University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics C (Physics 130C) Winter 2015
Assignment 2

Posted January 14, 2015 **Due 11am Thursday, January 22, 2015**

Please remember to put your name at the top of your homework.

Go look at the Physics 130C web site; there might be something new and interesting there.

Reading: Preskill's Quantum Information Notes, Chapter 2.1, 2.2.

1. **More linear algebra exercises.**

   (a) Show that an operator with matrix representation
   $$P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
   is a projector.

   (b) Show that a projector with no kernel is the identity operator.

2. **Complete sets of commuting operators.** In the orthonormal basis $\{|n\rangle\}_{n=1,2,3}$, the Hermitian operators $\hat{A}$ and $\hat{B}$ are represented by the matrices A and B:
   $$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \mathbf{i}b & 0 \\ -\mathbf{i}b & 0 & 0 \\ 0 & 0 & b \end{pmatrix},$$
   with a, b real.

   (a) Determine the eigenvalues of $\hat{B}$. Indicate whether its spectrum is degenerate or not.

   (b) Check that A and B commute. Use this to show that $\hat{A}$ and $\hat{B}$ do so also.

   (c) Find an orthonormal basis of eigenvectors common to A and B (and thus to $\hat{A}$ and $\hat{B}$) and specify the eigenvalues for each eigenvector.

   (d) Which of the following six sets form a complete set of commuting operators for this Hilbert space? (Recall that a complete set of commuting operators allow us to specify an orthonormal basis by their eigenvalues.)
   $$\{\hat{A}\}; \quad \{\hat{B}\}; \quad \{\hat{A}, \hat{B}\}; \quad \{\hat{A}^2, \hat{B}\}; \quad \{\hat{A}, \hat{B}^2\}; \quad \{\hat{A}^2, \hat{B}^2\}.$$

3. A *positive* operator is one whose eigenvalues are all positive. Show that sum of positive hermitian operators is positive, even if they don't commute.

4. **Measurement and time evolution.**

   Consider a quantum system governed by a Hamiltonian $\hat{H}$:

   $$\hat{H}|n\rangle = E_n|n\rangle, \quad n = 0, 1, 2, \ldots,$$

   with $\langle n|m\rangle = \delta_{nm}$ and non-degenerate energy levels. In addition, $\hat{A}$ is an observable of this system such that

   $$\hat{A}|a_\alpha\rangle = a_\alpha|a_\alpha\rangle, \quad \alpha = 0, 1, 2, \ldots,$$

   with $\langle a_\alpha|a_\beta\rangle = \delta_{\alpha\beta}$; the a_α are also non-degenerate. Finally, $[\hat{A}, \hat{H}] \neq 0$.

   (a) Prior to measurement of $\hat{A}$, the system is in state $|n\rangle$. What is the probability $P_n(a_\alpha)$ that measurement of $\hat{A}$ will yield a_α?

   (b) If the preceding measurement were carried out at time $t = 0$ and yielded the result a_α, what is the state of the system immediately following the measurement, *i.e.* what is $|\psi(t = 0)\rangle$?

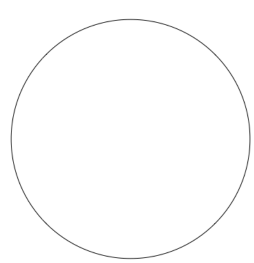
   (c) Following the $t = 0$ measurement of $\hat{A}$ that yielded a_α, $\hat{A}$ is again measured at $t > 0$. What is the probability $P(a_\alpha, t)$ that the value a_α will again be found? Express your answer in terms of the probabilities $P_n(a_\alpha)$.

   (d) Now let $\hat{H}$ be the Hamiltonian for the 1-dimensional harmonic oscillator with natural frequency ω_0, and set $t = 2\pi/\omega_0$. What is the numerical value of $P(a_\alpha, 2\pi/\omega_0)$? Note: None of your answers should contain $\hat{A}$ or $\hat{H}$.

The following three problems form a triptych, on the subject of resolving the various infinities involved in the quantum mechanics of a particle on the real line. There are two such infinities: one is the fact that the real line goes on forever; this is resolved in problem 5. The other is the fact that in between any two points there are infinitely many points; this is resolved in problem 6. In problem 7 we resolve both to get a finite-dimensional Hilbert space.

5. **Particle on a circle.**

   Consider a particle which lives on a circle:



That is, its coordinate x takes values in $[0, 2\pi R]$ and we identify $x \simeq x + 2\pi R$. The operator $\hat{p}$ generates translations of the particle's position, just as in the case of the particle on the line.

(a) Let's assume that the wavefunction of the particle is periodic in x:

$$\psi(x + 2\pi R) = \psi(x) \ .$$

What set of values can its momentum (that is, eigenvalues of the operator $\hat{p} = -i\hbar\partial_x$) take?

[Hint: demand that the translation operator $\hat{T}(2\pi R) = e^{\mathbf{i}2\pi R\hat{p}}$ acts like the identity operator.]

(b) Recall that the overall phase of the state vector is not physical data. This suggests the possibility that the wavefunction might not be periodic, but instead might acquire a phase when we go around the circle:

$$\psi(x + 2\pi R) = e^{i\varphi}\psi(x)$$

for some fixed φ. In this case what values does the momentum take?

6. **Particle on a lattice.**

Now consider a particle which lives on a lattice: its position can take only the discrete values $x = na, n \in \mathbb{Z}$ where a is some unit of length and n is an integer. We'll call the corresponding position eigenstates $|n\rangle$. The Hilbert space is still infinite-dimensional, but at least it's countably infinite.

Now only translation operators $\hat{T}^n = e^{\mathbf{i}\hat{p}na}$ which translate the particle by integer multiples of a map the Hilbert space to itself. In this problem we will determine: what is the spectrum of the momentum operator $\hat{p}$ in this system?

(a) Consider the state

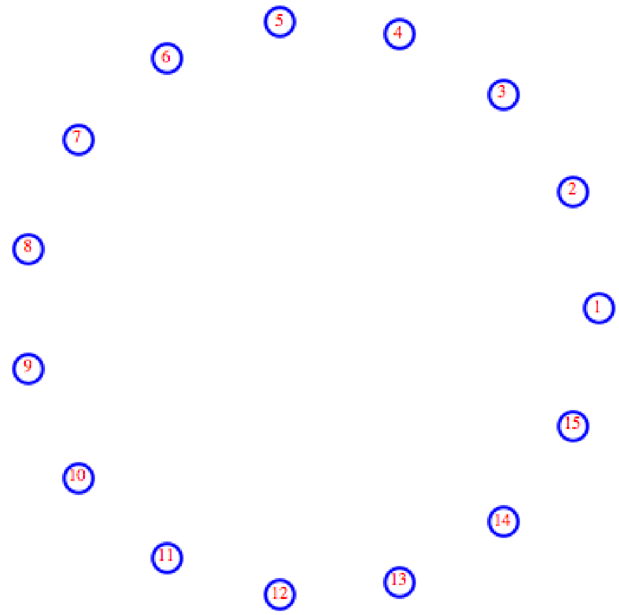
$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n\in\mathbb{Z}} e^{in\theta}|n\rangle.$$

Show that $|\theta\rangle$ is an eigenstate of $\hat{T}$. Why do I want to call θ momentum?

(b) What range of values of θ give different states $|\theta\rangle$? [Recall that n is an integer.]

7. **Discrete Laplacian.**

Consider again a particle which lives on a lattice, but now we'll wrap the lattice around a circle, in the following sense. Its position can take only the discrete values $x = a, 2a, 3a, ..., Na$ (where, again, a is some unit of length and again we'll call the corresponding position eigenstates $|n\rangle$). Suppose further that the particle lives on a circle, so that the site labelled $x = (N + 1)a$ is the same as the site labelled $x = a$. We can visualize this as in the figure:



In this case, the Hilbert space has finite dimension (N).

Consider the following $N \times N$ matrix representation of a Hamiltonian operator (a is a constant):

$$H = \frac{1}{a^2}\left(\left.\underbrace{\begin{matrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{matrix}}_{N}\right\} N\right)$$

(a) Convince yourself that this is equivalent to the following: Acting on an N-dimensional Hilbert space with orthonormal basis $\{|n\rangle, n = 1, \ldots, N\}$, $\hat{H}$ acts by

$$a^2\hat{H}|n\rangle = 2|n\rangle - |n+1\rangle - |n-1\rangle, \qquad \text{with } |N+1\rangle \simeq |1\rangle$$

that is, we consider the arguments of the ket to be integers modulo N.

(b) What are the symmetries of this system?
[Hint: what is $[\hat{H}, \hat{T}]$ where $\hat{T}$ is the 'shift operator' defined by $\hat{T} : |n\rangle \mapsto |n+1\rangle$?]

Consider again the state

$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta}|n\rangle.$$

(c) Show that $|\theta\rangle$ is an eigenstate of $\hat{T}$, for values of θ that are consistent with the periodicity $n \simeq n + N$.

(d) What values of θ give different states $|\theta\rangle$? [Recall that n is an integer.]

(e) Find the matrix elements of the unitary operator $\mathbf{U}$ which relates position eigenstates $|n\rangle$ to momentum eigenstates $|\theta\rangle$: $U_{\theta n} \equiv \langle n|\theta\rangle$.

(f) Find the spectrum of $\hat{H}$.

Draw a picture of $\epsilon(\theta)$: plot the energy eigenvalues versus the 'momentum' θ.

(g) Show that the matrix above is an approximation to (minus) the 1-dimensional Laplacian $-\partial_x^2$. That is, show (using Taylor's theorem) that

$$a^2 \partial_x^2 f(x) = -2f(x) + (f(x+a) + f(x-a)) + \mathcal{O}(a)$$

(where "$\mathcal{O}(a)$" denotes terms proportional to the small quantity a).

(h) In the expression for the Hamiltonian, to restore units, I should have written:

$$\hat{H}|n\rangle = \frac{\hbar^2}{2m} \frac{1}{a^2} \left(2|n\rangle - |n+1\rangle - |n-1\rangle\right), \qquad \text{with } |N+1\rangle \simeq |1\rangle$$

where a is the distance between the sites, and m is the mass. Consider the limit where $a \to 0, N \to \infty$ and look at the lowest-energy states (near $p = 0$); show that we get the spectrum of a free particle on the line, $\epsilon = \frac{p^2}{2m}$.