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Quantum Mechanics C (Physics 130C) Winter 2015
Worksheet 1

Announcements

- The 130C web site is:

  http://physics.ucsd.edu/~mcgreevy/w15/ .

  Please check it regularly! It contains relevant course information!

- Greetings everyone! This week is Dirac notation/linear algebra bootcamp.

Problems

1. **Unitary Operators Update**

   An operator U is unitary if (and only if) $U^\dagger U = \mathbb{1}$. Where the dagger of a matrix is a combination transpose (switch rows with columns) and complex conjugation. A restatement is $U^\dagger = U^{-1}$, the inverse matrix of U.

   (a) Unitary operators evolve states in quantum mechanics and are related to something familiar in classical mechanics.

   Consider a unitary matrix with only real valued entries. Denote this matrix R. What does this imply about R^{-1}?

   Take $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ Verify the above conclusion for this R.

   Act R on the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for your favorite value of θ to see what it does.

   Any such (real) matrices are called *orthogonal*.

   (b) Another important property U must satisfy, if it takes states to states, is that it doesn't change the laws of probability: the lengths of vectors should remain unchanged.

   Show that $\| U|\phi\rangle \| = \| |\phi\rangle \|$.

   Do this by showing the inner product of $U|\psi\rangle$ and $U|\phi\rangle$ is equal to $\langle\psi|\phi\rangle$. How do I relate norms to inner products?

   (c) Suppose I would like to measure an observable O after updating my state with U.

   Show that the quantity of interest is $O' = U^\dagger O U$.

   Prove that O' is Hermitian. Recall that $(AB)^\dagger = B^\dagger A^\dagger$

2. **Compatible Measurements**

   Consider the following matrices: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 5 & 3a \\ 6 & b \end{pmatrix}$

   For what values of a and b are these simultaneously diagonalizable? To solve this first determine the eigenvectors for X. Then demand these are also eigenvectors of Q.[1]

   We define the *commutator* $[M, N] = MN - NM$

   For these particular values of a and b what is $[X, Q]$?

   We say that two observables (Hermitian operators) are compatible if we the order in which we measure them doesn't matter.

   Translate this to a statement about commutators and diagonalization.

   Does $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ work?

3. **Projecting**

   An important type of operator is a projector. We say P is a projector if $P^2 = P$; projecting twice is the same as once.

   (a) Consider $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and prove P is a projector.

   What is the action on $\begin{pmatrix} x \\ y \end{pmatrix}$?

   Show P can be written as the *outer* product of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and itself.

   (b) Adopting Dirac's notation let $|i\rangle$ be a normalized ($\langle i|j\rangle = \delta_{i,j}$) state.
   Show that $P_i = |i\rangle\langle i|$ is a projector onto the state $|i\rangle$. Show it is Hermitian.
   What is $\langle j|P_i|j\rangle$?
   In this way projectors can represent *yes or no* questions relating to our system.

4. **Represent**

   Suppose the vectors $\{|a\rangle, |b\rangle, |c\rangle\}$ form an orthonormal basis for $\mathcal{H}$ where some operator $\hat{K}$ lives. Suppose we know also that these vectors are *eigenvectors* of $\hat{K}$:

   $$\hat{K}|a\rangle = 5|a\rangle \quad \hat{K}|b\rangle = -12|b\rangle \quad \hat{K}|c\rangle = 2\mathbf{i}|c\rangle \tag{1}$$

   (a) Could $\hat{K}$ represent a physical observable of the system?

   (b) Write the matrix $\hat{K}$ using Dirac notation. Hint: $\hat{K} = a\ |a\rangle\langle a| + b\ |b\rangle\langle b| + c\ |c\rangle\langle c|$

   (c) Consider $|\psi\rangle = \frac{1}{\sqrt{2}}(2|a\rangle + 5\mathbf{i}|c\rangle)$ and compute the expectation or average value of $\hat{K}$ in this state defined as $\langle\psi|\hat{K}|\psi\rangle$. Do this using Dirac notation and regular matrix multiplication.

[1] This would not work if either X or Q had repeated eigenvalues though you could still find a basis to diagonalize both.