University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics C (Physics 130C) Winter 2015
Assignment 3

Posted January 20, 2015 **Due 11am Thursday, January 29, 2015**

Please remember to put your name at the top of your homework.

Note that this problem set has been shortened relative to its initial version.

Reading: Reading: On two-state systems: Preskill §2.2, Schumacher §2.1. On composite quantum systems: Preskill §2.3, Le Bellac Chapter 6.

1. **Pauli spin matrix gymnastics.**

   (This problem is long but each part is pretty simple.) Recall the definition of the Pauli spin matrices:
   $$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
   (Occasionally I will write instead $\boldsymbol{\sigma}^x \equiv \boldsymbol{\sigma}^1 \equiv \mathbf{X}, \boldsymbol{\sigma}^y \equiv \boldsymbol{\sigma}^2 \equiv \mathbf{Y}, \boldsymbol{\sigma}^z \equiv \boldsymbol{\sigma}^3 \equiv \mathbf{Z}$.)

   (a) Show that the $\boldsymbol{\sigma}^i$ are Hermitian.

   (b) Find their eigenvalues and eigenvectors.

   (c) There are only so many two-by-two matrices. A product of sigmas can be written in terms of sigmas. Show that
   $$\boldsymbol{\sigma}^i \boldsymbol{\sigma}^j = \mathbf{i}\epsilon^{ijk}\boldsymbol{\sigma}^k + \delta^{ij}\mathbb{1} \tag{1}$$
   where ϵ^{ijk} is the completely antisymmetric object with $\epsilon^{123} = 1$ (that is: $\epsilon^{ijk} = 0$ if any of ijk are the same, $= 1$ if ijk is a cyclic permutation of 123 and $= -1$ if ijk is an odd permutation of 123, like 132)[1]. You may prefer to do the next two parts of the problem first.

   (d) Convince yourself that (1) implies
   $$[\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j] = 2\mathbf{i}\epsilon^{ijk}\boldsymbol{\sigma}^k,$$
   and that therefore $\mathbf{J}^i_{\frac{1}{2}} \equiv \frac{1}{2}\boldsymbol{\sigma}^i$ satisfy
   $$[\mathbf{J}^i_{\frac{1}{2}}, \mathbf{J}^j_{\frac{1}{2}}] = \mathbf{i}\epsilon^{ijk}\mathbf{J}^k_{\frac{1}{2}},$$

[1] Note the use of the Einstein summation convention here: the index k in equation (1) is summed.

the same algebra as the rotation generators on the 3-state system in §1.5 of the lecture notes. [Cultural note: this is the Lie algebra called $\mathbf{su}(2)$.]

(e) Convince yourself that (1) implies

$$\{\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j\} = 2\delta^{ij}$$

where $\{A, B\} \equiv AB + BA$ is called the *anti-commutator.*

[Cultural note: this is called the Dirac algebra or Clifford algebra.]

It may be useful to note that $\{A, B\} + [A, B] = 2AB$.

(f) Convince yourself that (1) is the same as

$$(\vec{\boldsymbol{\sigma}} \cdot \vec{a})(\vec{\boldsymbol{\sigma}} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + \mathbf{i}\vec{\boldsymbol{\sigma}} \cdot \left(\vec{a} \times \vec{b}\right) .$$

In particular, check that $(\vec{\boldsymbol{\sigma}} \cdot \hat{n})^2 = \mathbb{1}$ if $\hat{n}$ is a unit vector.

(g) Show that

$$e^{i\frac{\theta}{2}\vec{\boldsymbol{\sigma}}\cdot\hat{n}} = \mathbb{1}\cos\frac{\theta}{2} + \mathbf{i}\vec{\boldsymbol{\sigma}} \cdot \hat{n}\sin\frac{\theta}{2}$$

where $\hat{n}$ is a unit vector.

[Hint: use the Taylor expansion of the LHS (left-hand side): $e^A = 1 + A + A^2/2! + \ldots$ and the previous results for $(\vec{\boldsymbol{\sigma}} \cdot \hat{n})^2$.]

2. **Time evolution of a two-level system.** Le Bellac, Problem 4.4.4, parts 1-3.

Here's a summary, in case you don't have the book: Consider the Hamiltonian

$$H = \hbar \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

in the basis $|+\rangle \equiv |\uparrow\rangle, |-\rangle \equiv |\downarrow\rangle$, where A, B are real numbers. Set $\hbar = 1$. You know the eigensystem of this matrix from our discussion of Pauli gymnastics. It is useful to write it in terms of Ω, θ defined by

$$\Omega = 2\sqrt{A^2 + B^2}, \quad \tan\theta = \frac{B}{A} .$$

1. The state vector at time t can be written as

$$|\varphi(t)\rangle = c_+(t)|+\rangle + c_-(t)|-\rangle.$$

Find a system of ODEs for the components $c_\pm(t)$ that follow from the Schrödinger equation.

2. Decompose the initial state $|\varphi(t = 0)\rangle$ in the energy eigenbasis ($H|\epsilon_\pm\rangle = \epsilon_\pm|\epsilon_\pm\rangle$):

$$|\varphi(t = 0)\rangle = \sum_\pm a_\pm|\epsilon_\pm\rangle .$$

Express $c_+(t)$ in terms of the coefficients $a_\pm$ (Le Bellac calls them $a_+ = \lambda, a_- = \mu$.)

3. Suppose $c_+(0) = 0$. Find $a_\pm$ (up to a phase) in terms of A, B (or rather, Ω and θ). Find the probability of finding the system in the state $|\uparrow\rangle$ at time t.

4. You can skip this part if you prefer.

3. **Rotations of a spin-$\frac{1}{2}$.**

   Imagine you have under your control a spin-$\frac{1}{2}$ particle. By this I mean that the particle is fixed in space in your lab (so you can ignore the physics of its position degrees of freedom for this problem), and you have a machine by which you can rotate its spin by an arbitrary angle about an arbitrary axis, and measure its spin about the $\check{z}$ axis.

   Suppose at $t = 0$ you measure the spin to be up along the $\check{z}$ direction. You then apply a rotation by 90 degrees about the $\check{x}$ axis, $\mathbf{U}(\check{x}, \pi/2)$. What is the resulting state of the spin? What is the probability of measuring up along the $\check{z}$ axis now?

4. **A Mach-Zehnder interferometer.**

   Schumacher Exercise 2.7, page 23. (Electronic version here.)

5. **Maximally entangled spins.**

   Consider the following state in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ of two qbits A and B:

$$\sqrt{2}|\text{Bohm}\rangle = |\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \equiv |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle.$$

   Show that this state is a *singlet* of the total spin operator:

$$\vec{\boldsymbol{\sigma}}_T \equiv \vec{\boldsymbol{\sigma}}_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \vec{\boldsymbol{\sigma}}_B \ ,$$

   [2] meaning that

$$\vec{\boldsymbol{\sigma}}_T |\text{Bohm}\rangle = 0 \ .$$

   Notice that this means that in this Bohm state, a measurement of any component of $\boldsymbol{\sigma}_A$ always gives the minus the measurement of the same component of $\boldsymbol{\sigma}_B$, whichever component we choose to measure.

[2] We will sometimes abbreviate expressions like this as $\vec{\boldsymbol{\sigma}}_T = \vec{\boldsymbol{\sigma}}_A + \vec{\boldsymbol{\sigma}}_B$.