1. **Non-Abelian currents.** In previous homework, we studied a complex scalar field. Now, we make a big leap to two complex scalar fields, $\Phi_\alpha = 1, 2$, with

$$S[\Phi_\alpha] = \int d^dxd\tau \left( \frac{1}{2} \partial_\mu \Phi^*_\alpha \partial^\mu \Phi_\alpha - V(\Phi^*_\alpha \Phi_\alpha) \right)$$

Consider the objects

$$Q^i = \frac{1}{2} \int d^dxdy i \left( \Pi^\dagger_{\alpha} \sigma^i_{\alpha\beta} \Phi^\dagger_{\beta} \right) + h.c.$$ 

where $\sigma^{i=1,2,3}$ are the three Pauli matrices.

(a) What symmetries do these charges generate (i.e. how do the fields transform)? Show that they are symmetries of $S$.

$$\delta \epsilon \Phi_\alpha(x) = -i[\epsilon_i Q^i, \Phi_\alpha(x)] = \frac{1}{2} \epsilon_i \int d^dy \left[ \Pi_{\beta}(y) \sigma^i_{\alpha\beta} \Phi(y), \Phi_\alpha(x) \right]$$

where only the $+h.c.$ term in $Q$ contributes since $[\Pi^i, \Phi] = [\Phi^i, \Phi] = 0$; using $[\Phi(y), \Phi(x)] = 0, [\Pi_{\beta}(y), \Phi_\alpha(x)] = \delta_{\alpha\beta}(-i\delta^d(x-y))$, this is

$$\delta \epsilon \Phi_\alpha(x) = -\frac{1}{2} i \epsilon_i \sigma^i_{\alpha\beta} \Phi_\beta(x).$$

This is the infinitesimal version of an SU(2) rotation on a doublet:

$$\Phi \rightarrow \Phi - \frac{1}{2} i \hat{\epsilon} \cdot \vec{\sigma} \Phi + O(\epsilon^2) = e^{-\frac{1}{2} i \hat{\epsilon} \cdot \vec{\sigma}} \Phi$$

just like the rotation of a spin-$\frac{1}{2}$ wavefunction. This is indeed a symmetry of $S$ (note that $\epsilon$ is independent of space and time) since the fields only appear in the combination $\Phi^*_\alpha \Phi_\alpha$ which is an SU(2) singlet.

(b) If you want to, show that $[Q^i, H] = 0$, where $H$ is the Hamiltonian.
(c) Evaluate \([Q^i, Q^j]\). Hence, non-Abelian.

\[
[Q^i, Q^j] = \frac{1}{4} i^2 \int d^d x \int d^d y \left( \left[ \left( \Pi^i_{\alpha} \sigma_{\alpha \beta}^i \Phi^\dagger_{\beta} \right)(x), \left( \Pi^j_{\alpha'} \sigma_{\alpha' \beta'}^j \Phi^\dagger_{\beta'} \right)(y) \right] + \left[ \left(\Pi^i_{\alpha} \sigma_{\alpha \beta}^i \Phi^\dagger_{\beta} \right)(x), \left( \Pi^j_{\alpha'} \sigma_{\alpha' \beta'}^j \Phi^\dagger_{\beta'} \right) \right] \right)
\]

Using the identity \([AB, C] = A[B, C] + [A, C]B\), this is

\[
[Q^i, Q^j] = -\frac{1}{4} \int d^d x \left( \Pi^i_{\alpha}(x) \left( \sigma^i_{\alpha \beta} \sigma^j_{\beta \gamma} \right) \Phi^\dagger_{\gamma}(x) + h.c. \right) = -\frac{1}{2} i \epsilon^{ijk} Q^k
\]

This is the \(\text{su}(2)\) algebra.

(d) To complete the circle, find the Noether currents \(J^i_\mu\) associated to the symmetry transformations you found in part 1a.

(e) Generalize to the case of \(N\) scalar fields.

In this case the Pauli matrices are replaced by a basis of the \(N^2 - 1\) hermitian \(N \times N\) matrices. Only \(N - 1\) of them can be diagonal. The result is a representation of the Lie algebra \(\text{su}(N)\).

2. More about 0+0d field theory. Here we will study a bit more some field theories with no dimensions at all, that is, integrals.

Consider the case where we put a label on the field: \(q \rightarrow q_a, a = 1..N\). So we are studying

\[
Z = \int \prod_a dq_a \ e^{-S(q)}
\]

Let

\[
S(q) = \frac{1}{2} q_a K_{ab} q_b + T_{abcd} q_a q_b q_c q_d
\]

where \(T_{abcd}\) is a collection of couplings. Assume \(K_{ab}\) is a real symmetric matrix.

(a) Show that the propagator has the form:

\[
a - - - - - b \equiv \langle q_a q_b \rangle_{T=0} = (K^{-1})_{ab} = \sum_k \phi_a(k)^* \frac{1}{k} \phi_b(k)
\]

where \(\{k\}\) are the eigenvalues of the matrix \(K\) and \(\phi_a(k)\) are the eigenvectors in the \(a\)-basis.

This is the spectral decomposition of the operator \(K^{-1}\), written in the \(a, b\) basis. That is, the operator \(K = \sum_k |k\rangle \langle k| k\), and its matrix representation
in the \(ab\) basis is \(K_{ab} = \langle a \mid K \mid b \rangle\). The spectral representation of the inverse is \(K = \sum_k \langle k \mid k \rangle k^{-1}\) and its matrix elements in the \(ab\) basis are of the given form with \(\langle a \mid k \rangle \equiv \phi_a^*(k)\).

(b) Show that in a diagram with a loop, we must sum over the eigenvalue label \(k\). (For definiteness, consider the order-\(g\) correction to the propagator.) Compare the diagrammatic expansion of the propagator with the explicit expansion

\[
\langle q_a q_b \rangle = Z^{-1} \int \prod dqq_a q_b e^{-S_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (Tqqq)_{qqqq}^n.
\]

The leading correction to the propagator, for example, is proportional to

\[
\propto \sum_{cdef} T_{cdef} K_{ac}^{-1} K_{bd}^{-1} K_{ef}^{-1}.
\]

In the eigenbasis of the propagator, this is

\[
\langle q_k q_{k'} \rangle = \delta_{kk'} \frac{1}{k} - \sum_{pqrs} \delta_{kp} \frac{1}{k} \delta_{k'q} \frac{1}{q} T_{pqrs} \delta_{rs} \frac{1}{r} + O(T^2) = \delta_{kk'} \frac{1}{k} \frac{1}{kk'} \sum_r T_{kk'r} \frac{1}{r} + O(T^2)
\]

so there is a free sum over the eigenvalue label \(r\).

(c) Consider the case where \(K_{ab} = t \left( \delta_{a,b+1} + \delta_{a+1,b} \right)\), with periodic boundary conditions: \(a + N \equiv a\). Find the eigenvalues. Show that in this case if

\[
T_{abcd} q_a q_b q_c q_d = \sum_a gq_a^4
\]

the \(k\)-label is conserved at vertices, i.e. the vertex is accompanied by a delta function on the sum of the incoming eigenvalues.

This matrix \(K\) is the Hamiltonian for a particle hopping on a discrete circle, let's write this as

\[
K \ket{n} = \ket{n + 1} - \ket{n - 1},
\]

where \(\ket{n + N} = \ket{n}\). This has translation invariance, so \([K, T]\) where \(T \ket{n} = \ket{n + 1}\). More specifically, \(K = T + T^{-1}\). The eigenvectors of translation are \(\ket{\theta} = \sum_{n=1}^n e^{i n \theta} \ket{n}\) since

\[
T \ket{\theta} = e^{-i \theta} \ket{\theta}.
\]

Hence the eigenvalues of \(K\) are of the form \(2 \cos \theta\). Since \(n \in \mathbb{Z}\), \(\theta \equiv \theta + 2\pi\).

Since \(n \equiv n + N\), only \(\theta = 2\pi m / N, m \in \mathbb{Z}\) are allowed. Hence the eigenvalues of \(K\) are \(2 \cos 2\pi m / N, m = 1..N\).
(d) (Bonus question) What is the more general condition on $T_{abcd}$ in order that the $k$-label is conserved at vertices?

(e) (Bonus question) Study the physics of the model described in 2c.

Back to the case without labels.

(f) By a change of integration variable show that

$$Z = \int_{-\infty}^{\infty} dq \ e^{-S(q)}$$

with $S(q) = \frac{1}{2} m^2 q^2 + g q^4$ is of the form

$$Z = \frac{1}{\sqrt{m^2}} Z \left( \frac{g}{m^4} \right).$$

This means you can make your life easier by setting $m = 1$, without loss of generality.

(g) Convince yourself (e.g. with Mathematica) that the integral really is expressible as a Bessel function.

(h) It would be nice to find a better understanding for why the partition function of $(0+0)$-dimensional $\phi^4$ theory is a Bessel function. Then find a Schwinger-Dyson equation for this system which has the form of Bessel’s equation for

$$K(y) \equiv \frac{1}{\sqrt{y}} e^{-a/y} Z \left( \frac{1}{y} \right)$$

for some constant $a$. (Hint: I found it more convenient to set $g = 1$ for this part and use $\xi \equiv m^2$ as the argument. If you get stuck I can tell you what function to choose for the ‘anything’ in the S-D equation.)

You’ll get the right equation by studying

$$0 = \int dq \frac{\partial}{\partial q} (qe^{-S(q)})$$

and using $\int dq q^2 e^{-S} = \partial_{m^2} Z$, $\int dq q^4 e^{-S} = \partial_{m^2}^2 Z$. More explicitly: Letting $x = m^2$ and setting $g = 1$ (which we can do wlog by part (f)) gives

$$0 = (1 + 2x \partial_x - 16 \partial_x^2) Z(x).$$

If you plug this into mathematica it will tell you that the solution is what I said above, i.e. $Z(x) = e^{x^2/32} (x^2)^{1/4} K_{-1/4}(x^2/32)$ where $K$ is what Mathematica calls BesselK.
(i) Make a plot of the perturbative approximations to the ‘Green function’ $G \equiv \langle q^2 \rangle$ as a function of $g$, truncated at orders 1 through 6 or so. Plot them against the exact answer.

You can see that by $n = 6$th order, they are getting worse for quite small values of $g$.

(j) (Bonus problem) Show that $c_{n+1} \sim -\frac{2}{3}nc_n$ at large $n$ (by brute force or by cleverness).

3. Combinatorics from 0-dimensional QFT. [This is a bonus problem.]

Catalan numbers $C_n = \frac{(2n)!}{n!(n+1)!}$ arise as the answer to many combinatorics problems (beware: there is some disagreement about whether this is $C_n$ or $C_{n+1}$).

One such problem is: count random walks on a 1d chain with $2n$ steps which start at 0 and end at 0 without crossing 0 in between.

Another such problem is: in how many ways can $2n$ (distinguishable) points on a circle be connected by chords which do not intersect within the circle.

Consider a zero-dimensional QFT with the following Feynman rules:

- There are two fields $h$ and $l$.
- There is an $\sqrt{t}h^2l$ vertex in terms of a coupling $t$.
- The bare $l$ propagator is 1.
- The bare $h$ propagator is 1.
- All diagrams can be drawn on a piece of paper without crossing.\(^1\)

\(^1\)An annoying extra rule: All the $l$ propagators must be on one side of the $h$ propagators. You’ll see in part 3f how to justify this.
• There are no loops of $h$.

The last two rules can be realized from a lagrangian by introducing a large $N$
(below).

(a) Show that the full two-point green’s function for $h$ is

$$G(t) = \sum_n t^n C_n$$

the generating function of Catalan numbers.

(b) Let $\Sigma(t)$ be the sum of diagrams with two $h$ lines sticking out which may
not be divided into two parts by cutting a single intermediate line. (This
property is called 1PI (one-particle irreducible), and $\Sigma$ is called the “1PI
self-energy of $h$”. We’ll use this manipulation all the time later on.) Show
that $G(t) = \frac{1}{1-\Sigma(t)}$.

(c) Argue by diagrams for the equation (sometimes this is also called a Schwinger-
Dyson equation)

\[ \sum = \begin{array}{c} \includegraphics[scale=0.5]{diagram.png} \\ G \end{array} \]

where $\Sigma$ is the 1PI self-energy of $h$.

(d) Solve this equation for the generating function $G(t)$.

(e) If you are feeling ambitious, add another coupling $N^{-1}$ which counts the
crossings of the $l$ propagators. The resulting numbers can be called Touchard-
Riordan numbers.

(f) How to realize the no-crossings rule? Consider

$$L = \frac{\sqrt{t}}{\sqrt{N}} l_{\alpha\beta} h_{\alpha} h_{\beta} + \sum_{\alpha,\beta} l_{\alpha\beta}^2 + \sum_{\alpha} h_{\alpha}^2$$

where $\alpha, \beta = 1 \cdots N$. By counting index loops, show that the dominant
diagrams at large $N$ are the ones we kept above. Hint: to keep track of the
index loops, introduce ('t Hooft’s) double-line notation: since $l$ is a matrix,
it’s propagator looks like: $\beta \ldots \alpha$, while the $h$ propagator is just
one index line $\alpha \ldots \alpha$, and the vertex is __!__. If you don’t like my ascii diagrams, here are the respective pictures: $\langle l_{\alpha\beta} l_{\alpha\beta} \rangle = \begin{array}{c}

\end{array}$, $\langle h_{\alpha} h_{\alpha} \rangle = \begin{array}{c}

\end{array}$ and the $hhl$ vertex is: ___.

(g) Use properties of Catalan numbers to estimate the size of non-perturbative effects in this field theory. $C_n$ grows more slowly than $\frac{\alpha}{n^2}$ with $c > \frac{4}{3}\pi^2$. [van Lint and Wilson, A Course in Combinatorics, p. 138] So actually the series $\sum_n C_n t^n$ converges. No non-perturbative effects here.

(h) There are many other examples like this. Another similar one is the relationship between symmetric functions and homogeneous products. A more different one is the enumeration of planar graphs. For that, see BIPZ.

4. **Brain-warmer: the identity does nothing twice.** Check our relativistic state normalization by squaring the expression for the identity in the 1-particle sector:

$$\mathbb{1}^2 = \mathbb{1} = \int \frac{d^d p}{2\omega p} |\vec{p}\rangle \langle \vec{p}|.$$ 

$|\vec{p}\rangle = \sqrt{2\omega_p a^\dagger_p |0\rangle}$, so

$$\mathbb{1}^2 = \int \frac{d^d p}{2\omega p} |\vec{p}\rangle \langle \vec{p}| \int \frac{d^d p'}{2\omega p'} |\vec{p}'\rangle \langle \vec{p}'| = \int d^d p \int d^d p' a^\dagger_p |0\rangle \langle 0| a_p a^\dagger_p |0\rangle \langle 0| a_{p'}$$

$$= \int d^d p a^\dagger_p |0\rangle \langle 0| a_p = \int \frac{d^d p}{2\omega p} |\vec{p}\rangle \langle \vec{p}| = \mathbb{1}$$