University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2019
Assignment 6

Due 12:30pm Wednesday, February 20, 2019

1. **Brain-warmer.**

   Show that we did the right thing in the numerator of the electron self-energy: use the Clifford algebra to show that

$$\gamma^\mu \left(x\not{p} + m_0\right) \gamma_\mu = -2x\not{p} + 4m_0.$$

2. **An example of renormalization in classical physics.**

   Consider a classical field in $D+2$ spacetime dimensions coupled to an *impurity* (or defect or brane) in D dimensions, located at $X = (x^\mu, 0, 0)$. Suppose the field has a self-interaction which is localized on the defect. For definiteness and calculability, we'll consider the simple (quadratic) action

$$S[\phi] = \int d^{D+2}X \left(\frac{1}{2}\partial_\mu\phi(X)\partial^\mu\phi(X) + g\delta^2(\vec{x}_\perp)\phi^2(X)\right).$$

   (a) What is the mass dimension of the coupling g? This is why I picked a codimension[1]-two defect.

   (b) Find the equation of motion for ϕ. Where have you seen an equation like this before?

   (c) We will study the propagator for the field in a mixed representation:

$$G_k(x,y) \equiv \langle\phi(k,x)\phi(-k,y)\rangle = \int d^D z\ e^{\mathbf{i}k_\mu z^\mu} \langle\phi(z,x)\phi(0,y)\rangle$$

   – *i.e.* we go to momentum space in the directions in which translation symmetry is preserved by the defect. Find and evaluate the diagrams contributing to $G_k(x,y)$ in terms of the free propagator $D_k(x,y) \equiv \langle\phi(k,x)\phi(-k,y)\rangle_{g=0}$. (We will not need the full form of $D_k(x,y)$.) Sum the series.

   (d) You should find that your answer to part 2c depends on $D_k(0,0)$, which is divergent. This divergence arises from the fact that we are treating the defect as infinitely thin, as a pointlike object – the δ^2-function in the interaction involves arbitrarily short wavelengths. In general, as usual, we

[1] An impurity whose position requires specification of p coordinates has codimension p.

must really be agnostic about the short-distance structure of things. To reflect this, we introduce a regulator. For example, we can replace the fourier representation of $D_k(0,0)$ with the cutoff version

$$D_k(0,0;\Lambda) = \int_0^\Lambda đ^2 q \frac{e^{\mathbf{i}q\cdot 0}}{k^2+q^2}.$$

Do the integral.

(e) Now we renormalize. We will let the *bare coupling* g (the one which appears in the Lagrangian, and in the series from part 2c) depend on the cutoff $g = g(\Lambda)$. We wish to eliminate $g(\Lambda)$ in our expressions in favor of some measurable quantity. To do this, we impose a renormalization condition: choose some reference scale μ, and demand that

$$G_\mu(x,y) \overset{!}{=} D_\mu(x,y) - g(\mu) D_\mu(x,0) D_\mu(0,y). \tag{1}$$

This equation defines $g(\mu)$, which we regard as a physical quantity. Show that (1) is satisfied if we let $g(\Lambda) = g(\mu)Z$, with

$$Z = \frac{1}{1 - \frac{g(\mu)}{4\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right)}.$$

(f) Find the beta function for g,

$$\beta_g(g) \equiv \mu \frac{dg(\mu)}{d\mu},$$

and solve the resulting RG equation for $g(\mu)$ in terms of some initial condition $g(\mu_0)$. Does the coupling get weaker or stronger in the UV?

3. **Pauli-Villars practice.**

Consider a field theory of two scalar fields with

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\Phi\Box\Phi - \frac{1}{2}M^2\Phi^2 - g\phi\Phi^2 + \text{counterterms}.$$

Compute the one-loop contribution to the self-energy of Φ. Use a Pauli-Villars regulator – introduce a second copy of the ϕ field of mass Λ with the wrong-sign propagator.

Determine the counterterms required to impose that the Φ propagator has a pole at $p^2 = M^2$ with residue 1.