

Quantum Mechanics C (Physics 130C) Winter 2015 Assignment 4

Posted January 27, 2015

Due 11am Thursday, February 5, 2015

1. Projector onto up along \check{n} .

Show that the projector onto the state of a qbit $|\uparrow_{\check{n}}\rangle$ with spin up along an arbitrary unit vector \check{n}

$$\check{n} \cdot \boldsymbol{\sigma} |\uparrow_{\check{n}}\rangle = |\uparrow_{\check{n}}\rangle$$

can be written as

$$|\uparrow_{\check{n}}\rangle\langle\uparrow_{\check{n}}| = \frac{1}{2}(1 + \check{n} \cdot \boldsymbol{\sigma})$$

That is: check that the operator on the RHS is hermitian, squares to itself, and acts correctly on a basis.

2. The EPR state. [From Preskill.] [This problem is extra credit.]

This is the example considered in the paper by Einstein, Podolsky and Rosen in their famous paper pointing out that quantum mechanics is weird. Consider the Hilbert space of two (distinguishable) particles on a line,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = \text{span}\{|x_1\rangle \otimes |x_2\rangle, x_1, x_2 \in \mathbb{R}\} .$$

Define relative and center-of-mass coordinates and momenta by:

$$\mathbf{Q} = \mathbf{x}_1 + \mathbf{x}_2, \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \mathbf{q} = \mathbf{x}_1 - \mathbf{x}_2, \mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2.$$

[To be completely explicit: here \mathbf{x}_1 means $\mathbf{x}_1 \otimes \mathbb{1}_2$ and \mathbf{p}_2 means $\mathbb{1}_1 \otimes \mathbf{p}_2$ and so on. So *e.g.* $\mathbf{x}_2|x\rangle \otimes |x+q\rangle = (x+q)|x\rangle \otimes |x+q\rangle$ and $\mathbf{p}_1|x_1\rangle \otimes |x_2\rangle = -i\hbar\partial_{x_1}|x_1\rangle \otimes |x_2\rangle$.]

(a) Show that $[\mathbf{P}, \mathbf{q}] = 0$.

(b) Consider the state

$$|q, P\rangle \equiv \frac{1}{\sqrt{2\pi}} \int dx e^{iPx} |x\rangle \otimes |x+q\rangle ;$$

Show that this is an eigenstate of both \mathbf{P} and \mathbf{q} .

(c) Show that

$$\langle q', P' | q, P \rangle = \delta(q - q') \delta(P - P') .$$

(d) The states $\{|q, P\rangle\}$ form a basis for \mathcal{H} . We can expand a position eigenstate as

$$|x_1\rangle \otimes |x_2\rangle = \int dq dP |q, P\rangle \langle q, P| (|x_1\rangle \otimes |x_2\rangle).$$

Evaluate the coefficients $\langle q, P| (|x_1\rangle \otimes |x_2\rangle)$.

3. **A density matrix is a probability distribution on quantum states.** [from A. Manohar]

An electron gun produces electrons randomly polarized with spins up or down along one of the three possible randomly selected orthogonal axes 1, 2, 3 (i.e. x, y, z), with probabilities

$$p_{i,\uparrow} = \frac{1}{2}d_i + \frac{1}{2}\delta_i \quad p_{i,\downarrow} = \frac{1}{2}d_i - \frac{1}{2}\delta_i \quad i = 1, 2, 3 \quad (1)$$

Probabilities must be non-negative, so $d_i \geq 0$ and $|\delta_i| \leq d_i$.

- (a) Write down the resultant density matrix ρ describing our knowledge of the electron spin, in the basis $|\uparrow\rangle, |\downarrow\rangle$ with respect to the z axis.
- (b) Any 2×2 matrix ρ can be written as

$$\rho = a \mathbf{1} + \mathbf{b} \cdot \boldsymbol{\sigma} \quad (2)$$

in terms of the unit matrix and 3 Pauli matrices. Determine a and \mathbf{b} for ρ from part (a).

- (c) A second electron gun produces electrons with spins up or down along a single axis in the direction $\hat{\mathbf{n}}$ with probabilities $(1 \pm \Delta)/2$. Find $\hat{\mathbf{n}}$ and Δ so that the electron ensemble produced by the second gun is the same as that produced by the first gun.

4. **Measures of entanglement**

In the problems below, consider a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with the factors not necessarily of the same dimension. Consider a state of this system

$$|a\rangle = \sum_{i=1}^N \sum_{r=1}^M a_{ir} |i\rangle_A \otimes |r\rangle_B.$$

- (a) We defined the *Schmidt number* of the state $|a\rangle$ to be the rank of the matrix a_{ir} . Show that this is the same as the number of nonzero eigenvalues of

$$\rho_A = \text{tr}_B |a\rangle \langle a|.$$

Show that it's *also* the same as the number of nonzero eigenvalues of

$$\rho_B = \text{tr}_A |a\rangle\langle a| .$$

[Hint: To solve this problem, it's useful to take advantage of our ability to change basis. In particular, the problem does not depend on what basis we choose for \mathcal{H}_A and \mathcal{H}_B . If w were a Hermitian matrix, we could find a basis where it was diagonal, by using its eigenvectors as the basis elements. w is not Hermitian (and not even square in general), but there is still something we can do: such a matrix has right eigenvectors (elements of \mathcal{H}_B) and left eigenvectors (elements of \mathcal{H}_A) and they can be used to choose basis cleverly. In particular, any matrix has a *singular value decomposition* (SVD) of the form

$$w = U^T \Lambda V \quad w_{ir} = U_{ij}^T \Lambda_{js} V_{sr} .$$

where U and V are unitary (basis transformations) and Λ is a diagonal matrix (the entries are the 'singular values', which are in fact real and positive). Notice that Λ is not square. It looks like

$$\Lambda_{js} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{js} \quad \text{if } N > M \quad \text{or} \quad \Lambda_{js} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \end{pmatrix}_{js} \quad \text{if } M > N .$$

(In the examples I chose $(M, N) = (3, 5)$ and $(5, 3)$ respectively.) And U is $N \times N$ and V is $M \times M$. So we can choose a new basis for \mathcal{H}_a which is the image of our old one under U : $|i'\rangle \equiv U|i\rangle$. Similarly for \mathcal{H}_b : $|r'\rangle \equiv V|r\rangle$. In this basis, we have

$$|w\rangle = \sum_{i'r'} \Lambda_{i'r'} |i'\rangle \otimes |r'\rangle$$

and Λ is diagonal.]

Besides the Schmidt number, another measure of entanglement between two subsystems is the *entanglement entropy* (or von Neumann entropy), defined to be

$$S_A = -\text{tr} \rho_A \log \rho_A .$$

[The logarithm of a Hermitian operator can be defined by considering the spectral decomposition: if $\mathbf{A} = \sum_a a \mathbf{P}_a$ then $\log \mathbf{A} = \sum_a \log(a) \mathbf{P}_a$.]

(b) Show that if $|a\rangle$ is not an entangled state of A and B then $S_A = 0$.

Equivalently, show that

$$-\text{tr} \rho \log \rho = 0 \quad \text{if } \rho \text{ is a pure state.}$$

- (c) What's the maximum possible value of the entanglement entropy for a state of a qbit?
- (d) Show that if the whole system is in a pure state $|a\rangle\langle a|$, then the entanglement entropy for ρ_A is the same as that of ρ_B :

$$S_A = S_B .$$

[Hint: use your ability to choose the basis again.]

(e) **Entanglement cannot be created locally**

Define a *local unitary* to be an operator of the form $\mathbf{U}_A \otimes \mathbf{U}_B$ where $\mathbf{U}_{A,B}$ acts only on $\mathcal{H}_{A,B}$. These are the operations that can be done by actors with access only to *A or B*. Show that by acting on a state of \mathcal{H} with a local unitary we cannot change the Schmidt number or the entanglement entropy of either factor. Consider both the case of a pure state of \mathcal{H} and a mixed state of \mathcal{H} ; note that the action of a unitary \mathbf{U} on a density matrix ρ is

$$\rho \rightarrow \mathbf{U}\rho\mathbf{U}^\dagger.$$

[We conclude from this that to create an entangled state from an unentangled state, we must bring the two subsystems together and let them interact, resulting in a more general unitary evolution than a local unitary.]

5. Ambiguity of ensemble preparation

For this problem let A and B both be qbits. Show that the states

$$|c_n\rangle_{AB} = \frac{1}{\sqrt{2}} (|\uparrow_{\hat{n}}\rangle_A \otimes |\uparrow_{\hat{n}'}\rangle_B + |\downarrow_{\hat{n}}\rangle_A \otimes |\downarrow_{\hat{n}'}\rangle_B)$$

and

$$|c\rangle_{AB} = \frac{1}{\sqrt{2}} (|\uparrow_z\rangle_A \otimes |\uparrow_z\rangle_B + |\downarrow_z\rangle_A \otimes |\downarrow_z\rangle_B)$$

produce the *same* reduced density matrix for subsystem A .

[Hint: use the symmetries and the result of the previous problem.]

6. GHZM state

Consider a system of three qbits.

- (a) Show that the operators

$$\Sigma_a \equiv \sigma_x \otimes \sigma_y \otimes \sigma_y, \quad \Sigma_b \equiv \sigma_y \otimes \sigma_x \otimes \sigma_y, \quad \Sigma_c \equiv \sigma_y \otimes \sigma_y \otimes \sigma_x, \quad \Sigma \equiv \sigma_x \otimes \sigma_x \otimes \sigma_x$$

are all simultaneously diagonalizable.

- (b) Show that the state

$$|GHZM\rangle \equiv \frac{1}{\sqrt{2}} (|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)$$

is an eigenstate of the four operators Σ_i, Σ and find the respective eigenvalues.