University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Fall 2019
Assignment 5

Due 12:30pm Monday, November 4, 2019

1. **Distinguishability of distributions.** Suppose we sample N times a distribution P on a binary variable with $(p_0, p_1) = (p, 1-p)$. What is the probability that we mistake the distribution for Q with probabilities $(q, 1-q)$?

   Hint: the expected number of zeros $\langle n_0 \rangle_P$ is Np. The probability that we get it wrong is the probability that we get Nq zeros instead. Show that

   $$\text{Prob}(n_0 = Nq|P) \simeq 2^{-ND(Q||P)}$$

   where $D(Q||P)$ is the relative entropy, and the approximation is Stirling's.

2. **Brain-warmer: Entanglement cannot be created locally.** Consider a bipartite hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Define a *local unitary* to be an operator of the form $\mathbf{U}_A \otimes \mathbf{U}_B$ where $\mathbf{U}_{A,B}$ acts only on $\mathcal{H}_{A,B}$. These are the operations that can be done by actors with access only to A *or* B. Show that by acting on a state of $\mathcal{H}$ with a local unitary we cannot change the Schmidt number or the entanglement entropy of either factor. Consider both the case of a pure state of $\mathcal{H}$ and a mixed state of $\mathcal{H}$; note that the action of a unitary $\mathbf{U}$ on a density matrix $\boldsymbol{\rho}$ is

   $$\boldsymbol{\rho} \to \mathbf{U}\boldsymbol{\rho}\mathbf{U}^\dagger.$$

   [We conclude from this that to create an entangled state from an unentangled state, we must bring the two subsystems together and let them interact, resulting in a more general unitary evolution than a local unitary.]

3. **Copying classical information is OK.** Construct a *linear* operator $\mathcal{O}$ on a system of two qbits which acts as follows on the computational basis states of the first qbit:

   $$\mathcal{O}\,|0\rangle \otimes |a\rangle = |0\rangle \otimes |0\rangle\,, \quad \mathcal{O}\,|1\rangle \otimes |a\rangle = |1\rangle \otimes |1\rangle\,,$$

   for any computational-basis state $|a\rangle$ of the second qbit.

   What is $\mathcal{O}\,(\cos\theta\,|0\rangle + \sin\theta\,|1\rangle) \otimes |a\rangle$?

   [We can think of this operator as copying *classical* information: if we are forced (*e.g.* by decoherence) to remain in the computational basis, the information in the first qbit is just a classical bit; the operator $\mathcal{O}$ copies this classical bit into the second register. This shows that the quantum no-cloning theorem does not forbid the cloning of classical information.]

Extra credit: show that it is not possible to construct such a linear operator which acts as above for *arbitrary* states $|a\rangle$ of the second qbit.

4. **Phase-damping channel.**

   (a) Show that the Kraus operators given in lecture indeed reproduce the action of the phase-damping channel.

   (b) Show that the Kraus operators given in lecture indeed result from the given unitary action U_{AE} on the combined system and environment.

5. **Amplitude-damping channel.** This is a very simple model for a two-level atom, coupled to an environment in the form of a (crude rendering of a) radiation field.

   The atom has a groundstate $|0\rangle_A$; if it starts in this state, it stays in this state, and the radiation field stays in its groundstate $|0\rangle_E$ (zero photons). If the atom starts in the excited state $|1\rangle_A$, it has some probability p per time dt to return to the groundstate and emit a photon, exciting the environment into the state $|1\rangle_E$ (one photon). This is described by the time evolution

$$\mathbf{U}_{AE} |0\rangle_A \otimes |0\rangle_E = |0\rangle_A \otimes |0\rangle_E$$

$$\mathbf{U}_{AE} |1\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |1\rangle_A \otimes |0\rangle_E + \sqrt{p}\, |0\rangle_A \otimes |1\rangle_E \,.$$

   (a) Show that the evolution of the atom's density matrix can be written in terms of two Kraus operators $\mathcal{K}_i$, find those operators and show that they satisfy $\sum_i \mathcal{K}_i^\dagger \mathcal{K}_i = 1\!\!1_{\text{atom}}$.

   (b) Assuming that the environment is forgetful and resets to $|0\rangle_E$ after each time step dt, find the fate of the density matrix after time $t = ndt$ for late times $n \gg 1$, *i.e.* upon repeated application of the channel.

   (c) Evaluate the *purity* $\mathrm{tr}\boldsymbol{\rho}_n^2$ of the nth iterate. (Recall that the purity is 1 IFF the state is pure.)

6. **Phase-flipping decoherence channel.** Consider the following model of decoherence on an N-state Hilbert space, with basis $\{|k\rangle\,, k = 1..N\}$.

   Define the unitary operator

$$\mathbf{U}_\alpha \equiv \sum_k \alpha_k |k\rangle \langle k|$$

where α_k is an N-component vector of signs, ± 1 – it flips the signs of some of the basis states. There are 2^N distinct such operators.

Imagine that interactions with the environment act on any state of the system with the operator $\mathbf{U}_\alpha$, for some α, chosen randomly (with uniform probability from the 2^N choices).

[Hint: If you wish, set $N = 2$.]

(a) Warmup question: If the initial state is $|\psi\rangle$, what is the probability that the resulting output state is $\mathbf{U}_\alpha |\psi\rangle$?

(b) Write an expression for the resulting density matrix, $\mathcal{D}(\boldsymbol{\rho})$, in terms of $\boldsymbol{\rho}$.

(c) Think of $\mathcal{D}$ as a superoperator, an operator on density matrices. How does $\mathcal{D}$ act on a density matrix which is diagonal in the given basis,

$$\boldsymbol{\rho}_{\text{diagonal}} = \sum_k p_k |k\rangle \langle k| \quad ?$$

(d) The most general initial density matrix is not diagonal in the k-basis:

$$\boldsymbol{\rho}_{\text{general}} = \sum_{kl} \rho_{kl} |k\rangle \langle l| \quad .$$

what does $\mathcal{D}$ do to the off-diagonal elements of the density matrix?

7. **Brainwarmers on Kraus operators.**

(a) Check that the Kraus operators

$$\mathcal{K}_i = \langle i|U|0\rangle$$

(where U is a unitary on $A \otimes \bar{A}$, $\{|i\rangle\}$ is an ON basis of $\bar{A}$, and $|0\rangle$ is a reference state in $\bar{A}$) satisfy the condition

$$\sum_i \mathcal{K}_i^\dagger \mathcal{K}_i = \mathbb{1}_A \tag{1}$$

(b) Check that the condition (1) implies that the Kraus evolution $\boldsymbol{\rho} \to \sum_i \mathcal{K}_i \boldsymbol{\rho} \mathcal{K}_i^\dagger$ preserves the trace.

(c) Find a set of Kraus operators for the erasure (or reset) channel that takes $\boldsymbol{\rho} \mapsto |0\rangle\langle 0|$ for every $\boldsymbol{\rho}$. Check that they satisfy (1).

(d) **Stationary states of unital channels.** Check that the unital condition $\sum_i \mathcal{K}_i \mathcal{K}_i^\dagger = \mathbb{1}$ implies that the uniform density matrix $\mathbf{u} \equiv \mathbb{1}\frac{1}{|\mathcal{H}|}$ is a fixed point of the associated quantum channel, $\mathcal{E} : \mathcal{H} \to \mathcal{H}$,

$$\mathcal{E}(\mathbf{u}) = \mathbf{u}.$$

8. **Turtles all the way down.** [optional, open-ended]

A question you may have about our discussion of polarization-damping as a model of decoherence is: why does the environment reset to the reference state $|0\rangle_E$?

We can postpone the question a bit by coupling the environment to its own environment, according to an amplitude damping channel. On the previous problem set, you saw that the result of the repeated action of such a channel can set $\boldsymbol{\rho}_E = |0\rangle\langle 0|$. This statement in turn assumes a forgetful meta-environment. A thermodynamic limit is required to postpone the question indefinitely. Construct such a thermodynamic limit.