University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2019
Assignment 1

Due 12:30pm Monday, January 14, 2019

1. **Brain-warmer: Path integral for a free particle**

   Consider the path integral description of the quantum mechanics of a free particle in one dimension. The action is

   $$S[x] = \int_0^{t_f} dt \frac{m}{2}\dot{q}^2.$$

   (a) What is the equation of motion $0 = \frac{\delta S}{\delta q(t)}$?

   (b) Find the classical solution $\underline{q}(t)$ with $\underline{q}(t=0) = q_0$ and $\underline{q}(t = t_f) = q_f$.

   (c) Evaluate the action for the classical solution $S[\underline{q}]$, and evaluate the stationary-phase approximation to the path integral for the quantum propagator

   $$U(q_f, t_f; q_0, 0) \equiv \langle q_f | \mathbf{U}(t_f) | q_0 \rangle = \int [dq] e^{\mathbf{i}S[q]} \simeq U_{sc} \equiv e^{iS[\underline{q}]}.$$

   If you are feeling ambitious, include the next term in the semiclassical expansion: let the integration variable $q \equiv \underline{q} + y$ and treat the y integral as gaussian:

   $$\int [dq] e^{\mathbf{i}S[q]} = e^{iS[\underline{q}]} \int [dy] e^{\mathbf{i} \int ds \int dt \frac{\delta^2 S}{\delta q(t) \delta q(t)} |_{q=\underline{q}} y(t) y(s) + \cdots}.$$

   (d) Derive the Hamiltonian associated to the action $S = \int dt L$ above.
   [That is, find $p = \frac{\partial L}{\partial \dot{q}}$ and eliminate $\dot{q}$ in $H(q,p) = p\dot{q} - L$.]

   (e) Treating this Hamiltonian quantum mechanically, evaluate the exact quantum propagator,

   $$U(q_f, t_f; q_0, 0) = \langle q_f | \mathbf{U}(t_f) | q_0 \rangle = \langle q_f | e^{-\mathbf{i}t_f \mathbf{H}} | q_0 \rangle.$$

   Compare with the semiclassical approximation U_{sc} defined above.

2. **Gaussian integrals are your friends.**

   (a) Show that

   $$\int_{-\infty}^{\infty} dq e^{-\frac{1}{2}aq^2 + jq} = \sqrt{\frac{2\pi}{a}} e^{\frac{j^2}{2a}}.$$

   [Hint: square the integral and use polar coordinates.]

(b) Consider a collection of variables $q_i, i = 1..N$ and a real, symmetric matrix a_{ij}. Show that

$$\int \prod_{i=1}^{N} dq_i e^{-\frac{1}{2}q_i a_{ij} q_j + J^i q_i} = \frac{(2\pi)^{N/2}}{\sqrt{\det a}} e^{\frac{1}{2}J^i a_{ij}^{-1} J^j}.$$

(Summation convention in effect, as always.)

[Hint: change integration variables to diagonalize a. $\det a = \prod a_i$, where a_i are the eigenvalues of a.]

(c) For any function of the N variables, $f(q)$, let

$$\langle f(q)\rangle \equiv \frac{\int \prod_{i=1}^{N} dq_i e^{-\frac{1}{2}q_i a_{ij} q_j} f(q)}{Z[J=0]}, \quad Z[J] = \int \prod_{i=1}^{N} dq_i e^{-\frac{1}{2}q_i a_{ij} q_j + J^i q_i}.$$

Show that

$$\langle q_i q_j\rangle = \partial_{J_i}\partial_{J_j} \log Z[J]|_{J=0} = a_{ij}^{-1}$$

Also, convince yourself that

$$\left\langle e^{J_i q_i}\right\rangle = \frac{Z[J]}{Z[J=0]}.$$

(d) Note that the number N in the previous parts may be infinite. This is really the only path integral we know how to do.

(e) Now consider a Gaussian field q, governed by the (quadratic) euclidean action

$$S[q] = \int dt \frac{1}{2}\left(\dot{q}^2 + \Omega^2 q^2\right).$$

Show that

$$\left\langle e^{-\int ds J(s) q(s)}\right\rangle_q = \mathcal{N} e^{+\frac{1}{2}\int ds dt J(s) G(s,t) J(t)}$$

where G is the (Feynman) Green's function for q, satisfying:

$$\left(-\partial_s^2 + \Omega^2\right) G(s,t) = \delta(s-t).$$

Here $\mathcal{N}$ is a normalization factor which is independent of J. Note the similarity with the previous problem, under the replacement

$$a = -\partial_s^2 + \Omega^2, \quad a^{-1} = G.$$

3. **Gaussian identity.**

   Show that for a gaussian quantum system

   $$\left\langle e^{\mathbf{i}K\mathbf{q}}\right\rangle = e^{-A(K)\left\langle \mathbf{q}^2\right\rangle}$$

   and determine $A(K)$. Here $\langle ... \rangle \equiv \langle 0| ... |0\rangle$. Here by 'gaussian' I mean that $\mathbf{H}$ contains only quadratic and linear terms in both $\mathbf{q}$ and its conjugate variable $\mathbf{p}$ (but for the formula to be exactly correct as stated you must assume $\mathbf{H}$ contains only terms quadratic in $\mathbf{q}$ and $\mathbf{p}$; for further entertainment fix the formula for the case with linear terms in $\mathbf{H}$).

   I recommend using the path integral representation (with hints from the previous problem). Alternatively, you can use the harmonic oscillator operator algebra. Or, better, do it both ways.

   This result is useful for the following problem and in many other places.

4. **Zero-phonon process.**

   Here is an application of the harmonic chain that we studied briefly (it is also an application of the previous problem).

   We wish to understand the probability for a photon to hit (our crude model of) a crystalline solid without exciting any vibrational excitations.

   Fermi's golden rule says that the probability for a transition from one state of the lattice $|L_i\rangle$ to another $|L_f\rangle$ is proportional to

   $$W(L_i \to L_f) = |\langle L_f| \mathbf{H}_\mathrm{L} |L_i\rangle|^2.$$

   Here $\mathbf{H}_\mathrm{L}$ is the hamiltonian describing the interaction between the photon and an atom in the lattice. For the first parts of the problem, use the following form (to be justified in the last part of the problem):

   $$\mathbf{H}_\mathrm{L} = Ae^{\mathbf{i}K\mathbf{x}} + h.c. \tag{1}$$

   where $\mathbf{x}$ is the (center of mass) position operator of the atom in question; K is a constant (the photon wavenumber), and (for the purposes of the first parts of the problem) A is a constant. $+h.c.$ means 'plus the hermitian conjugate of the preceding stuff'.

   (a) Recalling that $\mathbf{x}$ (up to an additive constant) is part of a collection of coupled harmonic oscillators:

   $$\mathbf{x} = nx + \mathbf{q}_n$$

   evaluate the "vacuum persistence amplitude" $\langle 0| \mathbf{H}_\mathrm{L} |0\rangle$. This is the probability for the photon to be absorbed without production of any phonons. You will find the result of problem 3 useful.

(b) When the photon is absorbed in a zero-phonon process, where does its momentum go?

(c) From the previous calculation, you will find an expression that requires you to sum over wavenumbers. Show that in one spatial dimension, the probability for a zero-phonon transition is of the form

$$P_{\text{Mössbauer}} \propto e^{-\Gamma \ln L}$$

where L is the length of the chain and Γ is a function of other variables. Show that this infrared divergence is missing for the analogous model of crystalline solids with more than one spatial dimension. (Cultural remark: these amplitudes are called 'Debye-Waller factors').

(d) Convince yourself that a coupling $\mathbf{H}_{\text{L}}$ of the form (13) arises from the minimal coupling of the electromagnetic field to the constituent charges of the atom, after accounting for the transition made by the radiation field when the photon is absorbed by the atom. 'Minimal coupling' means replacing the momentum operator of the atom $\mathbf{p}$, with the gauge-invariant combination $\mathbf{p} \to \mathbf{p} + \mathbf{A}$. You will also need to recall the form of the quantized electromagnetic field in terms creation and annihilation operators for a photon of definite momentum K.