

Physics 217 Winter 2016 Assignment 5

Due 2pm Tuesday, March 1, 2016

1. Brain-warmer.

Show (using the definition of the Legendre transform $\Gamma[m] = (F[h] - \sum hm) |_{m=\partial_h F/V}$ and the chain rule) that the susceptibility $\chi = \partial_h m$ is the inverse of the curvature of the effective potential $\gamma(m) = \Gamma[m, \text{uniform}]/V$:

$$\chi^{-1} = \partial_m^2 \gamma.$$

Conclude that the correlation length diverges when $\partial_m^2 \gamma \rightarrow 0$. Think about this in terms of the spectrum of normal modes of the system.

2. Practice with systematically ignoring small things.

In doing perturbative RG, as we are going to do for the next few weeks, it is very useful to be systematic about ignoring corrections which are of the same size as corrections we are not computing. To do this, it is useful to introduce (or keep track of) an expansion parameter whose powers count orders of perturbation theory. By ‘of the same size’ I mean corrections that come with the same number of powers of g .

In lecture, we found the leading correction to the mean field free energy (down by one power of g), and found that the inverse susceptibility was

$$\chi^{-1} = r_0 + g_0 \delta(r_0) + \mathcal{O}(g_0^2)$$

where $\delta(r_0) \sim \int \frac{d^d q}{q^2 + r_0}$ is some known function of the bare coupling r_0 .

We assume that the parameter $r_0(T)$ is analytic in the temperature near T_c . This is the conservative assumption: the thing we are trying to explain is how physics can become non-analytic in T at some finite T ; we don’t want to put it in from the beginning. More precisely: we can rule out singular dependence of r_0 on T because non-analyticity requires the thermodynamic limit, and the microscopic couplings are properties of finite chunks of the system.

The definition of the critical temperature T_c is the value of T where the correlation length blows up. Use the susceptibility sum rule (you proved this on the last homework) to relate this condition to $\chi(T_c)$.

Use the previous two pieces of input to prove the expression I claimed in lecture which eliminates r and relates χ^{-1} directly to the deviation from the critical temperature $t \equiv \frac{T-T_c}{T_c}$:

$$\chi^{-1}(t) = c_1 t (1 + \partial_t \delta(t)) + \mathcal{O}(g^2)$$

where c_1 is a non-universal constant. You will have to ignore all errors of order g^2 . Determine the function $\partial_t \delta$.

3. When the higher-derivative terms in the LG free energy are important.

[Based on Goldenfeld Exercise 5-3]

Consider a system whose Landau-Ginzburg free energy looks like

$$F_{\text{LG}}[m] = \int dx \left(atm^2 + \frac{1}{2}bm^4 + \frac{1}{2}\gamma(\partial_x m)^2 + \frac{1}{2}\sigma(\partial_x^2 m)^2 \right).$$

For simplicity take the system to be one-dimensional and put it in a box of size L ; we'll use periodic boundary conditions, $m(x+L) = m(x)$. Suppose that the constants $\sigma > 0, b > 0, a > 0$ but allow that t and (here's the interesting bit) γ can be of either sign. The problem is to work out the phase diagram in the $t - \gamma$ plane.

(a) Fourier expand the magnetization

$$m(x) = \frac{1}{L} \sum_{n \in \mathbb{Z}} e^{iq_n x} m_n, \quad q_n = \frac{2\pi n}{L}.$$

Find the inverse transform for m_n in terms of $m(x)$ and check that all the (Kronecker and Dirac) delta functions work out.

(b) Write $F_{\text{LG}}[m]$ in terms of the Fourier modes m_n .

(c) By minimizing with respect to m_n for all n , show that the system exhibits three possible phases:

(1) a paramagnetic phase where $m = 0$

(2) a ferromagnetic phase where $m \neq 0$ but is uniform, and

(3) a *spatially modulated* phase where $m_q \neq 0$ for some nonzero wavenumber $q \neq 0$.

[Note that (because $b > 0$), when one mode m_q condenses, it becomes unfavorable for others to do so – so you should assume you only need one Fourier component.]

(d) What is the wavelength of the modulated mode which condenses? Find the phase boundaries and draw the phase diagram. What is the order of the transition at the various phase boundaries?

4. Legendre transforms and constrained integrals.

[Somewhat open-ended.] In lecture I claimed that the LG free energy could be regarded as a coarse-grained free energy in the sense that

$$e^{-\beta \Gamma_{\text{LG}}[m(r)]} = \sum_s e^{-\beta H(s)} \prod_r \delta \left(m(r) - \frac{1}{N(r)} \sum_{i \in r} s_i \right),$$

that is: it is the free energy computed at fixed magnetization. [It may be helpful to note that this is Goldenfeld's equation (5.49).]

Compare this expression to the definition in terms of the Legendre transform of the ordinary Helmholtz free energy $F[h]$:

$$\Gamma'_{\text{LG}}[m] = F[h] - \sum_r m_r h_r |_{m=\partial_h f}, \quad e^{-\beta F[h]} \equiv \sum_s e^{-\beta H(s)}.$$

Do they agree $\Gamma'_{\text{LG}}[m] \stackrel{N \rightarrow \infty}{\equiv} \Gamma_{\text{LG}}[m]$ in the thermodynamic limit?

For uniform-in-space magnetization, $\Gamma_{\text{LG}}[m] = V\gamma_{\text{LG}}(m)$ should be extensive. Can you prove this?

Must $\gamma_{\text{LG}}(m)$ be convex in m ?