University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 217 Winter 2016
Assignment 3

Due 2pm Thursday, February 4, 2016

1. **High temperature expansion for Ising model.**

   In lecture, we rewrote the partition function of the nearest-neighbor Ising model (on any graph) as a sum over closed loops. Without a magnetic field, the loops were weighted by their length, just like in our discussion of SAWs. If we turn on a magnetic field, how does it change the form of the sum?

2. **A case where mean field theory is right.**

   [from Goldenfeld, Ex 2-2] Consider the *infinite-range* Ising model, where the coupling constant is $J_{ij} = J$ for all i, j, with no notion of who is whose neighbor. That is:

   $$-H(s) = h \sum_i s_i + \frac{J_0}{2} \sum_{ij} s_i s_j .$$

   (a) Explain why it's a good idea to let $J_0 = J/N$ where N is the number of sites.

   (b) Prove that

   $$e^{\frac{a}{2N}x^2} = \int_{-\infty}^{\infty} dy \sqrt{\frac{Na}{2\pi}} e^{-\frac{Na}{2}y^2 + axy}, \quad \mathrm{Re}\, a > 0.$$

   (c) Use this to show that

   $$Z_\Lambda = \int_{-\infty}^{\infty} dy \sqrt{\frac{N\beta J}{2\pi}} e^{-N\beta L}$$

   where

   $$L(h, J, \beta, y) = \frac{J}{2} y^2 - T \log\left(2\cosh\left(\beta(h + Jy)\right)\right).$$

   When can this expression be non-analytic in β?

   (d) In the thermodynamic limit ($N \to \infty$), this integral can be evaluated exactly by the method of steepest descent. Show that

   $$Z = Z(\beta, h, J) \simeq \sum_i \sqrt{J(y_i)} e^{-\beta N L(h,J,\beta,y_i)}$$

   for some $J(y_i)$. Find the equation satisfied by the saddle point values y_i. Convince yourself that the $\sqrt{J}$ prefactor can be neglected in the thermodynamic limit.

What is the probability of the system being in the state specified by y_i? Use this to conclude that the magnetization is given by

$$m \equiv \lim_{N\to\infty} \frac{1}{\beta N} \partial_h \ln Z = y_0$$

where y_0 is the position of the global minimum of L.

(e) Now consider the case $h = 0$. Show by graphical methods that there is a phase transition and find the transition temperature T_c. What's the story with all the solutions, for T both below and above T_c?

(f) Calculate the (isothermal) susceptibility

$$\chi = \partial_h m.$$

For $h = 0$, show that χ diverges both above and below T_c, and find the leading and next-to-leading behavior of χ as a function of the reduced temperature $t \equiv \frac{T-T_c}{T_c}$.