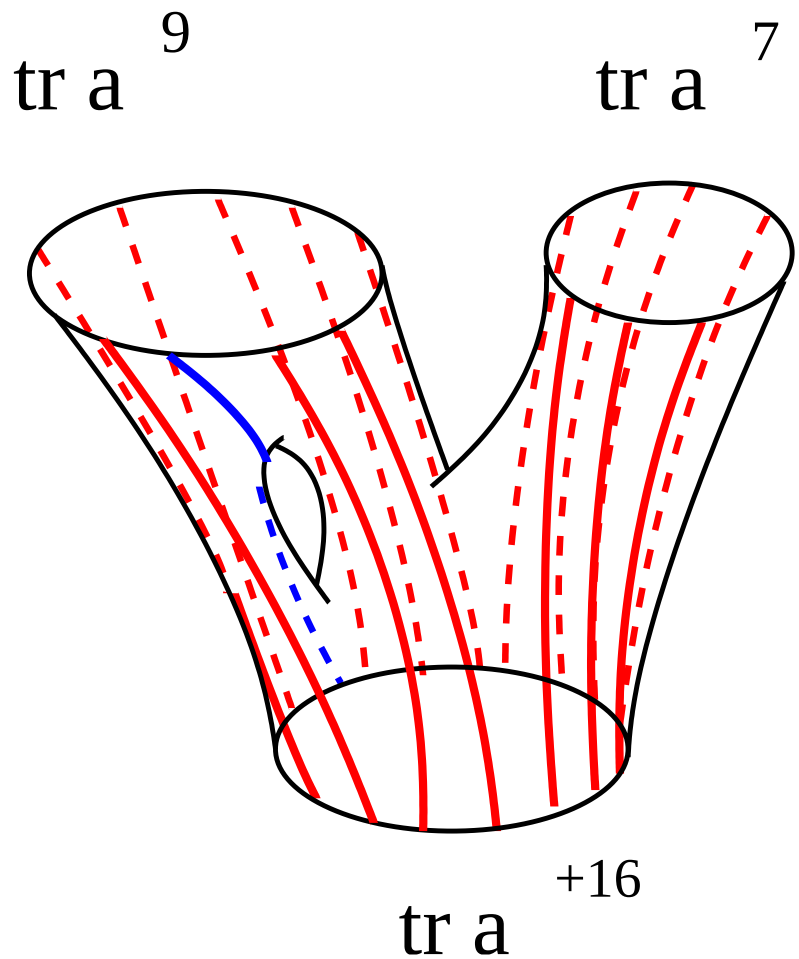
$$\langle \psi_+(t_+) | \psi_-(t_-) \rangle = \mathcal{O}(+;-)\ e^{i \sum k^+ t_+ - i \sum k^- t_-} .$$

$$\langle 0|\mathrm{tr}\ a^9 \mathrm{tr}\ a^7 \mathrm{tr}\ \left(a^\dagger\right)^{16}|0\rangle \sim N^{15} \quad :$$





$a^{j_1}_{j_9}$
$a^{j_2}_{j_1}$
$a^{k_1}_{k_7}$
$a^{k_2}_{k_1}$
$\sum_{j_1 \ldots j_9}^{N}$
$\sum_{k_1 \ldots k_7}^{N}$
$\sum_{i_1 \ldots i_{16}}^{N}$
j_9
j_1
j_2
j_3
k_6
k_7
k_1
k_2
i_5
i_{14}
i_4
i_{15}
i_{16}
i_1
i_2
i_3
a^{+i_1}
a^{+i_2}



$$\sim N^{15} \cdot \frac{1}{N^2}$$

N-counting

A "two-point function" is

$$\mathcal{O}(k_1;k_2) = \langle 0|\mathrm{tr}\ (a^{k_1})\mathrm{tr}\ ((a^\dagger)^{k_2})|0\rangle.$$

By counting index lines, the leading contribution in the large-N limit goes like

$$\mathcal{O}(k_1;k_2) \sim \delta_{k_1,k_2} N^{k_1} + \mathcal{O}\left(N^{-2}\right).$$

$1/N$ corrections truncate after a finite number of terms:

$$\mathcal{O}(k_1;k_2) = \delta_{k_1,k_2} \sum_{l=0}^{[k_1/2]} C_l(k_1) N^{k_1-2l},$$

C_l's depend on k_1 but not on N.

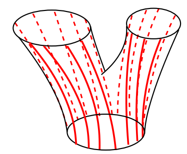
A handle must be traversed by at least one bit.

A direct consequence of the absence of interactions.

This is an interesting prediction for the string theory dual.

The analog of a three-point function is

$$\mathcal{O}(k;k_1,k_2) = \langle 0|\mathrm{tr}\ (a^k)\mathrm{tr}\ ((a^\dagger)^{k_1})\mathrm{tr}\ ((a^\dagger)^{k_2})|0\rangle.$$



the leading contribution at large-N is

$$\mathcal{O}(k;k_1,k_2) \sim \delta_{k,k_1+k_2} N^{k-1}.$$

Symmetries

At large N, a semi-classical description is useful:
the eigenvalues form a Fermi sea in phase space with coordinates

$$U = \frac{(X + iP)}{\sqrt{2}}, \qquad V = U^\dagger = \frac{(X - iP)}{\sqrt{2}},$$

which satisfy the Poisson bracket

$$\{U, V\}_{PB} = i.$$

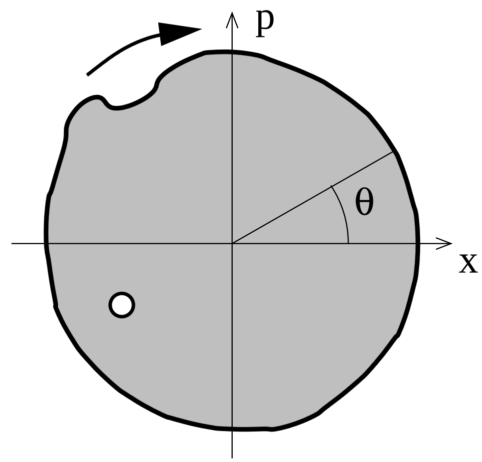
The Hamiltonian is $H = UV$, and thus

$$U(t) = e^{-it}U(0), \quad V(t) = e^{it}V(0).$$

The ground state is obtained by filling states in the phase space up to the Fermi surface

$$UV = N,$$

so that the area is equal to the number of fermions.

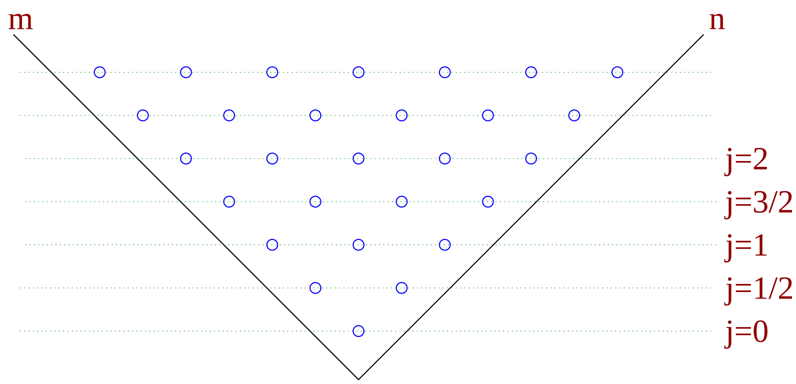


The following charges are conserved

$$Q_{n,m} = e^{i(n-m)t} U^n V^m.$$

These charges satisfy the w_∞ algebra

$$\{Q_{n,m}, Q_{n',m'}\}_{PB} = i(nm' - mn')\; Q_{n+n'-1,m+m'-1}.$$



Excitations above the ground state can be described using area-preserving transformations of the phase space.

The area is preserved since it is the number of fermions.

Area-preserving transformations can be described using a single scalar function $h(U, V)$, a basis for which are

$$h_{nm} = U^n V^m \longrightarrow \mathrm{tr}\ a^n \left(a^\dagger\right)^m ,$$

n and m must be integers in order to preserve the connectivity of the fermi surface. h_{nm} will be related to the 'discrete states' in the dual stringy description.

A small puzzle and its resolution

Quantumly there seem to be many more excitations than semi-classically.

There are many different single trace operators that at the semi-classical level are associated with the same h_{nm}.

e.g. tr $(aa^\dagger aa^\dagger)$ tr $(a^2 a^{\dagger\, 2})$

The fact that the theory is gauged resolves this issue:

in one dimension there is no electric or magnetic field

$\longrightarrow$ the current that couples to A must vanish on-shell,

$$0 = \frac{\delta S}{\delta (A_0)^i_j} = j^j_i = [X, D_0 X]^i_j = i[a, a^\dagger]^i_j .$$

$\longrightarrow$ operators which differ by such commutators actually give the same result when acting on physical states.

$\longrightarrow$ the inequivalent single-trace operators are in 1-to-1 correspondence with w_∞ generators : tr $a^n a^{\dagger\, m}$:

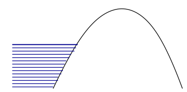
This gauss' law is reminiscent of a normal matrix constraint.

The dual string theory

Hint: Recall the well-studied duality between 2d string theory with a Liouville direction and matrix quantum mechanics.

This MQM is closely related to the one we are considering.

$V(X) = -\mathrm{tr}\ X^2$



The kinetic term is the same and the sign of the potential term is flipped.

The state with equal filling describes the 0B vacuuum.

Q: What does this sign flip mean on the string theory side?

The curvature of the potential in the matrix model (Klebanov-Maldacena-Seiberg) is related to the tension of the dual string by

$$U(x) = \frac{1}{2\alpha'} x^2.$$

So flipping the potential means that we have to take

$$\alpha' \to -\alpha'.$$

Massive modes become tachyonic??

In two dimensions there are no massive modes

(other than some discrete states that will play an important role shortly)

Or: keep α' positive and Wick rotate *all* dimensions.

In D dimensions $\longrightarrow$ $D-1$ time-like directions

But in two dimensions, time $\leftrightarrow$ space. no oscillators? no problem.

Apply to the $c=1$ worldsheet theory

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma \left(-\partial_\alpha X \partial^\alpha X + \partial_\alpha \varphi \partial^\alpha \varphi + \sqrt{\alpha'} R^{(2)} Q\varphi + \mu_0 e^{2b\varphi/\sqrt{\alpha'}}\right).$$

$$X \to iX, \quad \varphi \to i\varphi.$$

Notice that in the presence of the Liouville interaction this is not an *analytic* continuation (Polchinski).

$\longrightarrow$ a CFT whose stress tensor is

$$T(z) = \frac{1}{\alpha'} \left(: \partial\varphi\partial\varphi : - : \partial X \partial X :\right) + i\frac{Q}{\sqrt{\alpha'}}\partial^2\varphi.$$

T is complex $\longrightarrow$ world-sheet theory is non-unitary.

We will see that the target space theory is still ok.

This is like the Coulomb gas description of minimal models.

The central charge stays critical

$$c = 2 - 6i^2Q^2 = 26,$$

the minus sign comes from the fact that now the Liouville direction is time-like.

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\,\left(\partial_\alpha X\partial^\alpha X - \partial_\alpha\varphi\partial^\alpha\varphi + i\sqrt{\alpha'}R^{(2)}Q\varphi + \mu_0 e^{i2b\varphi/\sqrt{\alpha'}}\right).$$

$\alpha' = 1$ from now on.

$$g_s = e^{i2\varphi} \quad ?!$$

1. Unlike in the usual linear dilaton case there is no separation into weakly coupled and strongly coupled regions.

($|g_s| = 1$ everywhere.)

$1/\mu_0$ is the parameter that controls the genus expansion.

2. We can compactify φ. The allowed radii of compactification are

$$R = m/2, \quad m \in 2\mathbb{Z}$$

the '2' is a concession to the existence of open strings.

We will compactify at the smallest possible radius, $R = 1$ (self-dual)

$$\varphi \sim \varphi + 2\pi.$$

Proposal:

X should be identified with the quantum mechanics time and φ should be identified with the angular variable, θ, in phase-space.

Excitations in both descriptions travel at the same speed (1 in our units) regardless of their energy.

Puzzle: The string theory spectrum contains both left-movers and right-movers (in the target space), while on the quantum mechanics side there are only left-movers.

'Tachyons'

The ghost-number-two cohomology contains the tachyon vertex operator

$$T_k^{\pm} = c\bar{c}\; e^{-ik(X\pm\varphi)} e^{i2b\varphi},$$

where the factor $e^{i2b\varphi}$ is the string coupling that multiplies the tachyon wave function,

k is an integer due to the periodicity of φ.

Four kinds of modes:

$$\begin{array}{llll} T^{+}_{k>0}: & \text{incoming leftmover}, & T^{-}_{k>0}: & \text{incoming rightmover}, \\ T^{+}_{k<0}: & \text{outgoing leftmover}, & T^{-}_{k<0}: & \text{outgoing rightmover}. \end{array}$$

the energy in the X direction determines if a wave is incoming or outgoing.

Scattering

S-matrix amplitudes are constructed in the usual fashion:

$$A(k_1, k_2, ..., k_{n^+}; k_{n^++1}, ..., k_{n^++n^-}) =$$

$$\int \mathcal{D}X\mathcal{D}\varphi \;\; e^{-S} \prod_{i=1}^{n^+ + \, n^-} \int_{\Sigma} e^{ik\varphi \pm ikX} e^{i2b\varphi},$$

the first n^+ tachyons have positive chirality

the remaining n^- have negative chirality.

The effective string coupling is $\frac{1}{\mu_0}$

$$S = S_0 + \int_\Sigma \mu_0 e^{2ib\varphi}$$

Inside the path integral, expand the exponent in powers of μ_0:

$$A(k_1, k_2, ..., k_{n^+}; k_{n^++1}, ..., k_{n^++n^-}) =$$

$$\sum_{n=0}^{\infty} \frac{\mu_0^n}{n!} \int \mathcal{D}X\mathcal{D}\varphi \; e^{-S_0} \left(\int_\Sigma e^{i2b\varphi} \right)^n \prod_{i=1}^{n^++n^-} \int_\Sigma e^{ik_i\varphi \pm ik_i X} e^{i2b\varphi},$$

The S-matrix now takes the form of a sum over amplitudes in the free theory with extra insertions of the screening operator, $\int_\Sigma e^{i2b\varphi}$.

Following (Gupta et al) decompose $\varphi = \varphi_0 + \tilde{\varphi}$ and $X = X_0 + \tilde{X}$ and integrate first the zero modes, φ_0 and X_0.

The integral over X_0 is a delta function which imposes energy conservation

$$k^+_{tot} + k^-_{tot} = 0, \qquad k^+_{tot} = \sum_{i=1}^{n^+} k_i, \qquad k^-_{tot} = \sum_{i=n^++1}^{n^++n^-} k_i.$$

The integral over φ_0 is

$$\int_0^{2\pi} d\varphi_0 \;\exp\left(i\varphi_0 \left(-2 + 2g + n - \frac{1}{2}\sum_i^{n_+}(k_i^+ + 1) - \sum_j^{n_-}(k_j^- + 1)\right)\right)$$

imposes the relation between integers ('momentum conservation')

$$2 - 2g - (n + n_+ + n_-) + \frac{1}{2}(k^+_{tot} - k^-_{tot}) = 0,$$

n is the number of insertions of the screening operator (# of powers of μ_0).

g is the genus – this term comes from the coupling of φ_0 to the integrated ws curvature $\chi = 2 - 2g - h$.

Combining energy and momentum, (Knizhnik-Polyakov-Zamolodchikov)

$$k^-_{tot} = -k^+_{tot} = 2 - 2g - (n + n_+ + n_-) \qquad \text{(KPZ)}.$$

$\Longrightarrow 1/\mu_0$ is indeed the genus expansion parameter:
Given a certain amplitude determined by k_i, n^+ and n^-,
as we increase g, we decrease n, the power of μ_0.
The phases arising from the complex dilaton conspire to make μ_0^{-1} the string coupling constant.

Comparing to the 't Hooft counting of powers of N,

$$\mu_0 \sim N.$$

Comparison

Dictionary between the closed string modes and the quantum mechanics operators:

$$T^+_{k>0} \Leftrightarrow \mathrm{tr}\ ((a^\dagger)^k), \qquad T^-_{k<0} \Leftrightarrow \mathrm{tr}\ (a^k).$$

Comparison of $1 \to 1$ amplitudes At $g = 0$ the number of powers of μ is

$$n = k$$

$$\Longrightarrow A(k; -k) \sim \mu_0^k \sim N^k.$$

This amplitude does not vanish since (due to the insertions of the screening operators) it involves more than two closed string insertions on the sphere.

Agrees with the harmonic oscillator result.

Solution of puzzle

Two-point amplitudes that involve $T^+_{k<0}$ and $T^-_{k>0}$ vanish. This resolves the puzzle arising from the naive expectation that we should have twice as many stringy modes as excitations of the Fermi surface. just like in the harmonic oscillator, only tachyon wavefunctions with $p_\varphi = -i\partial_\varphi$ negative interact.

In this 'imaginary' version of Liouville theory, there is a sense in which the Seiberg bound becomes the fact that the target-space field is chiral.

We could have failed here:

Another way to satisfy E and M conservation is
$n = 0$ and $k^-_{tot} = k^+_{tot} = 0$.
This can be achieved by having two particles with the same chirality and opposite energy (say $n^+ = 2$ and $n^- = 0$).
This amplitude is zero since it involves only two closed string insertions on S^2.

Stringy exclusion effects

1. As usual in string theory there are corrections from higher-genus worldsheets.

Since $n \geq 0$ we find non-vanishing amplitudes only for

$$g \leq \left[\frac{k}{2}\right],$$

which exactly agrees with the matrix model prediction.

String perturbation theory is very finite in this theory.

2. Given the identification $T_k \Leftrightarrow \mathrm{tr}\, a^k$, the cutoff $k \leq N$ is a UV cutoff on the target space momentum at

$$k \sim \frac{1}{g_s}.$$

A similar phenomenon of a target space lattice with spacing of order g_s has recently been observed in the context of topological strings

Okounkov-Reshetikin-Vafa.

'Three-point functions'

Next $1 \to 2$ scattering amplitudes.

(KPZ) $\qquad k^-_{tot} = -k^+_{tot} = 2 - 2g - (n + n_+ + n_-)$

$\implies$ the sphere contribution to such amplitudes scales like

$$A(k; -k_1, -k_2) \sim \delta_{k,k_1+k_2} \mu_0^{k-1}.$$

Again we find agreement with the harmonic oscillator scaling.

At this level of detail, it seems possible to have $k_1 > 0$ and $k_2 < 0$ such that amplitudes are nonzero.

A closer look at the correlation functions

Back to the SHO to retrieve the factors: The $1 \to 1$ amplitude is

$$\mathcal{O}(k_1; k_2) = \delta_{k_1,k_2} k_1 N^{k_1} \left(1 + \mathcal{O}(N^{-2})\right).$$

There are k_1 planar ways to commute the a's through the $a^\dagger$.

$1 \to 2$ overlap amplitudes in the planar limit:

$$\mathcal{O}(k; k_1, k_2) = \delta_{k,k_1+k_2} k k_1 k_2 N^{k-1}.$$

The simplest way to calculate $1 \to m$ overlap amplitudes

$$\mathcal{O}(k; k_1, k_2, ...k_m) = \langle 0| \mathrm{tr}\ (a^k) \mathrm{tr}\ ((a^\dagger)^{k_1}) \mathrm{tr}\ ((a^\dagger)^{k_2}) ... \mathrm{tr}\ ((a^\dagger)^{k_m}) |0\rangle,$$

is to use the w_∞ algebra m times to move $\mathrm{tr}\ (a^k)$ to the right.

For terms leading in $\frac{1}{N}$, replace the Poisson bracket by a commutator.

$$\mathcal{O}(k; k_1, k_2, ...k_m) = \delta_{k,k_1+k_2+...k_m}$$

$$k(k-1)(k-2)...(k-m+2) k_1 k_2 ... k_m\ N^{k-m+1}$$

Reproduce this in the string theory:

$$A(k;-k_1,...,-k_m)_{\mu_0}=\sum_{n\in \mathrm{KPZ}}\frac{\mu_0^n}{n!}A(k;0,0,...,0,-k_1,-k_2,...,-k_m)_{free}\ ,$$

the 0s indicate the insertion of the screening operator $n=k-m+1$ times.

Calculations by (Polyakov, DiFrancesco-Kutasov, Klebanov) give the free answer.

The most general $1\to r$ amplitude is

$$A(k;-k_1,-k_2,...,-k_m)_{\mu_0}$$

$$=\frac{\pi^k}{(k-m+1)!(k-1)!}\left(\frac{\mu_0\Gamma(1)}{\Gamma(0)}\right)^{k-m+1}\ \prod_{i=1}^{m}\frac{\Gamma(1-k_i)}{\Gamma(k_i)}.$$

WARNING: $0\times\infty$ There are two kinds of zeros.

Important zeros. A vanishes when one of the outgoing momenta, $-k_i$, is positive.

$\Longrightarrow$ The $T^-_{k>0}$ decouple!

Silly zeros. $(1/\Gamma(0))^{k-m+1}$ is from the screening operators.

Even in the usual Liouville theory one needs to renormalize the cosmological constant when $b \to 1$ (*e.g.* Teschner's revisitation)

$\mu_0 \to \infty$ and $b \to 1$ while keeping

$$\mu = \pi \mu_0 \frac{\Gamma(b^2)}{\Gamma(1-b^2)} \text{ fixed.}$$

End result: Replace

$$\mu_0 \frac{\Gamma(1)}{\Gamma(0)} \longrightarrow \frac{\mu}{\pi}.$$

Infinities

Infinities arise from the Γ functions in $\prod_{i=1}^{m} \frac{\Gamma(1-k_i)}{\Gamma(k_i)}$ when $k_i > 0$.
The reason for these infinities is that the amplitudes are sitting on a resonance.

Introduce leg factors to match the two-point function on both sides.

$$f_{\text{ket}}(k_i) = -\frac{1}{\pi}\frac{\Gamma(k_i)}{\Gamma(-k_i)}, \qquad \text{and} \quad f_{\text{bra}}(k_i) = \pi\Gamma(k)\Gamma(k+1)$$

Relation between the string theory scattering amplitudes and the matrix model overlap amplitudes for the $1 \to m$ processes is

$$\mathcal{O}(k; k_1, k_2, \ldots k_m) = f_{\text{bra}}(k) \prod_{i=1}^{m} f_{\text{ket}}(k_i) A(k; -k_1, -k_2, \ldots - k_m)_\mu,$$

with $\mu = N$. Agrees $\forall\ k; k_1 \ldots k_m$.

Note: the coefficients are integers.

Discrete states

The closed-string ghost-number-two cohomology contains other states, known as discrete states, that are the remnants of the massive modes of the string.
In the usual $c = 1$ theory they appear as non-normalizable modes with imaginary energy and imaginary Liouville momentum.
In our case they become normalizable propagating modes that are as important as the tachyon modes discussed above.

Recall: chiral $SU(2)$ current algebra

$$J^{\pm}(z) = e^{\pm i2X(z)}, \qquad J^3(z) = i\partial X(z).$$

The highest weight fields with respect to this algebra are $\Psi_{j,j}(z) = e^{2ijX}$ where $j = 0, 1/2, 1....$ Using $J_0^- = \oint \frac{dw}{2\pi i} e^{-2iX(w)}$, form representations of the $SU(2)$ algebra

$$\Psi_{j,m}(z) \sim (J_0^-)^{j-m}\Psi_{j,j}(z), \qquad m = -j, -j+1, ..., j.$$

The resulting closed string vertex ops (with dimension $(1,1)$ and ghost number two) are

$$S_{j,m}(z,\bar{z}) = Y_{j,m}\bar{Y}_{j,m}, \qquad Y_{j,m} = c\Psi_{j,m}e^{i2(1-j)\varphi}.$$

The $S_{j,j}$'s are the tachyon vertex operators that we have already discussed.

The rest are new modes that are called discrete states.

Here this name is a bit misleading since the tachyon modes are discrete too.

The energy in the X direction of $S_{j,m}$ is $2m$

The momentum in the φ direction is $2j$.

For *e.g.* the $1 \to 1$ amplitude we get,

$$2m^{+} + 2m^{-} = 0, \qquad 2 - 2g - (n+1+1) + \frac{1}{2}(2j^{+} + 2j^{-}) = 0,$$

and so the sphere amplitude scales like

$$A(m^{-}, j^{-}; m^{+}, j^{+}) \sim \delta_{m^{+}+m^{-}} \ \mu_0^{j^{+}+j^{-}}.$$

On the quantum mechanics side we can follow the same steps. The $SU(2)$ generators are

$$J^+ = \frac{1}{2}\text{tr }((a^\dagger)^2), \qquad J^- = \frac{1}{2}\text{tr }(a^2), \qquad J^3 = \frac{1}{4}\text{tr }(aa^\dagger + a^\dagger a).$$

The highest-weight states with respect to this $SU(2)$ algebra are $\gamma_{j,j} = c_j \text{ tr }((a^\dagger)^{2j})$, where $j = 0, 1/2, 1, ...$
Again we can form a representation of the algebra by commuting $j - m$ times $\gamma_{j,j}$ with J^- to get $\gamma_{j,m}$.

$$\gamma_{j,m} \sim \text{tr }(a^{j-m}(a^\dagger)^{j+m}).$$

These operators satisfy a commutator algebra which at large N approaches the semi-classical w_∞ algebra.

Note: even at finite N, the algebra closes, using the gauge equivalence.

I don't know the structure constants, though.

The two-point function associated with the discrete states takes the form

$$\mathcal{O}(m^-, j^-; m^+, j^+) \sim \langle 0|\mathrm{tr}\ (a^{j^- + m^-}(a^\dagger)^{j^- - m^-})\mathrm{tr}\ (a^{j^+ - m^+}(a^\dagger)^{j^+ + m^+})|0\rangle$$

$$\sim \delta_{m^+ + m^-}\ N^{j^+ + j^-},$$

again in agreement with the string theory result.

Winding modes

Since φ is compactified, modular invariance implies that there must be winding modes in the theory.

But there are no candidate dual operators for the winding modes in the quantum mechanics. The same puzzle as with the momentum modes with opposite chirality. In that case the resolution was that scattering amplitudes with at least one opposite-momentum mode vanish.

Consider, for example, scattering amplitudes of m winding modes.

The number of screening insertions is determined by $2 - 2g - (n + m) = 0$, where again n is the number of insertions of the screening operator.

No amplitude with $m > 2$ can satisfy this relation.

For $m = 1$ and $m = 2$ the relation can be satisfied on the sphere with $n = 1$ and $n = 0$ respectively. But both cases vanish since they involve only two insertions of closed strings on the sphere.

Beyond tree level

Torus vacuum energy

On the string theory side the ground state energy is the expectation value of the zero momentum graviton, $S_{1,0}$.

The energy conservation condition is automatic (since $m = 0$) and the φ momentum conservation gives $n = 2 - 2g$.

$\longrightarrow$ Two nonzero perturbative contributions:

$g = 0, n = 2$ scales like N^2.

$g = 1, n = 0$ second comes from the torus with $n = 0$, $\sim N^0$.

On the quantum mechanics side the ground state energy is $N^2/2$ with no additional constant that scales like N^0.

This suggests that the one-loop vacuum energy in this string theory should be zero.

A simpler way to compute the ground state energy is to calculate the one particle irreducible contribution to the partition function with X compactified, $X \sim X + \beta$.

In the limit $\beta \to \infty$ we have

$$\ln Z = -\beta E_0,$$

from which we can read off E_0.

On the torus, the background charge is zero. $\Longrightarrow$

As far as this calculation is concerned X and φ are two free scalar fields X_1, X_2.

The result of the calculation with $X_1 \sim X_1 + 2\pi V_1$, and $X_2 \sim X_2 + 2\pi R$ is (Bershadsky-Klebanov)

$$\lim_{V_1 \to \infty} \frac{Z_{torus}}{V_1} = \frac{1}{12\sqrt{2}} \left(\frac{R}{\sqrt{\alpha'}} + \frac{\sqrt{\alpha'}}{R} \right), \quad (\star)$$

This calculation is for $V_1 \gg \ell_s$, and euclidean $X_{1,2}$.

For BK, $R =$ temperature, $V_1 = V_L$ was an IR cutoff on the Liouville direction.

For us, $V_1 = \beta$ is the temperature^{-1} and R is the imaginary liouville radius.

$$\lim_{\beta\to\infty} \frac{Z_{torus}}{\beta} = -\frac{i}{12\sqrt{2}} \left(\frac{R}{\sqrt{\alpha'}} - \frac{\sqrt{\alpha'}}{R} \right).$$

The i and the minus sign come from the fact that φ is a timelike direction.

Another way to see this is to take $\alpha' \to -\alpha'$ in $(\star)$.

Acting on a time-like direction, T-duality takes R to $-\alpha'/R$, in order that the partition function be invariant. $R = \sqrt{\alpha'} \quad \Longrightarrow$

$$\lim_{\beta\to\infty} \frac{Z_{torus}}{\beta} = 0,$$

as predicted by the duality with the quantum mechanics.

This zero obtained by summing over contributions of all of the physical states of the string.

In addition to the chiral momentum modes with nonzero S-matrix amplitudes, these include winding modes and momentum modes with the opposite chirality that, we argued, do not appear on external legs.

Hints for nonperturbative effects

$$\mathcal{A}(k) = \sum_{g=0}^{g_{\max}} \mathcal{A}_g(k) g_s^{2g-2}$$

is very convergent.

No D-instantons contribute to vacuum correlators.

Consider the matrix model at finite temperature.

$$F(q,N) = \ln Z_N(q) = \frac{1}{2}\beta N^2 - \sum_{n=1}^{N} \ln(1-q^n), \qquad q = e^{-\beta}.$$

$$F\left(q,N\right) = \frac{\beta N^2}{2} + f(\beta) + \sum_{i=1}^{\infty} C_w \left(e^{-w\beta N}\right), \qquad f(\beta) = \sum_{w=1}^{\infty} \ln(1-q^n).$$

The first term we recognize as the ground state energy.

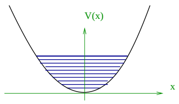
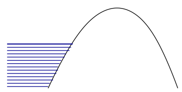
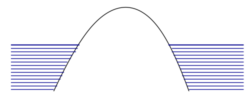
The second term is a prediction for the leading torus contribution to the free energy when β is finite.

There are no further perturbative corrections in accord with the scaling rule. There are, however, non-perturbative effects. The fact that these scale like $e^{-w\beta N}$ implies that there are D-particles in the spectrum which contribute to the partition sum when their worldlines wrap the thermal circle.

The fact that there are no terms of the form e^{-N} implies that, as was argued above, there are no D-instantons.

The D-particles in this string theory should should be related to the the matrix eigenvalues, and should be closely related to the 'instantonic' branes of the usual $c = 1$ theory which experience trajectories of the form $z = \tilde{\lambda} \cos t$.

How does this duality fit into the rest of string theory?

1.  $\xrightarrow{\varphi \to i\varphi}$  OR  ?

Claim: there is also a 0B theory with the same perturbative description.

Compactify φ at the "superaffine" self-dual radius.
$\longrightarrow$ every degenerate representation appears.
even momenta from NSNS, odd momenta from RR.
same ground ring (Douglas et al, "New Hat...").

$\longrightarrow$ RR backgrounds?

2. There is likely to be a topological string realization of this matrix model.
$\tau_{\text{Toda}}(t_k)$ generates correlators.

3. Higher-dimensional generalizations

I've tried not to focus on free fermions and w_∞ because I don't know how they generalize to higher dimensions. But:

$$H = \sum_{\alpha=1}^{L} \mathrm{tr}\ \left(a_\alpha^\dagger a_\alpha - \frac{1}{2} \right)$$

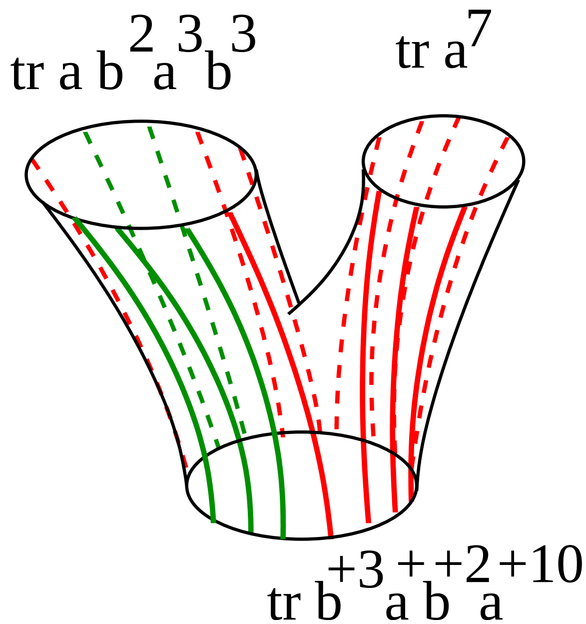
a. are also solvable

b. should also be string theories. (no need to find critical behavior $\Longrightarrow$ no in-principle $c = 1$ barrier)

c. have single-string hagedorn spectra for any $L > 1$ tr $a^2 b a^4 a^2 \ldots$

Think of α as indexing KK modes.

We can try to work our way up to higher-dimensional gauge theories:



THE END

Matrix CS

This matrix model is the $q = 0$ version of the Matrix Chern-Simons theory (Polychronakos, hep-th/0103...):

Add fundamental bosons ϕ, and 1d CS term $k \oint A_0$.

Gauss law:

$$[a, a^\dagger]_i^j - \phi^j \phi_i^\dagger = \delta_i^j q$$

At finite k there is a condensate of fundamentals: $\Longrightarrow$ boundaries

$$\phi^\dagger \cdot \phi = Nq$$

Conjecture: This quantized number q is RR flux $F_{RR}^{(1)} = q d\varphi$

$$2\pi q = \oint F_{RR}^{(1)}$$

turn on 't Hooft couplings

$$\delta S = \int dt \sum_k t_k \ \mathrm{tr} \ X^k(t)$$

still free fermions.

We know what operator tr X^k is in the string theory.

Normal matrix integral representation

A normal matrix model (NMM) is an integral over complex matrices Z such that $[Z, Z^\dagger] = 0$.
$\Longrightarrow Z, Z^\dagger$ are simultaneously diagonalizable.

Consider studied in relation with $c = 1$ (Alexandrov-Kazakov-Kostov)

$$\mathcal{Z}_{\mathrm{NMM}}(J, J^\dagger) = \int_{[Z,Z^\dagger]=0} d^{N^2}Z d^{N^2}Z^\dagger \; e^{-\mathrm{tr}\;(Z^\dagger Z + Z J^\dagger + Z^\dagger J)}$$

The normal-matrix constraint in the path integral can be imposed with a Lagrange multiplier matrix A,

$$\delta^{N^2}\left(i[Z, Z^\dagger]\right) = \int dA \; e^{\mathrm{tr}\;[Z,Z^\dagger]A}$$

$$\Longrightarrow \mathcal{Z}_{\mathrm{NMM}}(J, J^\dagger) = \int d^{N^2}A d^{N^2}Z d^{N^2}Z^\dagger \; e^{-\mathrm{tr}\;(Z^\dagger Z + [Z,Z^\dagger]A + Z J^\dagger + Z^\dagger J)}$$

In blue is the Hamiltonian of the gauged harmonic oscillator.

Comparing Wick contractions,

$$\prod_i \mathrm{tr}\, \left(\partial_J^{k_i} \partial_{J^\dagger}^{p_i}\right) |_{J=J^\dagger=0} \ln \mathcal{Z}_{\mathrm{NMM}}(J, J^\dagger) = \langle \mathbf{T} \left(\prod_i \mathrm{tr}\; a^{\dagger\ k_i} a^{p_i} \right) \rangle$$

$\mathbf{T}$ indicates a certain ordering prescription.

A way to think about this relationship: Wigner phase-space integral representation of expectation values in one-dimensional quantum mechanics:

$$\langle \psi | \hat{\mathcal{O}} | \psi \rangle = \int dx dp\; W_{\mathcal{O}}(x, p) W_\psi^\star(x, p)$$

where

$$W_{\mathcal{O}}(x, p) \equiv \int dy\; e^{ipy} \langle x + \frac{y}{2} | \hat{\mathcal{O}} | x - \frac{y}{2} \rangle$$

and the Wigner function of the state ψ is

$$W_\psi(x, p) \equiv W_{|\psi\rangle\langle\psi|} = \int dy\; e^{ipy} \psi^\star \left(x + \frac{y}{2}\right) \psi \left(x - \frac{y}{2}\right)$$

For the SHO, the ground state wave function is

$$\psi(x) = \psi_0(x) = e^{-x^2/2},$$

$$\langle\psi_0|\hat{\mathcal{O}}|\psi_0\rangle = \int dxdp\ e^{-H(x,p)}\ W_{\mathcal{O}}(x,p).$$

This quantum expectation value is equal to a classical statistical average at inverse temperature $\beta = 1$, the resonant frequency.

The generalization to $U(N)$ matrix quantum mechanics is $(\star)$.

The normal matrix constraint is the gauss law condition.

T above is Weyl ordering.

How the discrete states act on the vacuum

h_{nm} determines a vector field that is associated with an infinitesimal area-preserving transformation of the U-V plane,

$$\vec{B}_{nm} = \frac{\partial h_{nm}}{\partial U} \partial_V - \frac{\partial h_{nm}}{\partial V} \partial_U .$$

The Lie brackets of the $\vec{B}_{nm}$'s form a w_∞ algebra.

The vector field $\vec{B}_{nm}$ clearly depends both on n and m.
But when acting on the ground state, $\vec{B}_{nm}$ depends only on $|s| = |n-m|$.

h_{nm} generates the following infinitesimal deformation

$$\delta V = \epsilon \partial_U h = \epsilon n U^{n-1} V^m, \quad \delta U = -\epsilon \partial_V h = -\epsilon m U^n V^{m-1}.$$

To leading order in ϵ,

$$(V + \delta V)(U + \delta U) = UV + \epsilon (n-m) U^n V^m.$$

The associated deformation of the Fermi surface is

$$\delta(UV) \sim \epsilon \left(\frac{U}{V}\right)^{s/2}.$$

In polar coordinates $U = re^{i\theta}, \quad V = re^{-i\theta}$, the variation in the fermi level is

$$\delta(\theta) \sim \epsilon \mathrm{Re}\,(e^{is\theta}) = \epsilon \cos((n-m)\theta).$$

If we act more than once with $\vec{B}_{nm}$ on the ground state then both s and r matter

e.g. both 1 and tr $aa^\dagger$ act trivially on the ground state

only 1 acts trivially on every state.

Characters

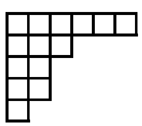
An orthonormal basis of states can be constructed as

$$|R(\{m_i\})\rangle = c_{\{m_i\}} \; \chi_{R(\{m_i\})}(a^\dagger)|0\rangle$$

(Corley-Jevicki-Ramgoolam, Berenstein) $R(\{m_i\})$ is the representation of $U(N)$ corresponding to the Young tableau with columns of lengths $(m_1, m_2, ..., m_r)$

$\chi_R(U)$ is the character of U in this representation.

The bound $m_i \leq N$ guarantees that this is indeed a tableau for a $U(N)$ representation.



$$\{m_i\} = \{5, 4, 2, 1, 1, 1\}, \; \{k_n\} = \{6, 3, 2, 2, 1\}, \; N \geq 5$$

In terms of the free-fermion description, the wavefunction for the state is a Slater determinant – Look at the tableau $\{m_i\}$ sideways, define k_n to be the row-lengths; note that there are at most N rows.

The many-fermion wavefunction is then

$$\langle z_1| \otimes \langle z_2| \cdots \langle z_N|R(\{m_i\})\rangle = \det_{n,l=1,..,N} \psi_{k_n+N-n+1}(z_l)$$

$\psi_n(z) = \langle n|z\rangle = H_n(z)e^{-|z|^2/2}$ are the single-particle harmonic-oscillator wavefunctions. *e.g.* for the empty tableau, this fills the lowest N energy levels with fermions.

The state tr $(a^\dagger)^k|0\rangle$ corresponds to exciting k fermions by one level each – it makes a hole in the fermi sea at level $N-k$.

For $m = N$, the hole is at the lowest state.

For $m > N$, there is no such single-particle description, in accord with the stringy exclusion principle.