1. **Brain-warmer.** Prove the Gordon identities

\[ \bar{u}_2 (q^\nu \sigma_{\mu\nu}) u_1 = i \bar{u}_2 ((p_1 + p_2)_\mu - (m_1 + m_2) \gamma_\mu) u_1 \]

and

\[ \bar{u}_2 ((p_1 + p_2)^\nu \sigma_{\mu\nu}) u_1 = i \bar{u}_2 ((p_2 - p_1)_\mu - (m_2 - m_2) \gamma_\mu) u_1 \]

where \( q \equiv p_2 - p_1 \) and \( \gamma_1 u_1 = m_1 u_1, \bar{u}_2 \gamma_2 = m_2 \bar{u}_2 \), using the definitions and the Clifford algebra.

2. **Numerator algebra.** Check that you understand the steps leading to the expression for the numerator of the integrand for the QED vertex correction (equation (7.30) of the lecture notes). It uses \( x + y + z = 1 \), the Dirac equation \( \bar{u}(p) = m_e u(p) \), \( \bar{u}(p') \gamma = \bar{u}(p') m_e \) and the Gordon identity.

There is a tricky step in getting the form of \( \Delta \) which involves remembering that

\[ (p + q)^2 = (p)^2 = m^2 \]

so that we may eliminate \( 2p \cdot q = -q^2 \).

3. **Symmetry is attractive.** [from Jared Kaplan] Consider a field theory in \( D = 3 + 1 \) with two massless (for simplicity) scalar fields which interact via the interaction Lagrangian

\[ V = -\frac{g}{4!} (\phi_1^4 + \phi_2^4) - \frac{2\lambda}{4!} \phi_1^2 \phi_2^2. \]

(a) Show that when \( \lambda = g \) the model possesses an \( O(2) \) symmetry.

At this special point, the potential is \( (\phi_1^2 + \phi_2^2)^2 \), which depends only on the distance from the origin of the field space.

(b) Will you need a counterterm of the form \( \phi_1 \phi_2 \) or \( \phi_1 \Box \phi_2 \)? If not, why not?

A very important point: such terms can’t be generated because they violate the \( \mathbb{Z}_2 \) symmetry which takes \( (\phi_1, \phi_2) \rightarrow (-\phi_1, \phi_2) \). In general, radiative effects (\( i.e. \) loops) will not violate symmetries of the bare action. Exceptions to this statement are called *anomalies*; this only happens when no regulator preserves the symmetry in question.
Renormalize the theory to one loop order by regularizing (for example with Pauli Villars), adding the necessary counterterms, and imposing a renormalization condition on the masses and $2 \rightarrow 2$ scattering amplitudes at some values of the kinematical variables $s_0, t_0, u_0$.

I’ll use a hard euclidean momentum cutoff since then we can reuse our results from $\phi^4$ theory. To save typing let me define $L(x) \equiv \frac{1}{16\pi^2} \log x$. Every loop integral we will encounter is the same as in the pure massless $\phi^4$ theory that we did in lecture.

The symmetry which interchanges $\phi_1 \leftrightarrow \phi_2$ guarantees that their self-couplings $g$ (and the masses) stay equal (using the same principle as above). This means we have only three counterterms to determine altogether: $\delta_{m_2}$ and two four-point counterterms $(\delta_g, \delta_\lambda)$ to determine. That is, we have to impose two renormalization conditions on the four-point functions.

First an annoying point: with the given normalization, the 1122 vertex is actually $-i\lambda/3$.

The self-energy for $\phi_1$ is

$$-i\Sigma(p^2) = \begin{array}{c} \hline \hline \end{array} + \begin{array}{c} \hline \hline \end{array} + .. = -i(g+\lambda/3)c\Lambda^2 + O(g,\lambda)^2$$

where $c$ is a numerical constant that I can’t remember right now and which we don’t need. To put the pole at $p^2 = m_p^2 = 0$, we need the bare mass to be

$$m_2^2(\Lambda) = -\Sigma(p^2 = 0) = (g + \lambda/4)c\Lambda^2.$$  

As in $\phi^4$ theory, there is no wavefunction renormalization at one loop because $\Sigma$ is independent of $p^2$.

There are three different $2 \rightarrow 2$ scattering processes to consider: 11 $\rightarrow$ 11, 11 $\rightarrow$ 22, 12 $\rightarrow$ 12. (The corrections to 22 $\rightarrow$ 22 are the same as those for 11 $\rightarrow$ 11, and similarly 22 $\rightarrow$ 11 is the same as 11 $\rightarrow$ 22, by the exchange symmetry.) Then using the notation $\begin{array}{c} \hline \hline \end{array} = \langle \phi_1 \phi_2 \rangle$ we have

$$M_{11 \rightarrow 11} = -g + (g^2 + \lambda^2)(L(s/\Lambda^2) + L(t/\Lambda^2) + L(u/\Lambda^2)) + \delta_g$$ \hspace{1cm} (1)

$$= \begin{array}{c} \hline \hline \end{array} + \begin{array}{c} \hline \hline \end{array} + \begin{array}{c} \hline \hline \end{array} + \begin{array}{c} \hline \hline \end{array} \hspace{1cm} (2)$$
The $\lambda^2$ term involves $\phi_2$ running in the loop. (Note that I am writing $i\mathcal{M} = -ig + (-ig)^2...$ and dividing the BHS by $i$)

$$\mathcal{M}_{22\rightarrow 11} = -\frac{\lambda}{3} + \frac{\lambda}{3}g^2L(s/\Lambda^2) + \left(\frac{\lambda}{3}\right)^2 \left(2L(t/\Lambda^2) + 2L(u/\Lambda^2)\right) + \delta\lambda \quad (3)$$

where the 2 in the s-channel term is from the fact that either $\phi_1$ or $\phi_2$ can run in the loop. The last two diagrams have a different symmetry factor from the others, since we can’t exchange the two propagators in the loop – so they get an extra factor of 2.

$$\mathcal{M}_{12\rightarrow 12} = -\frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 \left(2L(s/\Lambda^2) + 2L(u/\Lambda^2)\right) + 2\frac{\lambda}{3}gL(t/\Lambda^2) + \delta\lambda \quad (5)$$

Using the renormalization conditions $\mathcal{M}_{11\rightarrow 11}(s_0 = t_0 = u_0) = -g_P$ and $\mathcal{M}_{22\rightarrow 11}(s_0 = t_0 = u_0) = -\frac{\lambda_P}{3}$ we find

$$\lambda(\Lambda) \equiv \lambda + \delta\lambda = \lambda_P + \lambda_P^2g_PL + 4\left(\frac{\lambda_P}{3}\right)^2 L + \mathcal{O}(\lambda_P, g_P)^2 \quad (7)$$

$$g(\Lambda) \equiv g + \delta g = g_P + \left(g_P^2 + \left(\frac{\lambda_P}{3}\right)^2\right) 3L + \mathcal{O}(\lambda_P, g_P)^2 \quad (8)$$

where $L \equiv L(s_0/\Lambda^2)$. We’ve solved for the couplings perturbatively, to second order in both, which means we ignored the difference between e.g. $g$ and $g_P$ in the quadratic term, as we must. From now on I will drop the $P$ subscripts on the physical coupling.

Notice that we would get the same answer if we defined $\lambda_P$ by fixing a value of $\mathcal{M}_{12\rightarrow 12}$ instead.

(d) Consider the limit of low energies, i.e. when $s_0, t_0, u_0 \ll \Lambda^2$ where $\Lambda$ is the cutoff scale. Tune the location of the poles in both propagators to $p^2 = 0$. Show that the coupling goes to the $\mathcal{O}(2)$-symmetric value if it starts nearby (nearby means $\lambda/g < 3$).

A nice trick for doing this is found in the discussion from lecture one: compute the beta functions.

$$\beta_g \equiv 16\pi^2\Lambda^2\partial_{\Lambda^2}g(\Lambda) = -3\left(g^2 + \left(\frac{\lambda}{3}\right)^2\right), \quad \beta_\lambda \equiv 16\pi^2\Lambda^2\partial_{\Lambda^2}\lambda(\Lambda) = -\left(2\lambda g + 4\left(\frac{\lambda}{3}\right)^2\right)$$
where I’ve pulled out a factor of $4\pi^2$ in the definition of $\beta$ for convenience – it only affects how fast the flow happens. 

To look at the relative flow of $g$ and $\lambda$ let’s compute

$$
\beta_{\lambda/g} \equiv 8\pi^2 \Lambda^2 \partial_{\Lambda^2} \frac{\lambda}{g} = \frac{1}{g^2} (g\beta_\lambda - \lambda\beta_g) \propto (\lambda^3 - 4g\lambda + 3g^2) = \lambda(\lambda - g)(\lambda - 3g).
$$

This looks like this:

![Graph of $\beta_{\lambda/g}$ vs $\lambda/g$]

with the convention I’m using, positive $\beta$ means that as we increase $\Lambda$, the coupling decreases. This means that the couplings approach the point $g = \lambda$ as $\Lambda \to \infty$ fixing $g_P, \lambda_P$. This is the case as long as we start with $\lambda/g < 3$.

4. The Rosenbluth formula. [optional]

If you wish to experience the true suffering of the field theory student, do Peskin problem 6.1. I recommend undoing the use of the Gordon identity in the parametrization of the vertex.