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Quantum Mechanics C (Physics 130C) Winter 2015
Worksheet 4

Announcements

- The 130C web site is:

  http://physics.ucsd.edu/∼mcgreevy/w15/ .

  Please check it regularly! It contains relevant course information!

- My office hour is at 5pm today. Collect your homework! Inform me if things are wrong!

Problems

1. **Quis Custodiet Ipsos Custodes?**

   Projective measurements lead to some weird things.

   Consider a two state system with basis vectors $\{|0\rangle, |1\rangle\}$. We are going to evolve the system according the Hamiltonian $\hat{H} = \frac{\omega}{2} Y$ where Y is the Pauli matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

   (a) What is the unitary operator associated with time evolution? Given an initial prepared state of $|\psi_0\rangle = |0\rangle$. Write an expression for $|\psi(t)\rangle$.

   (b) What is the probability, as function of time, to measure $|0\rangle$?

   (c) Suppose we study the system over the time interval $[0, T]$ where $T \gg \delta t \equiv \frac{T}{N}$. We perform a measurement, in this basis, at every time $\frac{T}{N}, \frac{2T}{N}, \cdots$ where N is large. Assuming each measurement is independent from the other, what's the probability that the spin *never* flips to $|1\rangle$?

   (d) Evaluate this probability in the limit of $N \to \infty$.
   This is called the *quantum Zeno effect*.

2. **Getting Tensor Every Day**

   We use tensor products to describe Hilbert spaces of combined systems.

   Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces spanned by $\{|0\rangle, |1\rangle\}$ and $\{|\uparrow\rangle, |\downarrow\rangle\}$ respectively. One can construct the tensor product space:

   $\mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B = \text{span}\{|0\uparrow\rangle, |0\downarrow\rangle, |1\uparrow\rangle, |1\downarrow\rangle\}$

   where a vector in the basis above corresponds to specifying the eigenvalue of σ^z for each qubit. Let's get a feel for this.

(a) The tensor product is like multiplication. Consider the vector $|0\uparrow\rangle \equiv |0\rangle_A \otimes |\uparrow\rangle_B$ If one represented $|0\rangle_A \equiv \begin{pmatrix}1\\0\end{pmatrix}$ and $|\uparrow\rangle_B \equiv \begin{pmatrix}1\\0\end{pmatrix}$ on $\mathcal{H}_{A,B}$ respectively then: $|0\rangle_A \otimes |\uparrow\rangle_B \equiv \begin{pmatrix}|\uparrow\rangle_B\\0\end{pmatrix} \equiv \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and similar for the rest of the basis vectors.

Show that $\mathbb{1} \equiv \mathbb{1}_A \otimes \mathbb{1}_B$ works out similarly.

(b) Define X_B by $X_B|0\uparrow\rangle = |0\downarrow\rangle$ and $X_B|0\downarrow\rangle = |0\uparrow\rangle$. Show that $X_B \equiv \mathbb{1}_A \otimes \sigma_B^x$ accomplishes this and write it in matrix form.[1]

(c) Note that not every vector $\mathcal{H}_{AB}$ can be split like this. There exist $|\psi\rangle \in \mathcal{H}_{AB}$ such that $|\psi\rangle \neq |\Phi\rangle_A \otimes |\phi\rangle_B$. Such states are called *entangled*

Show that $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\uparrow\rangle + |1\downarrow\rangle)$ is such a state

(d) Now consider the following operator

$$U_{CNOT} \equiv |0_A\rangle\langle 0_A| \otimes \mathbb{1}_B + |1_A\rangle\langle 1_A| \otimes \sigma_B^x$$

Write a matrix representation of U_{CNOT}

What are the eigenstates of this operator? In which of these states are A and B entangled? Try to answer this question without relying on the matrix form you found.

[1] I will often suppress $\otimes$ and $\mathbb{1}$ when they are clear by context.