1. **Brain-warmer.** Explain why the Wilson-Fisher fixed point in \(2 < d < 4\) has only two relevant operators, which may be associated with \(\phi\) and \(\phi^2\). What happened to \(\phi^3\)?

2. **OPE.** Consider the Gaussian fixed point with \(O(n)\) symmetry. Compute the OPE coefficients for the operators \(O_2 \equiv \phi_a \phi_a\), \(O_4 \equiv (\phi_a \phi_a)^2\), and the identity operator. Use this information to compute the beta function, find the Wilson-Fisher fixed point and the correlation length critical exponent \(\nu\) there.

3. **A proof of the Mermin-Wagner-Hohenberg-Coleman theorem.**

   (a) **Bogoliubov-Schwartz inequality.** Convince yourself that
   \[
   \langle AA^*\rangle\langle BB^*\rangle \geq |\langle AB^*\rangle|^2
   \]  
   (1)
   where \(A, B\) are anything, and \(\langle \ldots \rangle\) means some statistical average.

   Now consider a microscopic realization of an XY model in \(d = 2\) on a lattice with \(N\) total sites. At each site we place a 2-component unit-normalized spin \(\vec{S}_i = (\cos \phi_i, \sin \phi_i)\). The Hamiltonian is
   \[
   -H = \sum_{\langle ij \rangle} J \cos (\phi_i - \phi_j) + h \sum_i \cos \phi_i.
   \]

   The magnetization is \(m = \langle \cos \phi_i \rangle\). The claim is that for \(d \leq 2\), \(\lim_{h \to 0} m\) must vanish in the thermodynamic limit – no spontaneous breaking of the continuous symmetry \(\phi_i \to \phi_i + \alpha\).

   To apply (1), we will (with cold-blooded foresight) choose
   \[
   A \equiv \frac{1}{N} \sum_j e^{-i\vec{q} \cdot \vec{r}_j} \sin \phi_j, \quad B \equiv \frac{1}{N} \sum_k e^{-i\vec{q} \cdot \vec{r}_k} \partial_{\phi_k} H.
   \]

   (b) Show that
   \[
   \sum_q \langle AA^* \rangle = \frac{1}{N} \sum_j \langle \sin^2 \phi_j \rangle \leq 1.
   \]
(c) Show that
\[ \langle AB^* \rangle = \frac{Tm}{N}. \]

Here’s a hint:
\[ 0 = \frac{1}{Z} \int_0^{2\pi} \prod_i d\phi_i \partial_{\phi_i} \left( e^{-H/T} \sin \phi_j \right). \]

(This kind of relation is sometimes called a Ward identity.)

(d) Show that
\[ \langle BB^* \rangle = \frac{T}{N^2} \sum_{ij} e^{-iq(r_i-r_j)} \langle \partial_{\phi_i} \partial_{\phi_j} H \rangle. \]

Hint: use the same trick as in the previous part.

(e) Show that (I have in mind a hypercubic lattice with coordination number \( z = 2d \))
\[ \langle BB^* \rangle \leq \frac{T}{N} \left( h + J \left( z - 2 \sum_{\mu=1}^d e^{-iq\mu} \right) \right) \leq \frac{T}{N} \left( h + J q^2 \right) \]

(f) Conclude that
\[ 1 \geq \sum_q \langle AA^* \rangle \geq \frac{Tm^2}{N^2} \sum_q \frac{1}{h + J q^2}. \]

Take the thermodynamic limit, and argue that the resulting inequality requires
\[ \lim_{h \to 0} m = 0 \text{ for } d \leq 2. \]

(g) [optional bonus question] Generalize the argument to the \( O(n) \) model.

4. **Long-range interactions and the lower critical dimension.** Consider perturbing an \( O(n) \) model by long-range interaction of the form
\[ \Delta H = g \int d^d q \Phi_a(q)|q|^r \Phi_a(-q). \]

(a) [optional] What does \( \Delta H \) look like in position space? What does
\[ \int d^d x d^d y \sum_a \frac{(\Phi_a(x) - \Phi_a(y))^2}{|x-y|^{d+r}} \]
look like in momentum space?

(b) Find the lower critical dimension as a function of \( r \).
5. Perturbative RG for worldsheet (Edwards-Flory) description of SAWs.

In this problem we make quantitative the analogy

unrestricted RW:SAW::gaussian fixed point:WF fixed point.

Parametrize a continuous-time random walk in $d$ dimensions by a trajectory $\bar{r}(t)$. Consider the Edwards hamiltonian

$$H[\bar{r}] = \frac{K}{2} \int_0^L ds \left( \frac{d\bar{r}}{ds} \right)^2 + \frac{u}{2} \int_{|s_1-s_2|>a} ds_1 ds_2 \delta^d[\bar{r}(s_1) - \bar{r}(s_2)]$$

with a self-avoidance coupling $u > 0$, a short-distance cutoff $a$, and an IR cutoff $L$. We would like to understand the large-$L$ scaling of the polymer size, $R$,

$$R^2(L) \equiv \langle |\bar{r}(L) - \bar{r}(0)|^2 \rangle \sim L^{2\nu}.$$ 

(a) Consider the probability density for two points a distance $|s_1 - s_2|$ along the chain to be separated in space by a displacement $\vec{x}$,

$$P(\vec{x}; s_1 - s_2) = \langle \delta^d[\bar{r}(s_1) - \bar{r}(s_2) - \vec{x}] \rangle.$$

Show that the polymer size $R$ can be obtained from its fourier transform $\tilde{P}(\vec{q}; s)$ by the relation

$$R^2(L) = -\nabla^2_{\vec{q}} \tilde{P}(\vec{q}; L)|_{q=0}.$$

(So far this does not involve a choice of hamiltonian.)

(b) For the free case $u = 0$, compute the equilibrium polymer size $R_0(L)$ in terms of $d, L, K$. It may be helpful to derive a relation of the form

$$\langle e^{\frac{1}{L} \int_0^L ds E(s) \cdot \bar{r}(s)} \rangle_0 = e^{\frac{1}{2\pi} \int_0^L ds ds' E(s) \cdot \bar{r}(s') G(s-s')}.$$

(c) Develop an expansion of $\tilde{P}(\vec{q}; L)$ to first order in $u$, using the cumulant expansion as in §6.6 of the lecture notes. You should find an expression of the form $R^2(L) = R_0^2(L) (1 + \delta R_1^2(L) + O(u^2))$ with

$$\delta R_1^2(L) = \frac{u}{L} \left( \frac{K}{2\pi} \right)^{d/2} \int_0^L ds_1 \int_{s_1+a}^L ds_2 \frac{A(s_1, s_2; L)}{|s_1 - s_2|^{d-2}}.$$

(d) Show that the integrals in the previous part diverge as $a/L \to 0$ below a certain dimension $d_c$. More precisely, by changing variables to $s = s_1 - s_2$ and $\bar{s} = (s_1 + s_2)/2$ (and ignoring stuff at the upper limit of integration, as appropriate for $L \gg a$) show that

$$\delta R_1^2(L) \simeq u \left( \frac{K}{2\pi} \right)^{d/2} \int_a^L ds s^{\epsilon - 1}$$

with $\epsilon = d_c - d$. 

3
(e) How does $\vec{r}$ scale with $s \mapsto bs$ if we demand that the free hamiltonian ($u = 0$) is a fixed point? What is $\nu$ at the free fixed point?

(f) Find $d_c$ by power counting.

(g) We wish to integrate out the short distance fluctuations with wavelengths between $a$ and $ba$, to find an effective Hamiltonian governing the remaining degrees of freedom:

$$\tilde{H}[\vec{r}] = \frac{\tilde{K}}{2} \int_0^L ds \left( \frac{d\vec{r}}{ds} \right)^2 + \frac{\tilde{u}}{2} \int_{|s_1 - s_2| > ba} ds_1 ds_2 \delta^d[\vec{r}(s_1) - \vec{r}(s_2)]$$

Using the first-order-in-$u$ result for $\delta R$ above, show that for small $\epsilon$ and small $\log b$, the coarse-grained ‘stiffness’ parameter is of the form

$$\tilde{K} = K(1 - \bar{v} \log b)$$

and find $\bar{v}$.

(h) A similar calculation yields $\bar{v} = v(1 - 2\bar{v} \log b)$. Do rescaling step of the RG procedure, redefining $s$ by a factor of $b = 1 + \ell + O(\ell^2)$ and rescaling the $\vec{r} \to Z(b)\vec{r}$ to put the Hamiltonian back in the original form with the original cutoff and renormalized parameters $K', v'$.

(i) Find the beta functions for $K(\ell)$ and $v(\ell)$. Find $\nu$ to first order in $\epsilon$.

6. Self-avoiding membranes?

[Slightly open-ended.] Consider redoing the Edwards-Flory analysis for a theory of membranes. The fields are now $\vec{r}(\sigma_1, \sigma_2, \sigma_D)$, vectors parametrizing the embedding of a $D$-dimensional object into $\mathbb{R}^d$. We might consider perturbing the Gaussian action

$$S_0[\vec{r}] = \int d^D\sigma \sum_{\alpha=1}^D (\partial_{\sigma_\alpha} \vec{r})^2$$

by a self-avoidance term

$$S_u[\vec{r}] = \int d^D\sigma \int d^D\sigma' \delta^d(\vec{r}(\sigma) - \vec{r}(\sigma')).$$

For various $d$ and $D$, what does the Flory argument predict for the scaling exponent of the brane size with the linear size $L$ of the base space? For which values is the excluded-volume term relevant?

Are there other terms we should consider in the action?

Try to resist googling before you think about this question.