University of California at San Diego – Department of Physics – Prof. John McGreevy

Quantum Mechanics C (130C) Winter 2013
Midterm exam

Friday February 8, 2013, 2-2:50pm

Please remember to put your name on your exam booklet. This is a closed-book exam. There are three problems, each with several parts, of varying levels of difficulty; make sure you try all of the parts. None of the problems require very extensive calculation; if you find yourself involved in a morass of calculation, step back and think. Good luck.

Possibly useful information:

$$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\uparrow_{\hat{n}}\rangle = e^{-\mathbf{i}\varphi/2} \cos\frac{\theta}{2} |\uparrow_{\hat{z}}\rangle + e^{+\mathbf{i}\varphi/2} \sin\frac{\theta}{2} |\downarrow_{\hat{z}}\rangle \text{ satisfies } \vec{\boldsymbol{\sigma}} \cdot \hat{n} |\uparrow_{\hat{n}}\rangle = |\uparrow_{\hat{n}}\rangle$$

$$e^{-i\alpha\hat{n}\cdot\vec{\boldsymbol{\sigma}}} = \mathbb{1} \cos\alpha - i\hat{n} \cdot \vec{\boldsymbol{\sigma}} \sin\alpha.$$

1. **Short answers and conceptual questions** [5 points each]

   (a) Write the matrix representation of the operator $\mathbf{X} = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|$ in the (orthonormal) basis $\{|\uparrow\rangle, |\downarrow\rangle\}$.

   (b) Suppose we have a quantum system in the state

   $$|\psi\rangle = a|1\rangle + b|2\rangle + c|3\rangle$$

   where $\mathcal{H} = \text{span}\{|1\rangle, |2\rangle, |3\rangle\}$, and the basis is orthonormal. What are the possible results one can get when measuring the observable whose matrix elements in this basis are

   $$\mathbf{Y} = \begin{pmatrix} 27 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

   and with what probabilities do they occur in the state $|\psi\rangle$?

   (c) What is the expectation value of the operator $\mathbf{Y}$ above in the mixed state specified by the density matrix

   $$\boldsymbol{\rho} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}.$$

(d) Is this matrix

$$\boldsymbol{\rho} \stackrel{?}{=} \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

a possible density matrix? If not, why not?

(e) Which of the following can be a representation of a symmetry operation? Which can be a representation of an observable?

$$\mathbf{A} \equiv |1\rangle\langle 2|, \quad \mathbf{B} \equiv |1\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 1|, \quad \mathbf{C} \equiv |1\rangle\langle 2| + |2\rangle\langle 1|$$

(f) How many (real) parameters must we specify to determine a pure state of one qbit?

(g) We are given a *single* qbit (*e.g.* one polarized photon, or one spin-$\frac{1}{2}$ particle) in a pure state. By doing measurements, how many of those (real) parameters can we determine?

(h) How many (real) parameters must we specify to determine a pure state of k qbits?

(i) What is the biggest possible value of the Schmidt number of a state of a pair of qbits? (Recall that the Schmidt number of a bipartite state is the rank of the reduced density matrix for one of the parts.)

(j) True or false: a superposition of maximally-entangled states is maximally entangled. (If true, give an argument; if false, give a counter-example.)

2. **Probabilities for quantum states.** [from Schumacher] [25 points]

Two boxes each produce a stream of qbits (if you like, suppose they are spin-$\frac{1}{2}$ particles).

Box A makes them in the state $|+\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$.

Box B randomly produces qbits in the states $|\uparrow\rangle$ or $|\downarrow\rangle$, each with probability 1/2.

(a) Write down the density matrices $\boldsymbol{\rho}_A$ and $\boldsymbol{\rho}_B$ describing the state of the qbits produced by A and B.

(b) Describe an experiment on the qbits that can tell the difference between box A and box B. (Also describe the expected outcome of the experiment.) You have a Stern-Gerlach apparatus at your disposal, and you can choose its orientation.

(c) Can you reliably tell the difference between the boxes by measuring only one qbit?

3. **Photons, polarizers, angular momentum.** [from Boccio] [25 points]

Photons propagating in the z direction are prepared in the polarization state

$$|\psi\rangle = a|x\rangle + b\mathbf{i}|y\rangle$$

where a and b are **real numbers** and $\mathbf{i} = \sqrt{-1}$.

(a) What is the probability that a photon in this state will pass through a polarizer with its transmission axis oriented in the y direction?

(b) What is the probability that a photon in this state will pass through a polarizer with its transmission axis y' making an angle φ with the y axis?

(c) A beam carrying N photons per second, each in the state $|\psi\rangle$ is totally absorbed by a black disk with its surface normal in the z direction. What is the torque exerted on the disk? (Specify the direction.)

[Reminders: The photon states

$$|R\rangle \equiv \frac{1}{\sqrt{2}}\left(|x\rangle + \mathbf{i}|y\rangle\right), \quad |L\rangle \equiv \frac{1}{\sqrt{2}}\left(\mathbf{i}|x\rangle + |y\rangle\right)$$

carry a unit $\hbar$ of angular momentum parallel and antiparallel, respectively, to the direction of propagation of the photon. Torque is the time rate of change of angular momentum.]