1. Brain-warmer.

Consider the field theory with action

\[ S[\phi] = \int d^{d+1}x \left( \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) - \frac{g}{3!} \phi^3 \right) . \]

(a) State the Feynman rules in position space for this theory.

(b) Draw the diagram which corrects the position-space two-point function of \( \phi \) at order \( g^2 \).

Actually, there’s a second diagram:

(c) Find the symmetry factor for this diagram.

The answer is 2 from exchanging the two internal lines. That is, if it were a momentum-space diagram, it would evaluate to

\[ = \frac{1}{2} \int d^{d+1}p \frac{1}{p^2 - m^2} \frac{i}{(k + p)^2 - m^2} . \]

To get the position-space Green’s function we can just Fourier transform.

(d) Check by explicit perturbation expansion that the symmetry factor is correct.

This is easier to do in position space.

\[ \langle \phi(x)\phi(0) \rangle = ... + \frac{1}{2!} \left( \frac{-ig}{3!} \right)^2 \int d^Dz_1 \int d^Dz_2 \langle \phi(x)\phi(0)\phi(z_1)\phi(z_2)^3 \rangle_0 + ... \]

The diagram is associated with the class of contractions where the external \( \phi \)s are contracted with \( \bar{\phi} \)s from the interaction vertex at different points; the remaining \( \phi(z) \)s are paired between the two points \( i.e.\) in this diagram,
we don’t contract \( \phi(z_1) \) with \( \phi(z_1) \). There are 6 choices for the partner of \( \phi(x) \), and then 3 choices for the partner of \( \phi(0) \). Then there are 2 ways to contract the remaining \( \phi(z_1)^2\phi(z_2)^2 \). This indeed gives

\[
\frac{1}{2} \cdot \frac{1}{6^2} \cdot 6 \cdot 3 \cdot 2 = \frac{1}{2}
\]

for the prefactor of the diagram.

2. **Propagator corrections in a solvable field theory.**

Consider a theory of a scalar field in \( D \) dimensions with action

\[
S = S_0 + S_1
\]

where

\[
S_0 = \int d^D x \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m_0^2 \phi^2 \right)
\]

and

\[
S_1 = - \int d^D x \frac{1}{2} \delta m^2 \phi^2.
\]

We have artificially decomposed the mass term into two parts. We will do perturbation theory in small \( \delta m^2 \), treating \( S_1 \) as an ‘interaction’ term. We wish to show that the organization of perturbation theory that we’ve seen lecture will correctly reassemble the mass term.

(a) Write down all the Feynman rules for this perturbation theory.

The propagator is the usual one, with amplitude \( \frac{1}{p^2 - m^2 + i\epsilon} \). The only ‘vertex’ is a 2-point vertex, which comes with amplitude \( -i\delta m^2 \).

(b) Determine the 1PI two-point function in this model, defined by

\[
i\Sigma \equiv \sum (\text{all 1PI diagrams with two nubbins}).
\]

The only Feynman diagrams are concatenations of free propagators and the mass insertion. The only 1PI diagram is the single mass insertion (with two nubbins sticking off).

\[
i\Sigma = -i\delta m^2.
\]

(c) Show that the (geometric) summation of the propagator corrections correctly produces the propagator that you would have used had we not split up \( m_0^2 + \delta m^2 \).
The geometric series for the two-point function results in

\[ G(p) = G_0(p) + G_0(p)i\Sigma G_0(p) + G_0(p)i\Sigma G_0(p)i\Sigma G_0(p) + ... = G_0(p) \left( \frac{1}{1 - i\Sigma G_0(p)} \right) \]

\[ = \frac{i}{p^2 - m_0^2 + i\epsilon} \left( \frac{1}{1 - i\Sigma p^2 - m_0^2 + i\epsilon} \right) \]

\[ = \frac{i}{p^2 - m_0^2 + \Sigma + i\epsilon} = \frac{i}{p^2 - m_0^2 - \delta m^2 + i\epsilon}. \]

3. **Reminder.** If you didn’t do the bonus problem on the last homework with the Catalan numbers, try it now.

Combining the relation

\[ i.e. \, \Sigma = tG \]

with the geometric series for the propagator in terms of the 1PI self-energy, \( G = \frac{1}{1 - \Sigma} \), we learn that

\[ G = \frac{1}{1 - tG}, \quad i.e. \, G(1 - tG) = 0 \]

which is a quadratic equation for \( G(t) \) whose solution is

\[ G = \frac{1 \pm \sqrt{1 - 4t}}{2t}. \]

How to decide which root is correct? When \( t \to 0 \), \( G \) is extremely, simple:

\[ G(t) = 1 \]

which means we must take the minus sign:

\[ G = \frac{1 - \sqrt{1 - 4t}}{2t}. \]
To see where the large-$N$ explains the pattern of diagrams we kept, compare the powers of $N$ between the following two diagrams:

\[ \propto \left( \frac{\sqrt{t}}{N} \right)^4 \cdot N^2 = t^2 \]

\[ \propto \left( \frac{\sqrt{t}}{N} \right)^4 \cdot N^0 = t^2 N^{-2}. \]

In the diagrams on the right, we regard the field as a matrix, draw its propagator with two lines, one for each index. In this double-line notation (due to ‘t Hooft), a loop represents a sum $\alpha = 1..N$ and hence a factor of $N$. These two diagrams have the same number of powers of the coupling $\sqrt{t/N}$, so the planar one (the one which can be drawn without crossing itself and has two more index loops) is bigger by two powers of $N$. With the coupling normalized as $\sqrt{t/N}$, the planar diagrams go like $N^0$, and all other diagrams vanish in the large-$N$ limit. Notice also that a diagram with an $h$ loop is suppressed by a power of $N$. This reproduces the apparently-artificial rules we introduced at the beginning of the problem.

4. Particle creation by an external source.

Consider the Hamiltonian

\[ H = H_0 + \int d^3x \left( -j(t, \vec{x}) \phi(x) \right) \]

where $H_0$ is the free Klein-Gordon Hamiltonian, $\phi$ is the Klein-Gordon field, and $j$ is a c-number scalar function. Here we will outline a calculation of particle-production probabilities (as a functional of $j$) using operator methods. Alternatively, if you prefer, you can do this whole problem using path integral methods. Even better, do both.

(a) Show that the probability that the source creates no particles is given by

\[ P(0) = \left| \langle 0 | T \{ e^{i \int d^4x j(x) \phi(x)} \} | 0 \rangle \right|^2. \]

The expression for the vacuum-to-vacuum amplitude follows from the Dyson formula for interaction-picture time evolution. (Or from the path integral.)

(b) Evaluate the term in $P(0)$ of order $j^2$, and show that $P(0) = 1 - \lambda + O(j^4)$ where

\[ \lambda = \int \frac{d^3p}{2E_p} |\tilde{j}(p)|^2. \]
We will show below that $\lambda = \langle N \rangle$ is the mean number of particles created by the source.

Expand the exponent to second order in $J$ The first order term vanishes. The second order term is the square of 

\[
\frac{i^2}{2!} \int d^{d+1}x_1 \int d^{d+1}x_2 J(x_1)J(x_2) \langle 0 | T \phi(x_1)\phi(x_2) | 0 \rangle \tag{1}
\]

\[
= -\frac{1}{2} \int d^{d+1}x_1 \int d^{d+1}x_2 J(x_1)J(x_2) \Delta_F(x_1 - x_2) \tag{2}
\]

\[
= -\frac{1}{2} \int d^{d+1}x_1 \int d^{d+1}x_2 J(x_1)J(x_2) \int d^{d+1}p \frac{i e^{ip(x_1-x_2)}}{p^2 - m^2 + i\epsilon} \tag{3}
\]

\[
= -\frac{1}{2} \int d^{d+1}x_1 \int d^{d+1}x_2 J(x_1)J(x_2) \int \frac{d^dp}{2E_p} e^{ip(x_1-x_2)}|_{p_0 = \omega_p} \tag{4}
\]

\[
= -\frac{1}{2} \int \frac{d^dp}{2E_p} \tilde{J}(p)\tilde{J}(-p)|_{p_0 = \omega_p} \tag{5}
\]

\[
= -\frac{1}{2} \int \frac{d^dp}{2E_p} |\tilde{J}(p)|^2. \tag{6}
\]

Therefore 

\[
P = |1 - \frac{1}{2} \lambda + \mathcal{O}(J^4)|^2 = 1 - \lambda + \mathcal{O}(J^4).
\]

(c) Represent the term computed in part 4b as a Feynman diagram. Now represent the whole perturbation series for $P(0)$ in terms of Feynman diagrams. Show that this series exponentiates, so that it can be summed exactly $P(0) = e^{-\lambda}$.

One way to do this is to use our formula from HW01 for the generating function of the gaussian distribution: $\langle e^{ikq} \rangle = e^{-k^2\langle qq \rangle/2}$. Alternatively, we can proceed as described in the next part.

(d) Compute the probability that the source creates one particle of momentum $k$. Perform this computation first to $\mathcal{O}(j)$ and then to all orders, using the trick of the previous part to sum the series.
First, the probability to produce zero particles is the square of

\[ A_0 = \left\langle 0 \left| e^{-i \int \phi V} \right| 0 \right\rangle \]

\[ = \sum_{n=0}^{\infty} \frac{(-i)^{2n}}{(2n)!} \int_{x_1} \cdots \int_{x_{2n}} J(x_1) \cdots J(x_{2n}) \left\langle 0 \right| \phi_1 \cdots \phi_{2n} \left| 0 \right\rangle \]

(Odd terms vanish.)

\[ = \sum_{n=0}^{\infty} \frac{(-i)^{2n}}{(2n)!} \frac{(2n)!}{n!2^n} \int d^{d+1} x \int d^{d+1} y J(x) \Delta_F(x - y)J(y) \]

\[ \equiv X \]

\[ = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} X^n = e^{-\frac{1}{2}X}. \]

In (9), we had to count the number of ways to pair 2n objects. Let me explain another way to get this answer. First choose a reference pairing. Others are obtained by permuting the 2n objects – there are (2n)! such permutations. A permutation is made from a list of swaps. If a permutation swaps two elements which are already paired, it does not produce a new pairing; half the permutations contain a particular swap. Similarly, if a permutation swaps pairs of objects which are already paired, it does nothing – there are n! of these. Altogether we conclude that there are \( \frac{(2n)!}{n!2^n} \) pairings. (This is sometimes called \( \frac{(2n)!}{n!2^n} \) = (2n)!!, double factorial. To see this, notice that once we choose one pair, there we are left with pairing 2n−2, so the number of pairs satisfies the recursion relation \( N_{2n} = 2n N_{2n-2} \) which is solved by the double factorial.)

Now, the probability amplitude to produce a finite number of particles m is

\[ A(k_1 \cdots k_m) = \left\langle k_1 \cdots k_m \left| e^{-i \int \phi V} \right| 0 \right\rangle \]

\[ = \sum_{\ell=m}^{\infty} \frac{(-i)^\ell}{(\ell)!} \int_{x_1} \cdots \int_{x_{\ell}} J(x_1) \cdots J(x_{\ell}) \prod_{i=1}^{m} \sqrt{2\omega_i} \left( 0 \right| a_{k_1} \cdots a_{k_m} \phi_1 \cdots \phi_{\ell} \left| 0 \right\rangle \]

Terms with \( \ell \neq 0 \) mod m vanish, and we need \( \ell \) at least m to get a nonzero answer. Here we have to choose m of the \( \ell \phi \)s to pair with the \( a_k \)s. There
are \( \binom{\ell}{m} = \frac{\ell(\ell-1)\cdots(\ell-(m+1))}{m!} \) ways to do this. Therefore, letting \( \ell \equiv m + 2n \)

\[
\mathcal{A}(k_1 \cdots k_m) = \sum_{n=0}^{\infty} \frac{(-i)^{2n} \ell(\ell-1)\cdots(\ell-(m+1))}{(2n+m)!} \frac{(2n)!}{m!} X^n.
\]

\[
= m! \sum_{n=0}^{\infty} \frac{(-i)^{2n}}{(2n)!} \frac{(2n)!}{n!2^n} X^n \prod_i \tilde{J}(k_i)
\]

\[
= e^{-\frac{X}{2}} \prod_i \tilde{J}(k_i).
\]

At step (15) I used

\[
\int d^{d+1}x J(x) \sqrt{2\omega_k} \langle 0 | a_k \phi(x) | 0 \rangle = \int d^{d+1}x J(x) \sqrt{2\omega_k} \int \frac{d^d p}{(2\omega_p)^{d/2}} e^{ipx} \langle 0 | a_k a_p^\dagger | 0 \rangle
\]

\[
= \int d^{d+1}x J(x) e^{ikx} = \tilde{J}(k).
\]

Actually, we should have known that the amplitude to make \( m \) particles has this form from the diagrams: there is just one diagram, which is just \( m \) propagators connecting the external \( k \)s to \( m \) sources (times the sum of vacuum bubbles, which is \( e^{-\frac{1}{2} \lambda} \)).

(e) Show that the probability of producing \( n \) particles is given by the Poisson distribution,

\[
P(n) = \frac{1}{n!} \lambda^n e^{-\lambda}.
\]

Now, the probability to produce \( m \) particles with momentum in the range \( k \) to \( k + dk \) is

\[
P_m(k) = \frac{1}{m!} |\mathcal{A}(k)|^2 \prod_{i=1}^{m} \frac{d^d k}{2E_{k_i}}.
\]

The \( \frac{1}{m!} \) is because the particles are identical – the configuration with momenta \( k_1, k_2 \ldots \) is the same as the one with \( k_2, k_1 \ldots \), so if we integrate over all \( k \), we must divide by \( m! \). Therefore, the probability to produce \( m \) particles
without specifying their momenta is
\[ P_m = \int_k P_m(k) = \int \frac{1}{m!} |A(k)|^2 \prod_{i=1}^m \frac{d^d k}{2E_{k_i}} = \frac{1}{m!} \lambda^m e^{-\lambda}. \]

(f) Prove the following facts about the Poisson distribution:
\[ \sum_{n=0}^\infty P(n) = 1, \quad \langle N \rangle \equiv \sum_{n=0}^\infty nP(n) = \lambda, \]
that is, \( P(n) \) is a probability distribution, and \( \langle N \rangle = \lambda \) as predicted. Compute the fluctuations in the number of particles produced \( \langle (N - \langle N \rangle)^2 \rangle \).

5. **A background field.** [This is a bonus problem.]
   Consider the following action for a real scalar field \( \Phi \):
   \[ S[\Phi] = \int d^{d+1}x \frac{1}{2} \left( \partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2 - g\phi(x)\Phi^2 \right). \]
   The last term here is a cubic coupling between \( \phi \) and \( \Phi \). But here we will treat \( \phi(x) \) as a fixed background field (analogous to \( j(x) \) on previous problems) which acts as a spacetime-dependent mass for the dynamical field \( \Phi \).

   (a) Show that the two-point Green’s function, \( G(x, y) \equiv \langle \Omega | T\Phi(x)\Phi(y) | \Omega \rangle \), satisfies the Schwinger-Dyson equation
   \[ \delta^{d+1}(x - y) = i \left( \partial^2 + m^2 + g\phi(x) \right) G(x, y). \]
   (19)

   (b) We would like to solve this differential equation. As a warmup, consider the case \( g = 0 \). Here is a trick: add a fictitious additional time direction \( T \)
   \[ (\partial_T + i \left( \partial^2 + m^2 \right) ) G(x, y, T) = \delta^{d+1}(x - y) \delta(T) \]
   (20)
   This is just a diffusion equation (in \( d+2 \) dimensions and with a funny factor of \( i! \)). Show that given a solution to (20), you can find the solution of (19) with \( g = 0 \) by
   \[ G(x, y) = \int_0^\infty dTG(x, y, T). \]
   (21)

   (c) Show that the solution to the diffusion equation (20) for an infinitesimal time step \( T \) is
   \[ G(x, y, T) = \frac{1}{(2\pi T)^\alpha} e^{a (x-y)^2 + b m^2 T}. \]
   (22)
   Find \( \alpha, a, b \). Use this to construct the path integral representation
   \[ G(x, y, T) = \int_{x(0) = x}^{x(T) = y} [Dx] e^{-\int_0^T d\tau (\dot{x}^\mu \dot{x}_\mu + m^2)}. \]
(d) For the case of constant $m^2$, the infinitesimal solution (22) actually works for finite $T$. Show by differentiation that plugging (22) into (21) gives an integral representation of the free Klein-Gordon propagator.

(e) Now let $g \neq 0$ and suppose that $\phi$ is slowly varying. Generalize the path integral representation to include the dependence on $\phi$.

(f) Consider a non-relativistic situation, where the spacetime points $x$ and $y$ are separated by a timelike distance large compared to $1/m$. Justify and use stationary-phase methods to show that the dominant contribution to the path integral is a straight-line trajectory between the two points $x$ and $y$. Evaluate the resulting amplitude as a functional of $\phi(x)$. This calculation shows that the heavy particle made by the field $\Phi$ can be treated as a source for $\phi$ propagating on a fixed path in spacetime.

(g) Redo the problem for a charged scalar field, $\Phi$ in the background of a vector potential $A_\mu$, with

$$S[\Phi] = \int d^{d+1}x \frac{1}{2} (D_\mu \Phi^* D^\mu \Phi - m^2 \Phi^2), \quad D_\mu \Phi \equiv \partial_\mu \Phi - iA_\mu \Phi.$$ 

It will help to recall that the action of a classical charged particle is $\int d\tau (\dot{x}^2 + \dot{x}^\mu A_\mu(x))$. See this review by Peskin.

6. Classical Maxwell theory. Classical electromagnetism follows from the action

$$S[A] = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu \right), \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

(a) Derive Maxwell’s equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables

$$0 = \frac{\delta S[A]}{\delta A_\mu(x)}.$$

Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

By the chain rule and product rule,

$$\frac{\delta}{\delta A_\mu(x)} \int d^4y \left( -\frac{1}{4} F_{\rho\sigma}(y) F^{\rho\sigma}(y) \right) = -\frac{1}{2} \int d^4y \frac{\delta F^{\rho\sigma}(y)}{\delta A_\mu(x)} F^{\rho\sigma}(y)$$

$$\frac{\delta F^{\rho\sigma}(y)}{\delta A_\mu(x)} = \partial_\rho \delta(x-y) \delta_{\mu\sigma} - (\rho \leftrightarrow \sigma)$$
So the EOM is
\[ \partial_\nu F^{\mu\sigma} = j^\mu. \]
The components of this are the four equations
\[ \nabla \cdot \vec{E} = j^0, \quad \nabla \times \vec{B} + \partial_t \vec{E} = \vec{j}. \]
The other four Maxwell equations are in the constraint
\[ 0 = \epsilon^{\mu\rho\sigma\nu} \partial_\rho F_{\sigma\nu} \]
(which follows automatically from \( F = dA \) by equality of mixed partials). Written in terms of \( E \) and \( B \), the \( \mu = t \) component gives \( \nabla \cdot \vec{v} B = 0 \) and the spatial components give \( \nabla \times \vec{E} = \partial_t \vec{B} \).

(b) Construct the energy-momentum tensor for this theory. Note that the usual procedure
\[ T^\mu_\nu = \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\nu \phi - L \delta^\mu_\nu \]
does not result in a symmetric tensor. To remedy that, we can add to \( T^\mu_\nu \) a term of the form \( \partial_\lambda K^{\lambda \mu\nu} \), where \( K^{\lambda \mu\nu} \) is antisymmetric in its first two indices. Such an object is automatically divergenceless, so
\[ \hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda \mu\nu} \]
is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with
\[ K^{\lambda \mu\nu} = F^{\mu\lambda} A_\nu, \]
leads to an energy-momentum tensor \( \hat{T} \) that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:
\[ \mathcal{E} = \frac{1}{2} (E^2 + B^2), \quad \vec{S} = \vec{E} \times \vec{B}. \]

(c) [Bonus question] What modification of the action will produce the improved stress tensor as the Noether current for translation symmetry?
Actually, I’m not sure that it is possible to accomplish this.