University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2019
Assignment 7

Due 12:30pm Wednesday, February 27, 2019

1. **Brain-warmer.** Prove the Gordon identities

$$\bar{u}_2\left(q^\nu \sigma_{\mu\nu}\right) u_1 = \mathbf{i}\bar{u}_2\left((p_1+p_2)_\mu - (m_1+m_2)\gamma_\mu\right) u_1$$

and

$$\bar{u}_2\left((p_1+p_2)^\nu \sigma_{\mu\nu}\right) u_1 = \mathbf{i}\bar{u}_2\left((p_2-p_1)_\mu - (m_2-m_2)\gamma_\mu\right) u_1$$

where $q \equiv p_2 - p_1$ and $\not{p}_1 u_1 = m_1 u_1, \bar{u}_2 \not{p}_2 = m_2 \bar{u}_2$, using the definitions and the Clifford algebra.

2. **Bosons have worse UV behavior than fermions.**

Consider the Yukawa theory

$$S[\phi,\psi] = -\int d^D x \left(\frac{1}{2}\phi\left(\Box + m_\phi\right)\phi + \bar{\psi}\left(-\not{\partial} + m_\psi\right)\psi + y\phi\bar{\psi}\psi + \frac{g}{4!}\phi^4\right) + \text{counterterms}.$$

(a) Show that the superficial degree of divergence for a diagram $\mathcal{A}$ with B_E external scalars and F_E external fermions is

$$D_{\mathcal{A}} = D + (D-4)\left(V_g + \frac{1}{2}V_y\right) + B_E\left(\frac{2-D}{2}\right) + F_E\left(\frac{1-D}{2}\right) \qquad (1)$$

where V_g and v_y are the number of ϕ^4 and $\phi\bar{\psi}\psi$ vertices respectively.

All the discussion below is about one loop diagrams.

(b) Draw the diagrams contributing to the self energy of both the scalar and the spinor in the Yukawa theory.

(c) Find the superficial degree of divergence for the scalar self-energy amplitude and the spinor self-energy amplitude.

(d) In the case of $D = 3+1$ spacetime dimensions, show that (with a cutoff on the Euclidean momenta) the spinor self-energy is actually only logarithmically divergent. (This type of thing is one reason for the adjective 'superficial'.)

Hint: the amplitude can be parametrized as follows: if the external momentum is p^μ, it is

$$\mathcal{M}(p) = A(p^2)\not{p} + B(p^2).$$

Show that $B(p^2)$ vanishes when $m_\psi = 0$.

3. **Dimension-dependence of dimensions of couplings.**

   (a) In what number of space dimensions does a four-fermion interaction such as $G\bar{\psi}\psi\bar{\psi}\psi$ have a chance to be renormalizable? Assume Lorentz invariance.

   [optional] Generalize the formula (1) for $D_{\mathcal{A}}$ to include a number V_G of four-fermion vertices.

   (b) If we violate Lorentz invariance the story changes. Consider a non-relativistic theory with kinetic terms of the form $\int dt d^d x \left(\psi^\dagger \left(\mathbf{i}\partial_t - D\nabla^2\right)\psi\right)$. (Here D is a dimensionful constant. In a relativistic theory we relate dimensions of time and space by setting the speed of light to one; here, there is no such thing, and we can choose units to set D to one.) For what number of space dimensions might the four-fermion coupling be renormalizable?

   (c) In the previous example, the scale transformation preserving the kinetic terms acted by $t \to \lambda^2 t, x \to \lambda x$. More generally, the relative scaling of space and time is called the *dynamical exponent* z ($z = 2$ in the previous example). Suppose that the kinetic terms are first order in time and quadratic in the fields. Ignoring difficulties of writing local quadratic spatial kinetic terms, what is the relationship between d and z which gives scale invariant quartic interactions? What if the kinetic terms are second order in time (as for scalar fields)?

4. **Symmetry is attractive.** Consider a field theory in $D = 3+1$ with two massless (for simplicity) scalar fields which interact via the interaction

$$V = -\frac{g}{4!}\left(\phi_1^4 + \phi_2^4\right) - \frac{2\lambda}{4!}\phi_1^2\phi_2^2.$$

   (a) Show that when $\lambda = g$ the model possesses an $\mathsf{O}(2)$ symmetry.

   (b) Will you need a counterterm of the form $\phi_1\phi_2$ or $\phi_1\Box\phi_2$ (for general g, λ)? If not, why not?

   (c) Renormalize the theory to one loop order by regularizing (for example with Pauli Villars), adding the necessary counterterms, and imposing a renormalization condition on the masses and $2 \to 2$ scattering amplitudes at some values of the kinematical variables s_0, t_0, u_0.

   (d) Consider the limit of low energies, *i.e.* when $s_0, t_0, u_0 \ll \Lambda^2$ where Λ is the cutoff scale. Tune the location of the poles in both propagators to $p^2 = 0$. Show that the coupling goes to the $\mathsf{O}(2)$-symmetric value if it starts nearby (nearby means $\lambda/g < 3$).

5. **The magnetic moment of a Dirac fermion.** [This problem is optional, but highly recommended, especially if you didn't get all of the problem with the γ^5 interactions.] In this problem we consider the hamiltonian density

$$\mathfrak{h}_I = q\bar{\Psi}\gamma^\mu\Psi A_\mu \ .$$

As we discussed, this describes a local, Lorentz invariant, and gauge invariant interaction between a Dirac fermion field Ψ and a vector potential A_μ. We will treat the vector potential, representing the electromagnetic field, as a fixed, classical background field.

Define single-particle states of the Dirac field by $\langle 0| \Psi(x) |\vec{p}, s\rangle = e^{-\mathbf{i}px} u^s(p)$. We wish to show that these particles have a magnetic dipole moment, in the sense that in their rest frame, their (single-particle) hamiltonian has a term $h_{NR} \ni \mu_B \vec{S} \cdot \vec{B}$ where $\vec{S} = \frac{1}{2}\vec{\sigma}$ is the particle's spin operator.

(a) q is a real number. What is required of A_μ for $H_I = \int d^3x \mathfrak{h}_I$ to be hermitian?

(b) How must A_μ transform under parity P and charge conjugation C in order for H_I to be invariant? How do the electric and magnetic fields transform? Show that this allows for a magnetic dipole moment but not an electric dipole moment.

(c) Show that in the non-relativistic limit

$$\bar{u}(p')\gamma^{\mu\nu}u'(p)F_{\mu\nu} = a\xi^\dagger\sigma \cdot \vec{B}\xi'$$

for some constant a (find a). Recall that $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Here u, u' are positive-energy solutions of the Dirac equation with mass m and

$$u = \begin{pmatrix} \sqrt{\sigma \cdot p}\xi \\ \sqrt{\bar{\sigma} \cdot p}\xi \end{pmatrix}, \ u' = \begin{pmatrix} \sqrt{\sigma \cdot p'}\xi' \\ \sqrt{\bar{\sigma} \cdot p'}\xi' \end{pmatrix}.$$

(d) Suppose that A_μ describes a magnetic field $\vec{B}$ which is uniform in space and time.

Show that in the non-relativistic limit

$$\langle \vec{p}', s'| H_I |\vec{p}, s\rangle = \not{\delta}^3 (\vec{p} - \vec{p}')\, h(\xi, \xi', \vec{B})$$

and find the function $h(\xi, \xi', \vec{B})$. You may wish to use the Gordon identity. Rewrite the result in terms of single-particle states with non-relativistic normalization (*i.e.* $\langle \vec{p}|\vec{p}'\rangle_{NR} = \not{\delta}^3(p - p')$). Interpret h as a non-relativistic hamiltonian term saying that the gyromagnetic ratio of the electron is $-g\frac{|q|}{2m}$ with $g = 2$.

(e) [more optional] How does the result change if we add the term

$$\Delta H = \frac{c}{M}\bar{\Psi}F_{\mu\nu}[\gamma^\mu, \gamma^\nu]\Psi \ \ ?$$