1. **Brain-warmer.**

Show that we did the right thing in the numerator of the electron self-energy: use the Clifford algebra to show that

\[ \gamma^\mu (xp + m_0) \gamma_\mu = -2xp + 4m_0. \]

2. **An example of renormalization in classical physics.**

Consider a classical field in \( D + 2 \) spacetime dimensions coupled to an impurity (or defect or brane) in \( D \) dimensions, located at \( X = (x^\mu, 0, 0) \). Suppose the field has a self-interaction which is localized on the defect. For definiteness and calculability, we’ll consider the simple (quadratic) action

\[ S[\phi] = \int d^{D+2}X \left( \frac{1}{2} \partial_\mu \phi(X) \partial^\mu \phi(X) + g\delta^2(x_\perp) \phi^2(X) \right). \]

(a) What is the mass dimension of the coupling \( g \)? This is why I picked a codimension\(^1\)-two defect.

(b) Find the equation of motion for \( \phi \). Where have you seen an equation like this before?

(c) We will study the propagator for the field in a mixed representation:

\[ G_k(x, y) \equiv \langle \phi(k, x) \phi(-k, y) \rangle = \int d^Dz \ e^{ik_\mu z^\mu} \langle \phi(z, x) \phi(0, y) \rangle \]

\( - \) i.e. we go to momentum space in the directions in which translation symmetry is preserved by the defect. Find and evaluate the diagrams contributing to \( G_k(x, y) \) in terms of the free propagator \( D_k(x, y) \equiv \langle \phi(k, x) \phi(-k, y) \rangle_{g=0} \).

We will not need the full form of \( D_k(x, y) \). Sum the series.

(d) You should find that your answer to part 2c depends on \( D_k(0, 0) \), which is divergent. This divergence arises from the fact that we are treating the defect as infinitely thin, as a pointlike object – the \( \delta^2 \)-function in the interaction involves arbitrarily short wavelengths. In general, as usual, we

\[^1\text{An impurity whose position requires specification of } p \text{ coordinates has codimension } p.\]
must really be agnostic about the short-distance structure of things. To reflect this, we introduce a regulator. For example, we can replace the Fourier representation of \(D_k(0,0)\) with the cutoff version

\[
D_k(0,0; \Lambda) = \int_0^\Lambda d^2q \frac{e^{i k \cdot 0}}{k^2 + q^2}.
\]

Do the integral.

(e) Now we renormalize. We will let the bare coupling \(g\) (the one which appears in the Lagrangian, and in the series from part 2c) depend on the cutoff \(g = g(\Lambda)\). We wish to eliminate \(g(\Lambda)\) in our expressions in favor of some measurable quantity. To do this, we impose a renormalization condition: choose some reference scale \(\mu\), and demand that

\[
G_\mu(x,y) = D_\mu(x,y) - g(\mu)D_\mu(x,0)D_\mu(0,y).
\]

This equation defines \(g(\mu)\), which we regard as a physical quantity. Show that (1) is satisfied if we let \(g(\Lambda) = g(\mu)Z\), with

\[
Z = \frac{1}{1 - \frac{g(\mu)}{4\pi} \ln \left( \frac{\Lambda^2}{\mu^2} \right)}.
\]

(f) Find the beta function for \(g\),

\[
\beta_g(g) \equiv \mu \frac{dg(\mu)}{d\mu},
\]

and solve the resulting RG equation for \(g(\mu)\) in terms of some initial condition \(g(\mu_0)\). Does the coupling get weaker or stronger in the UV?


Consider a field theory of two scalar fields with

\[
\mathcal{L} = -\frac{1}{2} \phi \Box \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \Phi \Box \Phi - \frac{1}{2} M^2 \Phi^2 - g \phi \Phi^2 + \text{counterterms}.
\]

Compute the one-loop contribution to the self-energy of \(\Phi\). Use a Pauli-Villars regulator – introduce a second copy of the \(\phi\) field of mass \(\Lambda\) with the wrong-sign propagator.

Determine the counterterms required to impose that the \(\Phi\) propagator has a pole at \(p^2 = M^2\) with residue 1.