

Physics 215B QFT Winter 2019 Assignment 2

Due 12:30pm Wednesday, January 23, 2019

1. **Complex scalar field and antiparticles, practice with Noether’s theorem.**

[This problem is related to Peskin problem 2.2.] So far we’ve discussed scalar field theory with one *real* scalar field. The particles created by this field are their own antiparticles.

To understand this statement better, consider a scalar field theory in $d + 1$ dimensions with *two* real fields ϕ_1, ϕ_2 . Organize them into one complex field $\Phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + \mathbf{i}\phi_2)$, with $\Phi^* = \frac{1}{\sqrt{2}}(\phi_1 - \mathbf{i}\phi_2)$, and let

$$S[\Phi, \Phi^*] = \int d^d x dt \left(\frac{1}{2} \mu \partial_t \Phi \partial_t \Phi^* - \frac{1}{2} \mu v^2 \vec{\nabla} \Phi \cdot \vec{\nabla} \Phi^* - V(\Phi^* \Phi) \right).$$

(a) Show that

$$S[\Phi, \Phi^*] = \int \left(\sum_{i=1,2} \left(A (\partial_t \phi_i)^2 - B \vec{\nabla} \phi_i \cdot \vec{\nabla} \phi_i \right) - V(\phi_1^2 + \phi_2^2) \right),$$

and where A, B are constants you must determine. If $V(q^2) = \frac{1}{2} m^2 q^2$, notice that the action is just the sum of two copies of the action of the theory we considered previously.

(b) Show by doing the Legendre transformation that the associated hamiltonian is

$$\mathbf{H} = \int d^d x \left(C \Pi \Pi^* + D \vec{\nabla} \Phi \cdot \vec{\nabla} \Phi^* + V(\Phi \Phi^*) \right)$$

where C, D are constants you must determine, and the canonical momenta are

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{1}{2} \mu \dot{\Phi}^*, \quad \Pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^*} = \frac{1}{2} \mu \dot{\Phi}$$

with the Lagrangian density \mathcal{L} defined by $S = \int dt d^d x \mathcal{L}$.

- (c) This theory has a continuous symmetry under which $\Phi \rightarrow e^{i\alpha}\Phi$, $\Phi^* \rightarrow e^{-i\alpha}\Phi^*$ with α a real constant. Show that the action S does not change if I make this replacement. ¹
- (d) The existence of a continuous symmetry means a conserved charge – a hermitian operator which commutes with the Hamiltonian, which generates the symmetry. Show that

$$\mathbf{q} \equiv \int d^d x \mathbf{i} (\Phi^* \Pi^* - \Pi \Phi)$$

generates this transformation, in the sense that the change in the field under a transformation with infinitesimal α is

$$\delta\Phi = \mathbf{i}\alpha\Phi = -\mathbf{i}\alpha[\mathbf{q}, \Phi], \quad \text{and} \quad \delta\Phi^* = -\mathbf{i}\alpha\Phi^* = -\mathbf{i}\alpha[\mathbf{q}, \Phi^*].$$

Show that $[\mathbf{q}, \mathbf{H}] = 0$.

- (e) For the case where $V(\Phi\Phi^*) = m^2\Phi\Phi^*$ the hamiltonian is quadratic. Diagonalize it in terms of *two sets* of creation operators and annihilation operators. You should find something of the form

$$\Phi = \sqrt{\frac{\hbar}{2\mu}} \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{ikx} \mathbf{a}_k + e^{-ikx} \mathbf{b}_k^\dagger \right). \quad (1)$$

- (f) Write the canonical commutators

$$[\Phi(x), \Pi(x')] = \mathbf{i}\hbar\delta(x - x'), \quad [\Phi(x), \Pi^*(x')] = 0$$

(and the hermitian conjugate expressions) in terms of \mathbf{a} and \mathbf{b} .

- (g) Rewrite \mathbf{q} in terms of the mode operators.
- (h) Evaluate the charge of each type of particle created by \mathbf{a}_k^\dagger and \mathbf{b}_k^\dagger (*i.e.* find $[\mathbf{q}, \mathbf{a}^\dagger]$).

I claim that the particle created by \mathbf{a}^\dagger is the antiparticle of that created by \mathbf{b}^\dagger in the sense that they have opposite quantum numbers. This means that we can add terms to the hamiltonian by which they can annihilate each other, without breaking any symmetries. What might such a term look like?

¹This is called a U(1) symmetry: it is a unitary rotation (hence ‘U’) on a one-dimensional (hence ‘(1)’) complex vector. Notice that on the real components ϕ_1, ϕ_2 it acts as a two-dimensional rotation:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

The name for this group is SO(2). So U(1) is the same as SO(2).

2. **Non-Abelian currents.** Now, we make a big leap to *two* complex scalar fields, $\Phi_{\alpha=1,2}$, with

$$S[\Phi_\alpha] = \int d^d x dt \left(\frac{1}{2} \partial_\mu \Phi_\alpha^* \partial^\mu \Phi_\alpha - V(\Phi_\alpha^* \Phi_\alpha) \right)$$

Consider the objects

$$Q^i \equiv \frac{1}{2} \int d^d x \mathbf{i} \left(\Pi_\alpha^\dagger \sigma_{\alpha\beta}^i \Phi_\beta^\dagger \right) + h.c.$$

where $\sigma^{i=1,2,3}$ are the three Pauli matrices, and Π_α is the canonical field momentum for Φ_α .

- What symmetries do these charges generate (*i.e.* how do the fields transform)? Show that they are symmetries of S .
- If you want to, show that $[Q^i, H] = 0$, where H is the Hamiltonian.
- Evaluate $[Q^i, Q^j]$. Hence, non-Abelian.
- To complete the circle, find the Noether currents J_μ^i associated to the symmetry transformations you found in part **2a**.
- Generalize to the case of N scalar fields.

3. Schwinger-Dyson equations.

Consider the path integral

$$\int [D\phi] e^{\mathbf{i}S[\phi]}.$$

Using the fact that the integration measure is independent of the choice of field variable, we have

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} (\text{anything})$$

(as long as ‘anything’ doesn’t grow at large ϕ). So this equation says that we can integrate by parts in the functional integral.

(Why is this true? As always when questions about functional calculus arise, you should think of spacetime as discrete and hence the path integral measure as simply the product of integrals of the field value at each spacetime point, $\int [D\phi] \equiv \int \prod_x d\phi(x)$, this is just the statement that

$$0 = \int d\phi_x \frac{\partial}{\partial\phi_x} (\text{anything})$$

with $\phi_x \equiv \phi(x)$, *i.e.* that we can integrate by parts in an ordinary integral if there is no boundary of the integration region.)

This trivial-seeming set of equations (I say ‘set’ because we get to pick the ‘anything’) can be quite useful and they are called Schwinger-Dyson equations (or sometimes Ward identities). Unlike many of the other things we’ll discuss, they are true non-perturbatively, *i.e.* are really true, even at finite coupling. They provide a quantum implementation of the equations of motion.

(a) Evaluate the RHS of

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} (\phi(y) e^{iS[\phi]})$$

to conclude that

$$\left\langle \mathcal{T} \frac{\delta S}{\delta\phi(x)} \phi(y) \right\rangle = +i\delta(x-y). \quad (2)$$

(b) These Schwinger-Dyson equations are true in interacting field theories; to get some practice with them we consider here a free theory. Evaluate (2) for the case of a free massive real scalar field to show that the (two-point) time-ordered correlation functions of ϕ satisfy the equations of motion, most of the time. That is: the equations of motion are satisfied away from other operator insertions:

$$\langle \mathcal{T} (+\square_x + m^2) \phi(x)\phi(y) \rangle = -i\delta(x-y), \quad (3)$$

with $\square_x \equiv \partial_{x^\mu} \partial^{x^\mu}$.

(c) Find the generalization of (3) satisfied by (time-ordered) three-point functions of the free field ϕ .

4. **More about 0+0d field theory.** Here we will study a bit more some field theories with no dimensions at all, that is, integrals.

Consider the case where we put a label on the field: $q \rightarrow q_a, a = 1..N$. So we are studying

$$Z = \int \int_{-\infty}^{\infty} \prod_a dq_a e^{-S(q)}.$$

Let

$$S(q) = \frac{1}{2} q_a K_{ab} q_b + T_{abcd} q_a q_b q_c q_d$$

where T_{abcd} is a collection of couplings. Assume K_{ab} is a real symmetric matrix.

(a) Show that the propagator has the form:

$$a \text{ --- } b \equiv \langle q_a q_b \rangle_{T=0} = (K^{-1})_{ab} = \sum_k \phi_a(k)^* \frac{1}{k} \phi_b(k)$$

where $\{k\}$ are the eigenvalues of the matrix K and $\phi_a(k)$ are the eigenvectors in the a -basis.

- (b) Show that in a diagram with a loop, we must sum over the eigenvalue label k . (For definiteness, consider the order- g correction to the propagator.)
- (c) Consider the case where $K_{ab} = t(2\delta_{a,b} - \delta_{a,b+1} - \delta_{a+1,b})$, with periodic boundary conditions: $a + N \equiv a$. (Relative to the first version of this homework, I added a constant to K and changed the sign to make the thermodynamic limit $N \rightarrow \infty$ a little nicer.) Find the eigenvalues. Show that in this case if

$$T_{abcd}q_a q_b q_c q_d = \sum_a g q_a^4$$

the k -label is conserved at vertices, *i.e.* the vertex is accompanied by a delta function on the sum of the incoming eigenvalues.

- (d) (Bonus question) What is the more general condition on T_{abcd} in order that the k -label is conserved at vertices?
- (e) (Bonus question) Study the physics of the model described in 4c.

Back to the case without labels.

- (f) By a change of integration variable show that

$$Z = \int_{-\infty}^{\infty} dq e^{-S(q)}$$

with $S(q) = \frac{1}{2}m^2 q^2 + \frac{g}{4!}q^4$ is of the form

$$Z = \frac{1}{\sqrt{m^2}} \mathcal{Z}\left(\frac{m^4}{g}\right).$$

This means you can make your life easier by setting $m = 1$, without loss of generality.

- (g) Convince yourself (*e.g.* with Mathematica) that the integral really is expressible as a Bessel function.
- (h) It would be nice to find a better understanding for why the partition function of $(0+0)$ -dimensional ϕ^4 theory is a Bessel function. By part 4f we can set $g = 1$ and use $x \equiv m^2$ as the argument of \mathcal{Z} . Find a Schwinger-Dyson equation for this system which has the form of Bessel's equation for

$$K(x) = e^{-ax^2} (x^2)^{-1/4} Z(x)$$

for some constant a .

- (i) Make a plot of the perturbative approximations to the 'Green function' $G \equiv \langle q^2 \rangle$ as a function of g , truncated at orders 1 through 6 or so. Plot them against the exact answer.

(j) (Bonus problem) Show that $c_{n+1} \sim -\frac{2}{3}nc_n$ at large n (by brute force or by cleverness).

5. **Combinatorics from 0-dimensional QFT.** [This is a bonus problem. I will wait to post the solution until next week, in case you want to think about it more.]

Catalan numbers $C_n = \frac{(2n)!}{n!(n+1)!}$ arise as the answer to many combinatorics problems (beware: there is some disagreement about whether this is C_n or C_{n+1}).

One such problem is: count random walks on a 1d chain with $2n$ steps which start at 0 and end at 0 without crossing 0 in between.



Another such problem is: in how many ways can $2n$ (distinguishable) points on a circle be connected by chords which do not intersect within the circle.



Consider a zero-dimensional QFT with the following Feynman rules:

- There are two fields h and l .
- There is an $\sqrt{t}h^2l$ vertex in terms of a coupling t .
- The bare l propagator is 1.
- The bare h propagator is 1.
- All diagrams can be drawn on a piece of paper without crossing.²
- There are no loops of h .

The last two rules can be realized from a lagrangian by introducing a large N (below).

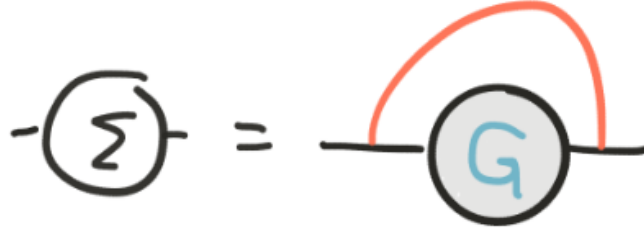
(a) Show that the full two-point green's function for h is

$$G(t) = \sum_n t^n C_n$$

the generating function of Catalan numbers.

²An annoying extra rule: All the l propagators must be on one side of the h propagators. You'll see in part 5f how to justify this.

- (b) Let $\Sigma(t)$ be the sum of diagrams with two h lines sticking out which may not be divided into two parts by cutting a single intermediate line. (This property is called 1PI (one-particle irreducible), and Σ is called the “1PI self-energy of h ”. We’ll use this manipulation all the time later on.) Show that $G(t) = \frac{1}{1-\Sigma(t)}$.
- (c) Argue by diagrams for the equation (sometimes this is also called a Schwinger-Dyson equation)



where Σ is the 1PI self-energy of h .

- (d) Solve this equation for the generating function $G(t)$.
- (e) If you are feeling ambitious, add another coupling N^{-1} which counts the crossings of the l propagators. The resulting numbers can be called Touchard-Riordan numbers.
- (f) How to realize the no-crossings rule? Consider

$$L = \frac{\sqrt{t}}{\sqrt{N}} l_{\alpha\beta} h_\alpha h_\beta + \sum_{\alpha,\beta} l_{\alpha\beta}^2 + \sum_{\alpha} h_\alpha^2$$

where $\alpha, \beta = 1 \dots N$. By counting index loops, show that the dominant diagrams at large N are the ones we kept above. Hint: to keep track of the index loops, introduce ('t Hooft's) double-line notation: since l is a matrix,

it's propagator looks like: $\alpha \text{ --- --- --- --- } \beta$, while the h propagator is just one index line $\alpha \text{ --- } \alpha$, and the vertex is ---!!--- . If you don't like my ascii diagrams, here are the respective pictures: $\langle l_{\alpha\beta} l_{\alpha\beta} \rangle = \begin{matrix} \alpha & \text{---} & \alpha \\ \beta & \text{---} & \beta \end{matrix}$,

$\langle h_\alpha h_\alpha \rangle = \alpha \text{ --- } \alpha$ and the hhl vertex is: $\begin{matrix} & & \parallel & & \\ \text{---} & & & & \text{---} \end{matrix}$.

- (g) Use properties of Catalan numbers to estimate the size of non-perturbative effects in this field theory.
- (h) There are many other examples like this. Another similar one is the relationship between symmetric functions and homogeneous products. A more different one is the enumeration of planar graphs. For that, see [BIPZ](#).