University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2017
Assignment 7

Due 11am Thursday, March 2, 2017

1. **Spectral representation at finite temperature.**

   In lecture we have derived a spectral representation for the two-point function of a scalar operator in the vacuum state

   $$\mathbf{i}\mathcal{D}(x) = \langle 0| \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) |0\rangle$$

   Derive a spectral representation for the two-point function of a scalar operator at a nonzero temperature:

   $$\mathbf{i}\mathcal{D}_\beta(x) \equiv \mathrm{tr}\frac{e^{-\beta\mathbf{H}}}{Z_\beta}\mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) = \frac{1}{Z_\beta}\sum_n e^{-\beta E_n} \langle n| \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) |n\rangle .$$

   Here $Z_\beta \equiv \mathrm{tr}e^{-\beta\mathbf{H}}$ is the thermal partition function. Check that the zero temperature ($\beta \to \infty$) limit reproduces our previous result.

2. **One more consequence of unitarity.** [From Aneesh Manohar]

   A general statement of the optical theorem is:

   $$-\mathbf{i}\left(\mathcal{M}(a \to b) - \mathcal{M}(b \to a)\right) = \sum_f \int d\Phi_f \mathcal{M}^\star(b \to f)\mathcal{M}(a \to f) .$$

   Consider again QED with electrons and muons.

   (a) Consider scattering of an electron (e^-) and a positron (e^+) into e^-e^+ (so $a = b$ in the notation above). We wish to consider the contribution to the imaginary part of the amplitude for this process which is proportional to $Q_e^2Q_\mu^2$ where Q_e and Q_μ are the electric charges of the electron and muon (which are in fact numerically equal but never mind that). Draw the relevant Feynman diagram, and compute the imaginary part of this amplitude (just the $Q_e^2Q_\mu^2$ bit) as a function of $s \equiv (k_1 + k_2)^2$ where $k_{1,2}$ are the momenta of the incoming e^+ and e^-. (You basically did this on the previous problem set.

(b) Use the optical theorem and the fact that the total cross section for $e^+e^- \to \mu^+\mu^-$ must be positive

$$\sigma(e^+e^- \to \mu^+\mu^-) \geq 0$$

to show that a Feynman diagram with a fermion loop must come with a minus sign.

3. **An application of effective field theory in quantum mechanics.**

[I learned this example from Z. Komargodski.]
Consider a model of two canonical quantum variables ($[\mathbf{x}, \mathbf{p}_x] = \mathbf{i} = [\mathbf{y}, \mathbf{p}_y], 0 = [\mathbf{x}, \mathbf{p}_y] = [\mathbf{x}, \mathbf{y}]$, etc) with Hamiltonian

$$\mathbf{H} = \mathbf{p}_x^2 + \mathbf{p}_y^2 + \lambda \mathbf{x}^2 \mathbf{y}^2.$$

(This is similar to the degenerate limit of the model studied in lecture with two QM variables where both natural frequencies are taken to zero.)

(a) Based on a semiclassical analysis, would you think that the spectrum is discrete?

(b) Study large, fixed x near $y = 0$. We will treat x as the slow (= low-energy) variable, while y gets a large restoring force from the background x value. Solve the y dynamics, and find the groundstate energy as a function of x:

$$V_{\text{eff}}(x) = E_{\text{g.s. of y}}(x).$$

(c) The result is not analytic in x at $x = 0$. Why?

(d) Is the spectrum of the resulting 1d model with

$$\mathbf{H}_{\text{eff}} = \mathbf{p}_x^2 + V_{\text{eff}}(\mathbf{x})$$

discrete? Is this description valid in the regime which matters for the semi-classical analysis?
[Bonus: determine the spectrum $\mathbf{H}_{\text{eff}}$.]

4. **All possible terms.**

Perturb the gaussian fixed point of the XY model in $D > 3$ by a term

$$\delta S_6 = \int d^D x \; g_6(x) (\phi^\star \phi)^3$$

where g_6 is a short-ranged coupling function. Only if you find *new* relevant perturbations (this does not include terms that may be absorbed in changes in the bare values of perturbations that we have already studied) must you keep track of the numerical coefficients.

What would you find if you included also δS_{2n} with $n > 3$?