University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics C (Physics 130C) Winter 2015 Assignment 5

Posted February 4, 2015 **Due 11am Thursday, February 12, 2015**

1. **More on von Neumann entropy**

   Consider the general density matrix for a qbit,

   $$\boldsymbol{\rho}_{\vec{n}} = \frac{1}{2}\left(\mathbb{1} + \vec{n}\cdot\boldsymbol{\sigma}\right).$$

   Recall that $\vec{n}$, $\vec{n}\cdot\vec{n} \leq 1$, specifies a point on the *Bloch ball.*

   Compute the von Neumann entropy $S(\boldsymbol{\rho}_{\vec{n}})$, and express it as a function of a single variable. Interpret this variable in terms of the geometry of the space of states.

   [Hint: use the fact that the von Neumann entropy is independent of the choice of basis.]

2. **Entropy and thermodynamics**

   Consider a quantum system with hamiltonian $\mathbf{H}$ and Hilbert space $\mathcal{H} = \text{span}\{|n\rangle, n = 0, 1, 2...\}$. Its behavior at finite temperature can be described using the *thermal density matrix*

   $$\boldsymbol{\rho}_\beta \equiv \frac{1}{Z}e^{-\beta\mathbf{H}}$$

   where $\beta \equiv \frac{1}{T}$ specifies the temperature and Z is a normalization factor. (We can think about this as the reduced density matrix resulting from coupling the system to a heat bath and tracing out the Hilbert space of the (inaccessible) heat bath.)

   (a) Find a formal expression for Z by demanding that $\boldsymbol{\rho}_\beta$ is normalized appropriately. (Z is called the *partition function.*)

   (b) Show that the von Neumman entropy of $\boldsymbol{\rho}_\beta$ can be written as

   $$S_\beta = E/T + \log Z \tag{1}$$

   where $E \equiv \langle\mathbf{H}\rangle_{\boldsymbol{\rho}} = \text{tr}\boldsymbol{\rho}\mathbf{H}$ is the expectation value for the energy. The expression above for S_β is the thermal entropy.

   (c) Evaluate Z and E for the case where the system is a simple harmonic oscillator

   $$\mathbf{H} = \hbar\omega\left(\mathbf{n} + \frac{1}{2}\right)$$

   with $\mathbf{n}|n\rangle = n|n\rangle$.

[For further inquiry in this direction I recommend Schumacher and Westmoreland sections 19.5, 19.6.]

3. **Purity test.** [from Chuang and Nielsen]

   Show that for any density matrix $\boldsymbol{\rho}$:

   (a) $\mathrm{tr}\boldsymbol{\rho}^2 \leq 1$

   (b) the inequality is saturated only if $\boldsymbol{\rho}$ is a pure state.

   [Hint: don't forget that the trace operation is basis-independent.]

4. **Polarization.** [from Boccio]

   Recall that in the expression for the general density matrix of a qbit

$$\boldsymbol{\rho} = \frac{1}{2}\left(\mathbb{1} + \vec{P} \cdot \vec{\boldsymbol{\sigma}}\right)$$

   we called $\vec{P}$ the *polarization*.

   (a) To justify this name, show that

$$\vec{P} = \langle \vec{\boldsymbol{\sigma}} \rangle.$$

   (b) Subject the qbit to a external magnetic field, which couples by

$$\mathbf{H} = -\vec{\boldsymbol{\mu}} \cdot \vec{B} = -\frac{\gamma}{2}\boldsymbol{\sigma} \cdot \vec{B}.$$

   (γ is called the *gyromagnetic ratio.*) Assuming the qbit is isolated, so that its time evolution is unitary, what is the time evolution of the polarization, $\partial_t \vec{P}$?

5. **Quantum gates**

   Schumacher and Westmoreland, Exercises 18.2, 18.6, 18.7, and 18.8, page 368. (Electronic version here.)

   Regarding the SWAP gate, Schumacher's wording is confusing and basis-dependent. This gate is defined to act on two qbits by

$$\mathbf{S} = |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|$$

   (so in words, it acts by swapping the labels of the basis states of the tensor-product Hilbert space). This implies (check this) that acting on *product* (*i.e.* unentangled) states $|a\rangle \otimes |b\rangle$ it gives $\mathbf{S}|a\rangle \otimes |b\rangle = |b\rangle \otimes |a\rangle$.